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# LIOUVILLE TYPE THEOREMS OF $f$-STATIONARY MAPS OF A FUNCTIONAL RELATED TO PULLBACK METRICS

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**Abstract:** In this paper, we investigate a generalized functional $\Phi_f$ related to the pullback metric. By using the method of $f$-stress energy tensor, we obtain the monotonicity formulas, vanishing theorems and the unique constant solution of the constant Dirichlet boundary value problem on some starlike domain for $f$-stationary maps.

**Keywords:** $f$-stationary map; $f$-stress-energy tensor; monotonicity formula; vanishing theorems

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## 1 Introduction

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds $(M^m, g)$ and $(N^n, h)$. Recently, Kawai and Nakauchi [1] introduced a functional related to the pullback metric $u^*h$ as follows:

$$\Phi(u) = \frac{1}{4}\int_M ||u^*h||^2 dv_g \tag{1.1}$$

(see [2–4]), where $u^*h$ is the symmetric 2-tensor defined by

$$(u^*h)(X, Y) = h(du(X), du(Y))$$

for any vector fields $X$, $Y$ on $M$ and $||u^*h||$, its norm as $||u^*h||^2 = \sum_{i,j=1}^{m} [h(du(e_i), du(e_j))]^2$ with respect to a local orthonormal frame $(e_1, \cdots, e_m)$ on $(M, g)$. The map $u$ is stationary for $\Phi$ if it is a critical point of $\Phi(u)$ with respect to any compact supported variation of $u$ and $u$ is stationary stable if the second variation for the functional $\Phi(u)$ is non-negative. They showed that the non-existence of nonconstant stable stationary map for $\Phi$, either from $S^m$ $(m \geq 5)$ to any manifold, or from any compact Riemannian manifold to $S^n$ ( $n \geq 5$).

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On the other hand, Lichnerowicz in [5] (also see [6]) introduced the $f$-harmonic maps, generalizing harmonic maps. Let $f : (M, g) \to (0, +\infty)$ be a smooth function. A smooth map $u : M \to N$ is said to be an $f$-harmonic map if it is a critical point of the following $f$-energy

$$E_f(u) = \int_M f \frac{||du||^2}{2} dv_g$$

with respect to any compactly supported variation of $u$, where $||du||$ is the Hilbert-Schmidt norm of the differential of $u$:

$$||du||^2 = \text{trace}_g u^* h = \sum_{i=1}^{m} h(du(e_i), du(e_i)).$$

The Euler-Lagrange equation gives the $f$-harmonic map equation (see [7–9])

$$\tau_f(u) = f\tau(u) + du(\text{grad} f) = f\text{trace}_g(\nabla du) + du(\text{grad} f). \tag{1.2}$$

Ara [10] introduced the $F$-harmonic maps, generalizing harmonic maps. Let $F : [0, \infty) \to [0, \infty)$ be a $C^2$ function such that $F(0) = 0$ and $F'(t) > 0$ for $t \in [0, \infty)$. A smooth map $u : M \to N$ is said to be an $F$-harmonic map if it is a critical point of the following $F$-energy functional $E_F$ given by

$$E_F(u) = \int_M F(\frac{||du||^2}{2}) dv_g$$

with respect to any compactly supported variation of $u$. The Euler-Lagrange equation gives the $F$-harmonic map equation

$$\begin{aligned} \tau_F(u) &= F'(\frac{||du||^2}{2})\tau(u) + du(\text{grad} F'(\frac{||du||^2}{2})) \\ &= F'(\frac{||du||^2}{2})\text{trace}_g(\nabla du) + du(\text{grad} F'(\frac{||du||^2}{2})). \end{aligned} \tag{1.3}$$

There were many results for $F$-harmonic maps such as [11–13]. From (1.2) and (1.3), we know that any $F$-harmonic map is a special $f$-harmonic map.

Recently, Dong and Ou in [14] introduced the stress energy tensor $S_f$ associated with $E_f$-energy as following:

$$S_f = f[\frac{||du||^2}{2} g - h(du(.), du(.))].$$

Via the stress-energy tensor $S_f$ of $E_f$, monotonicity formula and Liouville-type results were investigated in [14].

In this paper, we generalize and unify the concept of critical point of the functional $\Phi$. For this, we define the functional $\Phi_f$ by

$$\Phi_f(u) = \int_M f(x) \frac{||u^* h||^2}{4} dv_g, \tag{1.4}$$

which is $\Phi$ if $f = 1$. We call $u$ an $f$-stationary map for $\Phi_f(u)$, if $\frac{d}{dt}\Phi_f(u_t)|_{t=0} = 0$ for any compactly supported variation $u_t : M \to N$ with $u_0 = u$. We derive the first variation formula of $\Phi_f$ and we introduce the $f$-stress energy tensor $S_{\Phi_f}$ associated to $\Phi_f$. Then we use the $f$-stress energy tensor to obtain the monotonicity formula and vanishing theorems for $f$-stationary map under some conditions on $f$. The monotonicity formulas can also be used to investigate the constant Dirichlet boundary value problem. We also obtain the unique constant solution of the constant Dirichlet boundary value problem on some starlike domain for $f$-stationary map.

## 2 The First Variation Formula for $\Phi_f(u)$

Let $\nabla$ and ${}^N\nabla$ always denote the Levi-Civita connections of $M$ and $N$ respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_X W = {}^N\nabla_{du(X)}W$, where $X$ is a tangent vector of $M$ and $W$ is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ on $M$. We define the $f$-tension field $\tau_{\Phi_f}(u)$ of $u$ by

$$\tau_{\Phi_f}(u) = -\delta(f\sigma_u) = f\mathrm{div}\sigma_u + \sigma_u(\mathrm{grad}f), \tag{2.1}$$

where $\sigma_u = \sum_j h(du(.), du(e_j))du(e_j)$, which was defined in [1].

Under the notation above we have the following:

**Lemma 2.1** (The first variation formula) Let $u : M \to N$ be a $C^2$ map. Then

$$\frac{d}{dt}\Phi_f(u_t)|_{t=0} = -\int_M h(\tau_{\Phi_f}(u), V)dv_g, \tag{2.2}$$

where $V = \frac{d}{dt}u_t|_{t=0}$.

**Proof** Let $\Psi : (-\varepsilon, \varepsilon) \times M \to N$ be any smooth deformation of $u$ such that

$$\Psi(t, x) = u_t(x), \quad d\Psi(\frac{\partial}{\partial t})|_{t=0} = \frac{d}{dt}u_t|_{t=0} = V, \tag{2.3}$$

where $\varepsilon$ is a positive constant. Let $u_t(x) = \Psi(t, x)$ and then $u_0(x) = u(x)$. Now we compute

$$\begin{aligned}
\frac{d}{dt}\Phi_f(u_t)|_{t=0} &= \int_M f\frac{\partial}{\partial t}\frac{||u_t^*h||^2}{4}|_{t=0}dv_g \\
&= \frac{1}{4}\int_M f\frac{\partial}{\partial t}[\sum_{i,j} h(du_t(e_i), du_t(e_j))^2]|_{t=0}dv_g \\
&= \int_M f\sum_{i,j} h(\widetilde{\nabla}_{\frac{\partial}{\partial t}}d\Psi(e_i), d\Psi(e_j))h(d\Psi(e_i), d\Psi(e_j))|_{t=0}dv_g \\
&= \int_M f\sum_i h(\widetilde{\nabla}_{e_i}d\Psi(\frac{\partial}{\partial t}), \sigma_u(e_i))|_{t=0}dv_g \\
&= \int_M f\sum_i [e_i h(d\Psi(\frac{\partial}{\partial t}), \sigma_u(e_i)) - h(d\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{e_i}\sigma_u(e_i))]|_{t=0}dv_g,
\end{aligned}$$

where we use that

$$\widetilde{\nabla}_{\partial/\partial t}d\Psi(e_i) - \widetilde{\nabla}_{e_i}d\Psi(\frac{\partial}{\partial t}) = d\Psi([\frac{\partial}{\partial t}, e_i]) = 0$$

for the forth equality.

Let $X_t$ be a compactly supported vector field on $M$ such that $g(X_t, Y) = h(d\Psi(\frac{\partial}{\partial t}), \sigma_{u_t}(Y))$ for any vector field $Y$ on $M$. Then

$$\begin{aligned}
\frac{d}{dt}\Phi_f(u_t)|_{t=0} &= \int_M f\sum_i [e_i g(X_t, e_i) - h(d\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{e_i}\sigma_u(e_i))]|_{t=0} dv_g \\
&= \int_M f\sum_i [g(\nabla_{e_i}X_t, e_i) + g(X_t, \nabla_{e_i}e_i) - h(d\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{e_i}\sigma_u(e_i))]|_{t=0} dv_g \\
&= \int_M f[\mathrm{div}(X_t) - \sum_i h(d\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{e_i}\sigma_u(e_i) - \sigma_u(\nabla_{e_i}e_i))]|_{t=0} dv_g \\
&= \int_M [\mathrm{div}(fX_t) - h(d\Psi(\frac{\partial}{\partial t}), f\mathrm{div}(\sigma_u) + \sigma_u(\mathrm{grad}f))]|_{t=0} dv_g \\
&= -\int_M h(\tau_{\Phi_f}(u), V) dv_g,
\end{aligned}$$

where we use the Green's theorem for the last equation. This proves Lemma 2.1.

The first variation formula allows us to define the notion of $f$-stationary for the functional $\Phi_f$.

**Definition 2.2** A smooth map $u$ is called $f$-stationary map for the functional $\Phi_f$ if it is a solution of the Euler-Lagrange equation $\tau_{\Phi_f}(u) = 0$.

## 3 $f$-Stress Energy Tensor

Following Baird [15], for a smooth map $u : (M, g) \to (N, h)$, we associate a symmetric 2-tensor $S_{\Phi_f}$ to the functional $\Phi_f$ called the $f$-stress energy tensor

$$S_{\Phi_f}(X, Y) = f[\frac{||u^*h||^2}{4} g(X, Y) - h(\sigma_u(X), du(Y))], \tag{3.1}$$

where $X, Y$ are vector fields on $M$.

**Proposition 3.1** Let $u : (M, g) \to (N, h)$ be a smooth map and $S_{\Phi_f}$ be the associated $f$-stress energy tensor, then for all $x \in M$ and for each vector $X \in T_xM$,

$$(\mathrm{div}S_{\Phi_f})(X) = -h(\tau_{\Phi_f}, du(X)) + \frac{||u^*h||^2}{4} df(X). \tag{3.2}$$

**Proof** Let $\nabla$ and ${}^N\nabla$ always denote the Levi-Civita connections of $M$ and $N$ respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_X W = {}^N\nabla_{du(X)}W$, where $X$ is a tangent vector of $M$ and $W$ is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ on $M$ with $\nabla_{e_i}e_i|_x = 0$ at a point $x \in M$.

Let $X$ be a vector field on $M$. At $x$, we compute

$$\begin{aligned}
(\mathrm{div}S_{\Phi_f})(X) &= \sum_i (\nabla_{e_i} S_{\Phi_f})(e_i, X) \\
&= \sum_i \{e_i S_{\Phi_f}(e_i, X) - S_{\Phi_f}(\nabla_{e_i} e_i, X) - S_{\Phi_f}(e_i, \nabla_{e_i} X)\} \\
&= \sum_i \{e_i(f\frac{||u^*h||^2}{4} g(e_i, X)) - e_i(fh(\sigma_u(e_i), du(X))) \\
&\quad -f\frac{||u^*h||^2}{4} g(e_i, \nabla_{e_i} X) + fh(\sigma_u(e_i), du(\nabla_{e_i} X)\} \\
&= X(f\frac{||u^*h||^2}{4}) - h(\sigma_u(gradf), du(X)) \\
&\quad -f\sum_i e_i h(\sigma_u(e_i), du(X)) + f\sum_i h(\sigma_u(e_i), du(\nabla_{e_i} X) \\
&= X(f\frac{||u^*h||^2}{4}) - h(\sigma_u(gradf), du(X)) - f\sum_i h(\widetilde{\nabla}_{e_i}\sigma_u(e_i), du(X)) \\
&\quad -f\sum_i h(\sigma_u(e_i), \widetilde{\nabla}_{e_i} du(X)) + f\sum_i h(\sigma_u(e_i), du(\nabla_{e_i} X) \\
&= X(f\frac{||u^*h||^2}{4}) - h(\sigma_u(gradf), du(X)) \\
&\quad -fh(\mathrm{div}\sigma_u, du(X)) - f\sum_i h(\sigma_u(e_i), (\nabla_{e_i} du)(X)) \\
&= X(f\frac{||u^*h||^2}{4}) - h(\tau_{\Phi_f}(u), du(X)) - f\sum_i h(\sigma_u(e_i), (\nabla_{e_i} du)(X)) \\
&= \frac{||u^*h||^2}{4} df(X) + f\sum_{i,j} h(\widetilde{\nabla}_X du(e_i), du(e_j)) h(du(e_i), du(e_j)) \\
&\quad -h(\tau_{\Phi_f}(u), du(X)) - f\sum_i h(\sigma_u(e_i), (\nabla_{e_i} du)(X)) \\
&= \frac{||u^*h||^2}{4} df(X) - h(\tau_{\Phi_f}(u), du(X)) \\
&\quad +f\sum_i h((\nabla_X du)(e_i), \sigma_u(e_i)) - f\sum_i h(\sigma_u(e_i), (\nabla_{e_i} du)(X)).
\end{aligned}$$

Since $(\nabla_X du)(e_i) = (\nabla_{e_i} du)(X)$, we obtain

$$(\mathrm{div}S_{\Phi_f})(X) = -h(\tau_{\Phi_f}, du(X)) + \frac{||u^*h||^2}{4} df(X).$$

This proves this proposition.

From the above proposition, we know that if $u : M \to N$ is an $f$-stationary map, then

$$(\mathrm{div}S_{\Phi_f})(X) = \frac{||u^*h||^2}{4} df(X). \tag{3.3}$$

Recall that for two 2-tensors $T_1$, $T_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product defined as

follows:

$$\langle T_1, T_2\rangle = \sum_{ij} T(e_i, e_j)T_2(e_i, e_j), \tag{3.4}$$

where $\{e_i\}$ is an orthonormal basis with respect to $g$. For a vector field $X \in \Gamma(TM)$, we denote by $\theta_X$ is dual one form i.e. $\theta_X(Y) = g(X, Y)$. The covariant derivative of $\theta_X$ gives a 2-tensor field $\nabla\theta_X$:

$$(\nabla\theta_X)(Y, Z) = (\nabla_Z\theta_X)(Y) = g(\nabla_Z X, Y). \tag{3.5}$$

If $X = \nabla\phi$ is the gradient of some function $\phi$ on $M$, then $\theta_X = d\phi$ and $\nabla\theta_X = \mathrm{Hess}\phi$.

**Lemma 3.2** (see [11, 15]) Let $T$ be a symmetric $(0, 2)$-type tensor field and let $X$ be a vector field, then

$$\mathrm{div}(i_X T) = (\mathrm{div}T)(X) + \langle T, \nabla\theta_X\rangle = (\mathrm{div}T)(X) + \frac{1}{2}\langle T, L_X g\rangle. \tag{3.6}$$

Let $D$ be any bounded domain of $M$ with $C^1$ boundary. By using the Stokes' theorem, we immediately have the following integral formula:

$$\int_{\partial D} T(X, \nu)ds_g = \int_D [\langle T, \frac{1}{2}L_X g\rangle + \mathrm{div}(T)(X)]dv_g, \tag{3.7}$$

where $\nu$ is the unit outward normal vector field along $\partial D$.

By (3.3) and (3.7), we have

$$\int_{\partial D} S_{\Phi_f}(X, \nu)ds_g = \int_D [\langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle + \frac{||u^*h||^2}{4}df(X)]dv_g. \tag{3.8}$$

## 4 Monotonicity Formulas and Vanishing Theorems

Let $(M, g_0)$ be a complete Riemannian manifold with a pole $x_0$. Denote by $r(x)$ the $g_0$-distance function relative to the pole $x_0$, that is $r(x) = \mathrm{dist}_{g_0}(x, x_0)$. Set $B(r) = \{x \in M^m : r(x) \le r\}$. It is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $\mathrm{Hess}_{g_0}(r^2)$ associated to eigenvalue 2. Denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$ which appeared in [12] ) the maximum (resp. minimal) eigenvalues of $\mathrm{Hess}_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$. Let $(N^n, h)$ be a Riemannian manifold.

From now on, we suppose that $u : (M^m, g) \to (N, h)$ is an $f$-stationary map, where $g = \varphi^2 g_0$, $0 < \varphi \in C^\infty(M)$. Clearly the vector field $\nu = \varphi^{-1}\frac{\partial}{\partial r}$ is an outer normal vector field along $\partial B(r) \subset (M, g)$. Assume that $\varphi$ satisfies the following conditions:

$(\varphi_1)$ $\frac{\partial \log\varphi}{\partial r} \ge 0$.

$(\varphi_2)$ there is a constant $C_0 > 0$ such that

$$(m-4)r\frac{\partial \log\varphi}{\partial r} + \frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \ge C_0.$$

**Remark** If $\varphi(r) = r^2$, conditions $(\varphi_1)$ and $(\varphi_2)$ turn into the following:

$$2(m-4) + \frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \ge C_0.$$

Now we set $\mu = \sup_M r|\frac{\partial \log f}{\partial r}| < +\infty$.

**Theorem 4.1** Suppose $u : (M, \varphi^2 g_0) \to (N, h)$ is an $f$-stationary map, where $0 < \varphi \in C^\infty(M)$. If $C_0 - \mu > 0$ and $\varphi$ satisfies $(\varphi_1)$, $(\varphi_2)$, then

$$\frac{\int_{B(\rho_1)} f\frac{||u^*h||^2}{4}dv_g}{\rho_1^{C_0-\mu}} \leq \frac{\int_{B(\rho_2)} f\frac{||u^*h||^2}{4}dv_g}{\rho_2^{C_0-\mu}} \tag{4.1}$$

for any $0 < \rho_1 \leq \rho_2$. In particular, if $\int_{B(R)} f\frac{||u^*h||^2}{4}dv_g = o(R^{C_0-\mu})$, then $u$ is constant.

**Proof** We take $D = B(R)$ and $X = r\frac{\partial}{\partial r} = \frac{1}{2}\nabla^0 r^2$ in (3.8), where $\nabla^0$ denotes the covariant derivative determined by $g_0$. By a direct computation, we have

$$\frac{1}{2}L_X g = \frac{1}{2}L_X(\varphi^2 g_0) = r\varphi\frac{\partial\varphi}{\partial r}g_0 + \frac{1}{2}\varphi^2 L_X g_0 = r\frac{\partial \log\varphi}{\partial r}g + \frac{1}{2}\varphi^2 L_X g_0,$$

and thus

$$\begin{aligned}\langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle &= \langle S_{\Phi_f}, r\frac{\partial\log\varphi}{\partial r}g\rangle + \langle S_{\Phi_f}, \frac{1}{2}\varphi^2 L_X g_0\rangle \\ &= r\frac{\partial\log\varphi}{\partial r}\langle S_{\Phi_f}, g\rangle + \frac{1}{2}\varphi^2\langle S_{\Phi_f}, \mathrm{Hess}_{g_0}(r^2)\rangle. \end{aligned}\tag{4.2}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis with respect to $g_0$ and $e_m = \frac{\partial}{\partial r}$. We may assume that $\mathrm{Hess}_{g_0}(r^2)$ becomes a diagonal matrix w.r.t. $\{e_i\}$. Then $\{\widetilde{e_i} = \varphi^{-1}e_i\}$ is an orthonnormal basis with respect to $g$.

$$\begin{aligned}&\frac{1}{2}\varphi^2\langle S_{\Phi_f}, \mathrm{Hess}_{g_0}(r^2)\rangle = \frac{1}{2}\varphi^2\sum_{i,j=1}^m S_{\Phi_f}(\widetilde{e_i},\widetilde{e_j})\mathrm{Hess}^{r^2}_{g_0}(\widetilde{e_i},\widetilde{e_j}) \\ =\ &\frac{1}{2}\varphi^2\{\sum_{i=1}^m f\frac{||u^*h||^2}{4}\mathrm{Hess}^{r^2}_{g_0}(\widetilde{e_i},\widetilde{e_i}) - \sum_{i,j=1}^m fh(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))\mathrm{Hess}^{r^2}_{g_0}(\widetilde{e_i},\widetilde{e_j})\} \\ =\ &\frac{1}{2}f\frac{||u^*h||^2}{4}\sum_{i=1}^m \mathrm{Hess}^{r^2}_{g_0}(e_i,e_i) - \frac{1}{2}f\sum_{i=1}^m h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_i}))\mathrm{Hess}^{r^2}_{g_0}(e_i,e_i) \\ \geq\ &\frac{1}{2}f\frac{||u^*h||^2}{4}[(m-1)\lambda_{\min}+2] - \frac{1}{2}\max\{2,\lambda_{\max}\}f\sum_{i=1}^m h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_i})) \\ =\ &\frac{1}{2}f\frac{||u^*h||^2}{4}[(m-1)\lambda_{\min}+2] - \frac{1}{2}\max\{2,\lambda_{\max}\}f||u^*h||^2 \\ \geq\ &\frac{1}{2}[(m-1)\lambda_{\min}+2-4\max\{2,\lambda_{\max}\}]f\frac{||u^*h||^2}{4}, \end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\langle S_{\Phi_f}, g\rangle &= mf\frac{||u^*h||^2}{4} - f\sum_{i,j=1}^m h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))g(\widetilde{e_i},\widetilde{e_j}) \\ &= mf\frac{||u^*h||^2}{4} - f||u^*h||^2 \geq (m-4)f\frac{||u^*h||^2}{4}. \end{aligned}\tag{4.4}$$

From (4.2), (4.3), (4.4), $(\varphi_1)$ and $(\varphi_2)$, we have

$$\langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle \geq [r\frac{\partial \log\varphi}{\partial r}(m-4)+\frac{m-1}{2}\lambda_{\min}+1-2\max\{2,\lambda_{\max}\}]f\frac{||u^*h||^2}{4} \geq C_0 f\frac{||u^*h||^2}{4},$$

i.e.,

$$\langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle \geq C_0 f\frac{||u^*h||^2}{4}. \tag{4.5}$$

On the other hand, by the coarea formula and $|\nabla r|_g = \varphi^{-1}$, we have

$$\begin{aligned}
&\int_{\partial B(r)} S_{\Phi_f}(X,\nu)ds_g = \int_{\partial B(r)}[f\frac{||u^*h||^2}{4}g(X,\nu)-fh(\sigma_u(X),du(\nu))]ds_g\\
=\ & r\int_{\partial B(r)} f\frac{||u^*h||^2}{4}\varphi ds_g - \int_{\partial B(r)} fr\varphi^{-1}h(\sigma_u(\frac{\partial}{\partial r}),du(\frac{\partial}{\partial r}))ds_g\\
=\ & r\int_{\partial B(r)} f\frac{||u^*h||^2}{4}\varphi ds_g - \int_{\partial B(r)} fr\varphi^{-1}\sum_{i=1}^{m} h(du(\frac{\partial}{\partial r}),du(\widetilde{e_i}))^2 ds_g\\
\leq\ & r\int_{\partial B(r)} f\frac{||u^*h||^2}{4}\varphi ds_g = r\frac{d}{dr}\int_0^r\{\int_{\partial B(t)}\frac{f\frac{||u^*h||^2}{4}}{|\nabla r|}ds_g\}dt\\
=\ & r\frac{d}{dr}\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g.
\end{aligned} \tag{4.6}$$

From (3.8), (4.5) and (4.6), we have

$$C_0\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g + \int_{B(r)}\frac{||u^*h||^2}{4}rdf(\frac{\partial}{\partial r})dv_g \leq r\frac{d}{dr}\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g,$$

so

$$C_0\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g - \mu\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g \leq r\frac{d}{dr}\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g,$$

i.e.,

$$\frac{d}{dr}\frac{\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g}{r^{C_0-\mu}} \geq 0. \tag{4.7}$$

Therefore

$$\frac{\int_{B(\rho_1)} f\frac{||u^*h||^2}{4}dv_g}{\rho_1^{C_0-\mu}} \leq \frac{\int_{B(\rho_2)} f\frac{||u^*h||^2}{4}dv_g}{\rho_2^{C_0-\mu}}$$

for any $0<\rho_1\leq\rho_2$. This proves this theorem.

From the proof of Theorem 4.1, we immediately get the following:

**Theorem 4.2** Suppose $u:(M,\varphi^2 g_0)\to(N,h)$ is an $f$-stationary map, where $0<\varphi\in C^\infty(M)$. If $\frac{\partial f}{\partial r}\geq 0$ and $\varphi$ satisfies $(\varphi_1)$, $(\varphi_2)$, then

$$\frac{\int_{B(\rho_1)} f\frac{||u^*h||^2}{4}dv_g}{\rho_1^{C_0}}\leq\frac{\int_{B(\rho_2)} f\frac{||u^*h||^2}{4}dv_g}{\rho_2^{C_0}} \tag{4.8}$$

for any $0<\rho_1\leq\rho_2$. In particular, if $\int_{B(R)} f\frac{||u^*h||^2}{4}dv_g=o(R^{C_0})$, then $u$ is constant.

**Lemma 4.3** [11, 16] Let $(M^m,g)$ be a complete Riemannian manifold with a pole $x_0$. Denote by $K_r$ the radial curvature of $M$.

(i) if $-\alpha^2\leq K_r\leq-\beta^2$ with $\alpha\geq\beta>0$, then

$$\beta\coth(\beta r)[g-dr\otimes dr]\leq \mathrm{Hess}(r)\leq\alpha\coth(\alpha r)[g-dr\otimes dr];$$

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}}\leq K_r\leq\frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon>0$, $A\geq 0$ and $0\leq B<2\varepsilon$, then

$$\frac{1-B/2\varepsilon}{r}[g-dr\otimes dr]\leq\mathrm{Hess}(r)\leq\frac{e^{A/2\varepsilon}}{r}[g-dr\otimes dr];$$

(iii) if $-\frac{a^2}{1+r^2}\leq K_r\leq\frac{b^2}{1+r^2}$ with $a\geq 0$ and $b^2\in[0,\frac14]$, then

$$\frac{1+\sqrt{1-4b^2}}{2r}[g-dr\otimes dr]\leq\mathrm{Hess}(r)\leq\frac{1+\sqrt{1+4a^2}}{2r}[g-dr\otimes dr].$$

**Lemma 4.4** Let $(M^m,g)$ be a complete Riemannian manifold with a pole $x_0$. Denote by $K_r$ the radial curvature of $M$.

(i) if $-\alpha^2\leq K_r\leq-\beta^2$ with $\alpha\geq\beta>0$ and $(m-1)\beta-4\alpha\geq 0$, then

$$[(m-1)\lambda_{\min}+2-4\max\{2,\lambda_{\max}\}]\geq 2(m-\frac{4\alpha}{\beta});$$

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}}\leq K_r\leq\frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon>0$, $A\geq 0$ and $0\leq B<2\varepsilon$, then

$$[(m-1)\lambda_{\min}+2-4\max\{2,\lambda_{\max}\}]\geq 2[1+(m-1)(1-\frac{B}{2\varepsilon})-4e^{A/2\varepsilon}];$$

(iii) if $-\frac{a^2}{1+r^2}\leq K_r\leq\frac{b^2}{1+r^2}$ with $a\geq 0$ and $b^2\in[0,\frac14]$, then

$$\begin{aligned}&[(m-1)\lambda_{\min}+2-4\max\{2,\lambda_{\max}\}]\\ \geq\ & 2[1+(m-1)\frac{1+\sqrt{1-4b^2}}{2}-4\frac{1+\sqrt{1+4a^2}}{2}].\end{aligned}$$

**Proof** If $K_r$ satisfies (i), then by Lemma 4.3, we have on $B(r)-\{x_0\}$, for every $r>0$,

$$\begin{aligned}&[(m-1)\lambda_{\min}+2-4\max\{2,\lambda_{\max}\}]\\ \geq\ & (m-1)2\beta r\coth(\beta r)+2-4\times 2\alpha r\coth(\alpha r)\\ =\ & 2[1+\beta r\coth(\beta r)(m-1-\frac{4\alpha}{\beta}\frac{\coth(\alpha r)}{\coth(\beta r)})]\\ \geq\ & 2[1+1\times(m-1)-\frac{4\alpha}{\beta})]\\ =\ & 2[m-\frac{4\alpha}{\beta}],\end{aligned}$$

where the second inequality is because the increasing function $\beta r\coth(\beta r)\to 1$ as $r\to 0$, and $\frac{\coth(\alpha r)}{\coth(\beta r)}<1$, for $0<\beta<\alpha$. Similarly, from Lemma 4.3, the above inequality holds for cases (ii) and (iii) on $B(r)$.

**Theorem 4.5** Let $(M,g)$ be an $m$-dimensional complete manifold with a pole $x_0$. Assume that the radial curvature $K_r$ of $M$ satisfies one of the following three conditions:

(i) if $-\alpha^2\le K_r\le -\beta^2$ with $\alpha\ge\beta>0$ and $(m-1)\beta-4\alpha\ge 0$;

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}}\le K_r\le\frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon>0$, $A\ge 0$, $0\le B<2\varepsilon$ and $1+(m-1)(1-\frac{B}{2\varepsilon})-4e^{A/2\varepsilon}>0$;

(iii) if $-\frac{a^2}{1+r^2}\le K_r\le\frac{b^2}{1+r^2}$ with $a\ge 0$, $b^2\in[0,\frac{1}{4}]$ and $1+(m-1)\frac{(1+\sqrt{1-4b^2})}{2}-4\frac{(1+\sqrt{1+4a^2})}{2}>0$.

If $u:(M,g)\to(N,h)$ is an $f$-stationary map and $\Lambda-\mu>0$, then

$$\frac{\displaystyle\int_{B(\rho_1)} f\frac{||u^*h||^2}{4}dv_g}{\rho_1^{\Lambda-\mu}}\le\frac{\displaystyle\int_{B(\rho_2)} f\frac{||u^*h||^2}{4}dv_g}{\rho_2^{\Lambda-\mu}} \tag{4.9}$$

for any $0<\rho_1\le\rho_2$, where

$$\Lambda=\begin{cases} m-\frac{4\alpha}{\beta}, & \text{if } K_r \text{ satisfies (i)},\\ 1+(m-1)(1-\frac{B}{2\varepsilon})-4e^{A/2\varepsilon}, & \text{if } K_r \text{ satisfies (ii)},\\ 1+(m-1)\frac{1+\sqrt{1-4b^2}}{2}-4\frac{1+\sqrt{1+4a^2}}{2}, & \text{if } K_r \text{ satisfies (iii)}.\end{cases} \tag{4.10}$$

In particular, if $\displaystyle\int_{B(R)} f\frac{||u^*h||^2}{4}dv_g=o(R^{\Lambda-\mu})$, then $u$ is constant.

**Proof** From the proof of Theorem 4.1 for $\varphi=1$ and Lemma 4.4, we have

$$\frac{d}{dr}\frac{\displaystyle\int_{B(r)} f\frac{||u^*h||^2}{4}dv_g}{r^{\Lambda-\mu}}\ge 0.$$

Therefore we get the monotonicity formula

$$\frac{\displaystyle\int_{B(\rho_1)} f\frac{||u^*h||^2}{4}dv_g}{\rho_1^{\Lambda-\mu}}\le\frac{\displaystyle\int_{B(\rho_2)} f\frac{||u^*h||^2}{4}dv_g}{\rho_2^{\Lambda-\mu}}$$

for any $0<\rho_1\le\rho_2$.

**Theorem 4.6** Let $M$, $K_r$ and $\Lambda$ be as in Theorem 4.5. If $u:(M,g)\to(N,h)$ is an $f$-stationary map and $\frac{\partial f}{\partial r}\ge 0$, then

$$\frac{\displaystyle\int_{B(\rho_1)} f\frac{||u^*h||^2}{4}dv_g}{\rho_1^{\Lambda}}\le\frac{\displaystyle\int_{B(\rho_2)} f\frac{||u^*h||^2}{4}dv_g}{\rho_2^{\Lambda}} \tag{4.11}$$

for any $0<\rho_1\le\rho_2$. In particular, if $\displaystyle\int_{B(R)} f\frac{||u^*h||^2}{4}dv_g=o(R^{\Lambda})$, then $u$ is constant.

**Proof** From Theorem 4.2 and $\varphi = 1$, we know that formula (4.11) is true.

We say the functional $\Phi_f(u)$ of $u$ is slowly divergent if there exists a positive function $\psi(r)$ with $\int_{R_0}^{\infty} \frac{dr}{r\psi(r)} = +\infty$ $(R_0 > 0)$, such that

$$\lim_{R\to\infty} \int_{B(R)} \frac{f\frac{||u^*h||^2}{4}}{\psi(r(x))} dv_g < \infty. \tag{4.12}$$

**Theorem 4.7** Suppose $u : (M, \varphi^2 g_0) \to (N, h)$ is an $f$-stationary map. If $C_0 - \mu > 0$, $\varphi$ satisfies $(\varphi_1)$, $(\varphi_2)$ and $\Phi_f(u)$ of $u$ is slowly divergent, then $u$ is constant.

**Proof** From the proof of Theorem 4.1, we have

$$(C_0 - \mu) \int_{B(R)} f\frac{||u^*h||^2}{4} dv_g \le R \int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi ds_g. \tag{4.13}$$

Now Suppose that $u$ is a nonconstant map, so there exists $R_0 > 0$ such that for $R \ge R_0$,

$$\int_{B(R)} f\frac{||u^*h||^2}{4} dv_g \ge c_1, \tag{4.14}$$

where $c_1$ is a positive constant. From (4.13) and (4.14), we have

$$\int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi ds_g \ge \frac{c_1(C_0 - \mu)}{R} \tag{4.15}$$

for $R \ge R_0$.

$$\begin{aligned} \lim_{R\to\infty} \int_{B(R)} \frac{f\frac{||u^*h||^2}{4}}{\psi(r(x))} dv_g &= \int_0^{\infty} \frac{dR}{\psi(R)} \int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi ds_g \\ &\ge \int_{R_0}^{\infty} \frac{dR}{\psi(R)} \int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi ds_g \\ &\ge c_1(C_0 - \mu) \int_{R_0}^{\infty} \frac{dR}{R\psi(R)} = \infty, \end{aligned}$$

which contradicts (4.12), therefore $u$ is constant.

From the proof of Theorem 4.7, we immediately get the following.

**Theorem 4.8** Suppose $u : (M, \varphi^2 g_0) \to (N, h)$ is an $f$-stationary map. If $\frac{\partial f}{\partial r} \ge 0$, $\varphi$ satisfies $(\varphi_1)$, $(\varphi_2)$ and $\Phi_f(u)$ of $u$ is slowly divergent, then $u$ is constant.

**Theorem 4.9** Let $M$, $K_r$ and $\Lambda$ be as in Theorem 4.5. If $u : (M, g) \to (N, h)$ is an $f$-stationary map, $\Lambda - \mu > 0$ and $\Phi_f(u)$ of $u$ is slowly divergent, then $u$ is constant.

**Theorem 4.10** Let $M$, $K_r$ and $\Lambda$ be as in Theorem 4.5. If $u : (M, g) \to (N, h)$ is an $f$-stationary map, $\frac{\partial f}{\partial r} \ge 0$ and $\Phi_f(u)$ of $u$ is slowly divergent, then $u$ is constant.

## 5 Constant Dirichlet Boundary-Value Problems

To investigate the constant Dirichlet boundary value problems for $f$-stationary map, we begin with

**Definition 5.1** (see [11]) A bounded domain $D \subset M$ with $C^1$ boundary $\partial D$ is called starlike if there exists an interior point $x_0 \in D$ such that

$$g(\frac{\partial}{\partial r_{x_0}}, \nu) \geq 0, \tag{5.1}$$

where $\nu$ is the unit outer normal to $\partial D$, and the vector field $\frac{\partial}{\partial r_{x_0}}$ is the unit vector field such that for any $x \in (D - \{x_0\}) \cup \partial D$, $\frac{\partial}{\partial r_{x_0}}$ is the unit vector tangent to the unique geodesic joining $x_0$ and pointing away form $x_0$.

It is obvious that any convex domain is starlike.

**Theorem 5.2** Suppose $u : (M, \varphi^2 g_0) \to (N, h)$ is an $f$-stationary map and $D \subset M$ is a bounded starlike domain with $C^1$ boundary with the pole $x_0 \in D$. Assume that $C_0 - \mu > 0$ on $D$ and $\varphi$ satisfies $(\varphi_1)$, $(\varphi_2)$. If $u|_{\partial D} \equiv P \in N$, then $u$ must be constant in $D$.

**Proof** Take $X = r\frac{\partial}{\partial r}$, where $r = r_{x_0}$. From the proof of Theorem 4.1, we have

$$\langle S_{\Phi_f}, \frac{1}{2} L_X g \rangle \geq (C_0 - \mu) f \frac{||u^* h||^2}{4}, \tag{5.2}$$

where $C_0$ is a positive constant. Since $u|_{\partial D} = P$, $du(\eta) = 0$ for any tangent vector $\eta$ of $\partial D$. We can derive the following on $\partial D$:

$$\begin{aligned} S_{\Phi_f}(X, \nu) &= r S_{\Phi_f}(\frac{\partial}{\partial r}, \nu) \\ &= r[f \frac{||u^* h||^2}{4} g(\frac{\partial}{\partial r}, \nu) - f h(\sigma_u(\frac{\partial}{\partial r}), du(\nu))] \\ &= r g(\frac{\partial}{\partial r}, \nu)[f \frac{||u^* h||^2}{4} - f||u^* h||^2] \\ &\leq -3 r g(\frac{\partial}{\partial r}, \nu) f \frac{||u^* h||^2}{4} \leq 0. \end{aligned} \tag{5.3}$$

From (3.8), (5.2) and (5.3), we have

$$0 \leq \int_{B(r)} (C_0 - \mu) f \frac{||u^* h||^2}{4} dv_g \leq 0, \tag{5.4}$$

which implies that $u(D) \equiv P$.

From the proof of Theorem 5.2, we immediately get the following.

**Theorem 5.3** Suppose $u : (M, \varphi^2 g_0) \to (N, h)$ is an $f$-stationary map and $D \subset M$ is a bounded starlike domain with $C^1$ boundary with the pole $x_0 \in D$. Assume that $\frac{\partial f}{\partial r} \geq 0$ on $D$ and $\varphi$ satisfies $(\varphi_1)$, $(\varphi_2)$. If $u|_{\partial D} \equiv P \in N$, then $u$ must be constant in $D$.

**Theorem 5.4** Let $M$, $K_r$ and $\Lambda$ be as in Theorem 4.5. Suppose $u : (M, g) \to (N, h)$ is an $f$-stationary map, $D \subset M$ is a bounded starlike domain with $C^1$ boundary with the pole $x_0 \in D$ and $\Lambda - \mu > 0$. If $u|_{\partial D} \equiv P \in N$, then $u$ must be constant in $D$.

**Theorem 5.5** Let $M$, $K_r$ and $\Lambda$ be as in Theorem 4.5. Suppose $u : (M, g) \to (N, h)$ is an $f$-stationary map, $D \subset M$ is a bounded starlike domain with $C^1$ boundary with the pole $x_0 \in D$ and $\frac{\partial f}{\partial r} \geq 0$. If $u|_{\partial D} \equiv P \in N$, then $u$ must be constant in $D$.

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# 与拉回度量相关泛函的$f$-稳态映射的刘维尔型定理


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**摘要**: 本文研究了与拉回度量相关的广义泛函$\Phi_f$. 利用$f$-应力能力张量的方法, 得到$f$-稳态映射的单调公式, 消灭定理以及常Dirichlet边值问题在星型区域上的唯一常值解.



**关键词**: $f$-稳态映射; $f$-应力能量张量; 单调公式; 消灭定理

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