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SOME NEW TOPOLOGICAL PROPERTIES IN CONE METRIC SPACES

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Abstract: In this paper, we study some properties with respect to cones in cone metric spaces. By using the notions of completeness, we obtain nested closed-ball theorem in such spaces, which improves some previous conclusions in metric spaces.

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1 Introduction

Nonlinear functional analysis, especially ordered normed spaces, had some applications in optimization theory [1]. In these cases a partial ordering "$\leq$" by means of which certain elements can be compared better than crude estimates in terms of a norm, is introduced by using vector space cones. In 2007, Huang and Zhang [3] defined the cone metric spaces with a different view. In fact, they substituted a normed space instead of the real line, but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. Since then, many scholars focused on the investigations in such spaces [4–5]. In recent years, some topological properties in cone metric spaces became the center of strong research activities [6–8]. Throughout this paper, we give some properties on cones. Furthermore, we present nested closed-ball theorem in cone metric spaces. All results directly generalize and replenish some assertions in metric spaces and some previous results in cone metric spaces.

We need the following definitions and results, consistent with [3], in the sequel.

Let E be a real Banach spaces and P a subset of E. By θ we denote the zero element of E and by int$P$ the interior of P. The subset P is called a cone if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

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Based on this, a partial ordering $\leq$ with respect to P is defined as $x \leq y$ if and only if $y - x \in P$. $x < y$ stands for $x \leq y$ but $x \neq y$, while $x \ll y$ indicates $y - x \in \mathrm{int}P$. Write $\|\cdot\|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose that E is a Banach space, P is a cone in E with $\mathrm{int}P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 1.1 [3] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.2 [4] Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$).

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

2 Main Results

In this section, we firstly introduce some useful lemmas which are often appeared in the articles. It is usual about their proofs. Secondly, we display some meaningful assertions such as necessary and sufficient conditions about normal cone, nested closed-ball theorem and so on in cone metric spaces.

Lemma 2.1 [11] Let P be a cone and $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.

Remark 2.1 Following an argument similar to that given in Lemma 2.1, letting P be a cone and $x, y, z \in E$, we suffice to prove that

(1) if $x \ll y$ and $y \leq z$, then $x \ll z$;

(2) if $x \ll y$ and $y \ll z$, then $x \ll z$.

Lemma 2.2 [1] Let P be a cone, then $\theta \notin \mathrm{int}P$.

Lemma 2.3 [4] Let P be a cone. We have the following properties:

(1) If $x \in P$ and $y \in \mathrm{int}P$, then $x + y \in \mathrm{int}P$, that is, $P + \mathrm{int}P \subseteq \mathrm{int}P$. In particular, $\mathrm{int}P + \mathrm{int}P \subseteq \mathrm{int}P$.

(2) If $x \in \mathrm{int}P$, $\lambda > 0$, then $\lambda x \in \mathrm{int}P$, that is, $\lambda\, \mathrm{int}P \subseteq \mathrm{int}P$.

Remark 2.2 It is easy to see that Lemma 2.1 is equivalent to (1) of Lemma 2.3. In fact, if $x \leq y$ and $y \ll z$, then $y - x \in P$ and $z - y \in \mathrm{int}P$. Taking advantage of (1) of Lemma 2.3, we obtain $z - x = (y - x) + (z - y) \in \mathrm{int}P$, i.e., $x \ll z$.

Lemma 2.4 [2] If $c \in \mathrm{int}P$ and $\theta \leq a_n \leq b_n \to \theta$ (as $n \to \infty$), then there exists N such that, for all $n > N$, we have $a_n \ll c$.

Proof Since $c \in \mathrm{int}P$, there exists $\delta \geq 0$ such that $U(c, \delta) = \{x \in E : \|x - c\| < \delta\} \subseteq P$. Note that $b_n \to \theta$, then for the above δ, there is N such that for all $n > N$, $\|b_n\| < \delta$. Thus $\|c - b_n - c\| = \|b_n\| < \delta$, i.e., $c - b_n \in U(c, \delta) \subseteq P$. It is clear that $b_n \ll c$ all $n > N$. So by Lemma 2.1, we arrive at $a_n \ll c$ for all $n > N$.

Theorem 2.1 Let P be a cone in E, then P is normal if and only if the following conditions are satisfied: for arbitrary sequence $\{a_n\}$ in P, and for each $c \gg \theta$, if there exists N such that for all $n > N$, it satisfies $a_n \ll c$, then $a_n \to \theta$ (as $n \to \infty$).

Proof Suppose that P is a normal cone with the normal constant K. For any $\varepsilon > 0$, choose $c \in E$ with $c \gg \theta$ and $K\|c\| < \varepsilon$. By the given conditions, we have $\|a_n\| \leq K\|c\| < \varepsilon$ for all $n > N$, then $a_n \to \theta$ (as $n \to \infty$).

Conversely, we suppose for absurd that P is not normal. There exist $u_n, v_n \in P$ such that $\theta \leq u_n \leq v_n$ and $\|u_n\| > n\|v_n\|$ $(n = 1, 2, \cdots)$. Put

$$a_n = -\frac{u_n}{\|u_n\|} + \frac{v_n}{n\|v_n\|}.$$

It concludes that $a_n \in P$ and $\theta \leq a_n \leq \dfrac{v_n}{n\|v_n\|}$. Because of $\dfrac{v_n}{n\|v_n\|} \to \theta$ (as $n \to \infty$), we deduce by Lemma 2.4 that, for all $c \gg \theta$, there exists N such that for all $n > N$, $a_n \ll c$. But $\|a_n\| \geq 1 - \dfrac{1}{n}$, which leads to $a_n \nrightarrow \theta$ (as $n \to \infty$). This is a contradiction.

Remark 2.3 For each $c \gg \theta$, if P is non-normal and there exists N, then we do not demonstrate by Theorem 2.1 that $a_n \to \theta$ (as $n \to \infty$) even if $a_n \ll c$ for all $n > N$. Based on this statement, if (X, d) is a cone metric space, $x \in X$ and $\{x_n\} \subseteq X$ satisfy $x_n \to x$ (as $n \to \infty$), then it does not always get $d(x_n, x) \to \theta$ (as $n \to \infty$) unless P is normal. Therefore, Lemma 1.1 of [9] or Lemma 1.2 of [10] is incorrect. It need to remind that this lemma does not exist in its given reference [6]. The following example illustrates our conclusions.

Example 2.1 Let $E = C^1_{\mathbb{R}}([0,1])$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$, $P = \{x \in E : x(t) \geq 0\}$. This cone is non-normal. As a matter of fact, choose $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \leq x_n \leq y_n$, and $y_n \to \theta$ (as $n \to \infty$). By virtue of Lemma 2.4, for any $c \gg \theta$, it is easy to see that there exists N such that for all $n > N$, $x_n \ll c$. But

$$\|x_n\| = \max_{t \in [0,1]} |\frac{t^n}{n}| + \max_{t \in [0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1,$$

hence $x_n \nrightarrow \theta$ (as $n \to \infty$). Further, define the cone metric $d : P \times P \to E$ by $d(x, y) = x + y$, $x \neq y$, $d(x, x) = \theta$. Then (P, d) is a cone metric space. We have $d(x_n, \theta) = x_n \ll c$ for each $c \gg \theta$, that is, $x_n \to \theta$ in P, but $d(x_n, \theta) = x_n \nrightarrow \theta$ in E.

Theorem 2.2 Let (X, d) be a cone metric space, $\{x_n\}$ and $\{y_n\}$ be two different Cauchy sequences in X. If $\{d(x_n, y_n)\}$ is not convergent in E, then P must be a non-normal cone.

Proof Suppose that P is a normal cone with the normal constant K. Set $\varepsilon > 0$ and choose $c \in E$ with $c \gg \theta$ and $\|c\| < \frac{\varepsilon}{4K+2}$. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, there is N such that for all $n, m > N$, it satisfies $d(x_n, x_m) \ll c$ and $d(y_n, y_m) \ll c$. Note that

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \le d(x_m, y_m) + 2c, \tag{2.1}$$

$$d(x_m, y_m) \le d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \le d(x_n, y_n) + 2c. \tag{2.2}$$

It follows immediately from (2.1) and (2.2) that

$$\theta \le d(x_m, y_m) + 2c - d(x_n, y_n) \le d(x_n, y_n) + 2c + 2c - d(x_n, y_n) = 4c. \tag{2.3}$$

Because of the normality of P, (2.3) means that

$$\|d(x_m, y_m) + 2c - d(x_n, y_n)\| \le 4K\|c\|.$$

Hence, it ensures us that

$$\|d(x_m, y_m) - d(x_n, y_n)\| \le \|d(x_m, y_m) + 2c - d(x_n, y_n)\| + \|2c\| \le (4K+2)\|c\| < \varepsilon,$$

which implies that $\{d(x_n, y_n)\}$ is Cauchy. Making full use of the completeness of $(E, \|\cdot\|)$, we speculate $\{d(x_n, y_n)\}$ is convergent. This is a contradiction.

Theorem 2.3 Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X, then the following is equivalent:

(1) (X, d) is complete.

(2) Suppose $S_n = \{x : d(x, x_n) \le r_n\} \subseteq X$ and $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supseteq \cdots$, if $r_n \to \theta$ (as $n \to \infty$), then there exists a unique point $x \in \bigcap\limits_{n=1}^{\infty} S_n$.

Proof Assume that (X, d) is complete. If $m \ge n$, then the fact $x_m \in S_m \subseteq S_n$ implies $d(x_n, x_m) \le r_n \to \theta$ (as $n \to \infty$). It includes by Lemma 2.4 that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there is $x \in X$ such that $x_n \to x$ (as $n \to \infty$). Thus for each $c \gg \theta$, there exists N_1 such that for all $m > N_1$ and any $k > 0$, $d(x_m, x) \ll \frac{c}{k}$. Noticing that

$$d(x, x_n) \le d(x_n, x_m) + d(x_m, x) \le r_n + d(x_m, x) \le r_n + \frac{c}{k},$$

we speculate $r_n + \frac{c}{k} - d(x, x_n) \in P$. Since P is closed, letting $k \to \infty$, we have

$$r_n - d(x, x_n) \in P,$$

that is, $d(x, x_n) \le r_n$, i.e., $x \in S_n$. Hence $x \in \bigcap\limits_{n=1}^{\infty} S_n$. In addition, if there is another point $y \in \bigcap\limits_{n=1}^{\infty} S_n$, then $d(y, x_n) \le r_n$. For each $c \gg \theta$, since $r_n \to \theta$ and $x_n \to x$, there exists N_2 such that for all $n > N_2$ and any $k > 0$, we get $r_n \ll \frac{c}{2k}$ and $d(x_n, x) \ll \frac{c}{2k}$. Note that

$$d(y, x) \le d(y, x_n) + d(x_n, x) \le r_n + d(x_n, x) \ll \frac{c}{2k} + \frac{c}{2k} = \frac{c}{k},$$

so by Lemma 2.1 it establishes $d(y, x) \ll \frac{c}{k}$. This leads to $\frac{c}{k} - d(y, x) \in P$. Because P is closed, set $k \to \infty$, it follows $-d(y, x) \in P$, that is to say, $y = x$.

On the contrary, suppose (2) is satisfied and $\{x_n\}$ is a Cauchy sequence in X. Put $c_0 > \theta$, there exist $n_1 < n_2 < \cdots < n_k < \cdots$, such that for all $n, m \geq n_k$, $d(x_n, x_m) \ll \frac{c_0}{2^{k+1}}$. Write a class of balls as

$$S(x_{n_k}, \frac{c_0}{2^k}) = \{x : d(x, x_{n_k}) \leq \frac{c_0}{2^k}\} \subseteq X,$$

$k = 1, 2, \cdots$. Choose $y \in S(x_{n_{k+1}}, \frac{c_0}{2^{k+1}})$. Owing to

$$d(x_{n_k}, y) \leq d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, y) \leq \frac{c_0}{2^{k+1}} + \frac{c_0}{2^{k+1}} = \frac{c_0}{2^k},$$

we obtain

$$S(x_{n_{k+1}}, \frac{c_0}{2^{k+1}}) \subseteq S(x_{n_k}, \frac{c_0}{2^k}).$$

In view of (2), there exists a unique point $x \in \bigcap\limits_{k=1}^{\infty} S(x_{n_k}, \frac{c_0}{2^k})$, moreover,

$$d(x, x_{n_k}) \leq \frac{c_0}{2^k}.$$

Now that $\{x_n\}$ is a Cauchy sequence, for each $c \gg \theta$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll \frac{c}{2}$. Since $d(x, x_{n_k}) \leq \frac{c_0}{2^k} \to \theta$ (as $k \to \infty$), by Lemma 2.4, for the above c and N, there exists k_0 such that $n_{k_0} > N$ and $d(x_{n_{k_0}}, x) \ll \frac{c}{2}$. Thus for all $n > N$, we deduce that

$$d(x_n, x) \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) \ll \frac{c}{2} + \frac{c}{2} = c,$$

which means $x_n \to x$ (as $n \to \infty$). Therefore (X, d) is complete.

Remark 2.4 Theorem 2.3 is called as nested closed-ball theorem, which generalizes the theorem of nested interval from metric spaces to cone metric spaces. The results may bring us more convenience in applications. Indeed, it tell us a necessary and sufficient condition on completeness for cone metric spaces.

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锥度量空间中的一些新的拓扑性质

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摘要: 本文给出了锥度量空间中的锥的一些性质. 利用完备性概念, 得到了此空间中的闭球套定理, 改进了前人在度量空间中的相应结果.

关键词: 正规锥; 锥度量空间; 闭球套定理

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