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ALGEBRAIC SOLUTIONS OF SYSTEMS OF COMPLEX DIFFERENTIAL EQUATIONS

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Abstract: This paper investigates the problem of existence of algebraic solutions of system of complex differential equations. Using maximum modulus principle and the Nevanlinna theory of the value distribution of meromorphic functions, a new result is obtained, and some existing results are improved and generalized. Examples show that our result is precise.

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1 Introduction and Main Results

In this paper, we assume that the reader is familiar with the standard notation and basic results of the Nevanlinna theory of meromorphic functions, for example, see [1, 2].

In 1988, Toda N considered the existence of algebroid solutions of algebraic differential equation of the form

$$(w')^n = \frac{\sum\limits_{i=0}^{p} a_i w^i}{\sum\limits_{j=0}^{q} b_j w^j}, a_p b_q \neq 0.$$

He obtained the following result:

Theorem A (see [3]) Let $w(z)$ be a nonconstant ν-valued algebroid solution of the above differential equation and all $\{a_i\}, \{b_j\}$ are polynomials. If $p < n + q$, then $w(z)$ is algebraic.

Since the 1990s, many authors, such as Tu Zhenhan, Xiao Xiuzhi, Song Shugang, Li Kamshun, Gao Lingyun, using the Nevanlinna theory of the value distribution of meromorphic functions, studied the problem of the existence and the growth of solutions of systems of complex differential equations and obtained many new and interesting results, for example, see [4–11].

The purpose of this paper is to study the system of complex algebraic differential equations on the base of Toda N's paper with the aid of maximum modulus principle and Nevanlinna theory.

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We will study the existence of algebraic solutions of system of complex differential equations of the following form

$$\begin{cases} (w_1')^{n_1} = \frac{P_1(z,w_1,w_2)}{Q_1(z,w_1)}, \\ (w_2')^{n_2} = \frac{P_2(z,w_1,w_2)}{Q_2(z,w_2)}, \end{cases} \tag{1.1}$$

where

$$P_1(z,w_1,w_2) = \sum_{i=0}^{p_{11}}\sum_{j=0}^{p_{12}} a_{ij}w_1^i w_2^j, \quad Q_1(z,w_1) = \sum_{i=0}^{q_1} b_i w_1^i,$$
$$P_2(z,w_1,w_2) = \sum_{i=0}^{p_{21}}\sum_{j=0}^{p_{22}} c_{ij}w_1^i w_2^j, \quad Q_2(z,w_2) = \sum_{i=0}^{q_2} d_i w_2^i,$$

$\{a_{ij}\}, \{b_i\}, \{c_{ij}\}, \{d_i\}$ are entire functions, n_1, n_2 are positive integer numbers, $p_{11}, p_{12}, q_1, p_{21}, p_{22}, q_2$ are non-negative integer numbers.

We will prove

Theorem 1.1 Let $(w_1(z), w_2(z))$ be a nonconstant transcendental meromorphic solution of system (1.1) and all $\{a_{ij}\}, \{b_i\}, \{c_{ij}\}, \{d_i\}$ are polynomials. If there exists a positive constant K such that $\min\{n_1, n_1+q_1-p_{11}\}\min\{n_2, n_2+q_2-p_{22}\} > Kp_{12}p_{21}$, and one of the following conditions is satisfied

(i) $n_1+q_1 > p_{11}+p_{12}$, $n_2+q_2 > p_{21}+p_{22}$,

(ii) $n_1+q_1 > p_{11}+p_{21}$, $n_2+q_2 > p_{22}+p_{12}$,

then $(w_1(z), w_2(z))$ is algebraic.

In this paper, we denote by E is a subset of $[0,\infty)$ for which $m(E) < \infty$ and by K is a positive constant, where $m(E)$ denotes the linear measure of E. E or K does not always mean the same one when they appear in the following.

2 Some Lemmas

Lemma 2.1 (see [2]) Let $w(z)$ be a transcendental meromorphic function such that $w(z), w'(z)$ has only finite number of poles. Then, for some constants $C_i > 0, i = 1, 2$, it holds

$$M(r,w) \le C_1 + C_2 r M(r,w'),$$

where $M(r,w) = \max\limits_{|z|=r}\{|w(z)|\}$.

Lemma 2.2 (see [2]) Let $w(z)$ be a transcendental meromorphic function such that $w(z)$ has only finite number of poles. Then, for $\alpha > 0$, it holds

$$M(r,w') \le 2^{\frac{1}{\alpha}}[M(r,w)]^{\alpha+1}, (r \notin E).$$

3 Proof of Theorem 1.1

Let S be the set of zeros of $\{a_{ij}\}, \{b_i\}, \{c_{ij}\}, \{d_i\}$.

First, we will prove the poles of w_1 and w_2 are contained in S.

By the conditions of Theorem 1.1, if z_0 is a pole of w_1 or w_2, $z_0 \notin S$, then z_0 is also a pole of w_2 or w_1. Suppose that z_0 is a pole of w_1 of order τ_1, a pole of w_2 of order τ_2, but $z_0 \notin S$. Then it follows from (1.1), we have

$$\begin{cases} n_1(\tau_1+1) \le p_{11}\tau_1 + p_{12}\tau_2 - q_1\tau_1, \\ n_2(\tau_2+1) \le p_{21}\tau_1 + p_{22}\tau_2 - q_2\tau_2, \end{cases}$$

that is

$$\begin{cases} n_1(\tau_1+1) \le (p_{11}-q_1)\tau_1 + p_{12}\tau_2, \\ n_2(\tau_2+1) \le p_{21}\tau_1 + (p_{22}-q_2)\tau_2. \end{cases}$$

Case (i): Noting that $p_{11}+p_{12} < n_1+q_1,\ p_{21}+p_{22} < n_2+q_2$, we get

$$\begin{cases} n_1(\tau_1+1) < (n_1-p_{12})\tau_1 + p_{12}\tau_2, \\ n_2(\tau_2+1) < p_{21}\tau_1 + (n_2-p_{21})\tau_2. \end{cases}$$

That is

$$n_1 < p_{12}(\tau_2-\tau_1), n_2 < p_{21}(\tau_1-\tau_2).$$

This is a contradiction because $p_{12} \ge 0, p_{21} \ge 0$.

Case (ii): Noting that $p_{11}+p_{21} < n_1+q_1,\ p_{12}+p_{22} < n_2+q_2$, we obtain

$$n_1(\tau_1+1) + n_2(\tau_2+1) \le (p_{11}+p_{21}-q_1)\tau_1 + (p_{22}+p_{12}-q_2)\tau_2 < n_1\tau_1 + n_2\tau_2.$$

Then, we have $n_1+n_2 < 0$. This is a contradiction.

Combining Case (i) and Case (ii), we obtain the poles of w_1 and w_2 are contained in S, that is

$$N(r,w_1) \le K_1[N(r,\frac{1}{a_{ij}}) + N(r,\frac{1}{b_i}) + N(r,\frac{1}{c_{ij}}) + N(r,\frac{1}{d_i})], \tag{3.1}$$

$$N(r,w_2) \le K_2[N(r,\frac{1}{a_{ij}}) + N(r,\frac{1}{b_i}) + N(r,\frac{1}{c_{ij}}) + N(r,\frac{1}{d_i})], \tag{3.2}$$

where K_1, K_2 are positive constants.

Next, we will estimate $m(r,w_1), m(r,w_2)$.

We rewrite the system of complex differential equation (1.1) as follows:

$$\begin{cases} b_{q_1}^{n_1}(\overline{Q_1}(w_1)w_1')^{n_1} = P_1(z,w_1,w_2)Q_1(z,w_1)^{n_1-1}, \\ d_{q_2}^{n_2}(\overline{Q_2}(w_2)w_2')^{n_2} = P_2(z,w_1,w_2)Q_2(z,w_2)^{n_2-1}, \end{cases} \tag{3.3}$$

where $\overline{Q_1}(w_1) = \dfrac{Q_1(z,w_1)}{b_{q_1}}, \overline{Q_2}(w_2) = \dfrac{Q_2(z,w_2)}{d_{q_2}}$.

Let

$$U_1(z) = \frac{w_1^{q_1+1}}{q_1+1} + \sum_{i=0}^{q_1-1} \frac{b_i}{b_{q_1}} \frac{w_1^{i+1}}{i+1},\ V_1(z) = \sum_{i=0}^{q_1-1} (\frac{b_i}{b_{q_1}})' \frac{w_1^{i+1}}{i+1},$$

$$U_2(z) = \frac{w_2^{q_2+1}}{q_2+1} + \sum_{i=0}^{q_2-1} \frac{d_i}{d_{q_2}} \frac{w_2^{i+1}}{i+1},\ V_2(z) = \sum_{i=0}^{q_2-1} (\frac{d_i}{d_{q_2}})' \frac{w_2^{i+1}}{i+1}.$$

Then system (3.3) becomes

$$\begin{cases} b_{q_1}^{n_1}(U_1'(z)-V_1(z))^{n_1}=P_1(z,w_1,w_2)Q_1(z,w_1)^{n_1-1}, \\ d_{q_2}^{n_2}(U_2'(z)-V_2(z))^{n_2}=P_2(z,w_1,w_2)Q_2(z,w_2)^{n_2-1}. \end{cases} \tag{3.4}$$

We can easily prove the poles of $U_i(z)$, $U_i'(z)$ $(i=1,2)$ are contained in S.

Because S is a finite set, it follows from Lemma 2.1 that

$$M(r,U_1)\le C_1+C_2rM(r,U_1'), r\notin E,$$
$$M(r,U_2)\le C_3+C_4rM(r,U_2'), r\notin E.$$

According to the definitions of $U_i, i=1,2$, we obtain

$$M(r,U_1)\ge \frac{M(r,w_1)^{q_1+1}}{q_1+1}-\frac{K_3M(r,w_1)^{q_1}\{\sum\limits_{i=0}^{q_1-1}M(r,b_i)\}}{r^{\overline{d_1}}}, \tag{3.5}$$

$$M(r,U_2)\ge \frac{M(r,w_2)^{q_2+1}}{q_2+1}-\frac{K_4M(r,w_2)^{q_2}\{\sum\limits_{i=0}^{q_2-1}M(r,d_i)\}}{r^{\overline{d_2}}}, \tag{3.6}$$

$$(\frac{b_i}{b_{q_1}})'=\frac{b_i'b_{q_1}-b_{q_1}'b_i}{b_{q_1}^2}, \quad (\frac{d_i}{d_{q_2}})'=\frac{d_i'd_{q_2}-d_{q_2}'d_i}{}d_{q_2}^2,$$

where $\overline{d_1}, \overline{d_2}$ are respectively the degrees of polynomials b_{q_1}, d_{q_2} and K_3, K_4 are positive constants.

From Lemma 2.2, we obtain

$$M(r,(\frac{b_i}{b_{q_1}})')\le K_5\frac{r[M(r,b_i)]^2+M(r,b_i)}{r^{\overline{d_1}+1}}, r\notin E,$$
$$M(r,(\frac{d_i}{d_{q_2}})')\le K_6\frac{r[M(r,d_i)]^2+M(r,d_i)}{r^{\overline{d_2}+1}}, r\notin E,$$

where K_5, K_6 are positive constants.

Therefore

$$M(r,V_1)\le \frac{K_7M(r,w_1)^{q_1}\sum\limits_{i=0}^{q_1-1}\{r[M(r,b_i)]^2+M(r,b_i)\}}{r^{\overline{d_1}+1}}, \tag{3.7}$$

$$M(r,V_2)\le \frac{K_8M(r,w_2)^{q_2}\sum\limits_{i=0}^{q_2-1}\{r[M(r,d_i)]^2+M(r,d_i)\}}{r^{\overline{d_2}+1}}, \tag{3.8}$$

where K_7, K_8 are positive constants.

By (3.5) and (3.7), we have

$$\begin{aligned}
&M(r,U_1')-M(r,V_1)\geq \frac{M(r,U_1)-C_1}{C_2r}-M(r,V_1)\\
\geq\ &\frac{K_9M(r,w_1)^{q_1+1}}{r}-\frac{K_{10}M(r,w_1)^{q_1}\sum\limits_{i=0}^{q_1-1}\{r[M(r,b_i)]^2+M(r,b_i)\}}{r^{\overline{d_1}+1}}, \qquad (3.9)\\
&M(r,U_2')-M(r,V_2)\geq \frac{M(r,U_2)-C_3}{C_4r}-M(r,V_2)\\
\geq\ &\frac{K_{11}M(r,w_2)^{q_2+1}}{r}-\frac{K_{12}M(r,w_2)^{q_2}\sum\limits_{i=0}^{q_2-1}\{r[M(r,d_i)]^2+M(r,d_i)\}}{r^{\overline{d_2}+1}}, \qquad (3.10)
\end{aligned}$$

where $K_9,K_{10},\ K_{11},K_{12}$ are positive constants.

Let z_r be a point such that $M(r,U'(z))=|U'(z_r)|,|z_r|=r,(r\notin E)$. Then

$$\begin{aligned}
(M(r,U_1')-M(r,V_1))^{n_1}\ &\leq\ |U_1'(z_r)-V_1(z_r)|^{n_1}\\
&\leq\ M(r,\frac{P_1(z,w_1,w_2)Q_1(z,w_1)^{n_1-1}}{b_{q_1}^{n_1}}), \qquad (3.11)\\
(M(r,U_2')-M(r,V_2))^{n_2}\ &\leq\ |U_2'(z_r)-V_2(z_r)|^{n_2}\\
&\leq\ M(r,\frac{P_2(z,w_1,w_2)Q_2(z,w_2)^{n_2-1}}{d_{q_2}^{n_2}}). \qquad (3.12)
\end{aligned}$$

Since

$$\begin{aligned}
&M(r,\frac{P_1(z,w_1,w_2)Q_1(z,w_1)^{n_1-1}}{b_{q_1}^{n_1}})\\
\leq\ &\frac{K_{13}M(r,w_2)^{p_{12}}M(r,w_1)^{p_{11}+q_1(n_1-1)}\{\sum\limits_{i=0}^{p_{11}}\sum\limits_{j=0}^{p_{12}}M(r,a_{ij})\}\{\sum\limits_{i=0}^{q_1}M(r,b_i)\}^{n_1-1}}{r^{n_1\overline{d_1}}}, \qquad (3.13)\\
&M(r,\frac{P_2(z,w_1,w_2)Q_2(z,w_2)^{n_2-1}}{d_{q_2}^{n_2}})\\
\leq\ &\frac{K_{14}M(r,w_1)^{p_{21}}M(r,w_2)^{p_{22}+q_2(n_2-1)}\{\sum\limits_{i=0}^{p_{21}}\sum\limits_{j=0}^{p_{22}}M(r,c_{ij})\}\{\sum\limits_{i=0}^{q_2}M(r,d_i)\}^{n_2-1}}{r^{n_2\overline{d_2}}}, \qquad (3.14)
\end{aligned}$$

where K_{13},K_{14} are positive constants.

Combining (3.9)–(3.14), we obtain

$$\begin{cases}
\frac{M(r,w_1)^{q_1+1}}{r}\leq \frac{K_{15}M(r,w_2)^{p_{12}/n_1}M(r,w_1)^{(p_{11}+q_1(n_1-1))/n_1}\{\sum\limits_{i=0}^{p_{11}}\sum\limits_{j=0}^{p_{12}}M(r,a_{ij})\}^{1/n_1}\{\sum\limits_{i=0}^{q_1}M(r,b_i)\}^{(n_1-1)/n_1}}{r^{\overline{d_1}}}\\
\qquad +\frac{K_{16}M(r,w_1)^{q_1}\sum\limits_{i=0}^{q_1-1}\{r[M(r,b_i)]^2+M(r,b_i)\}}{r^{\overline{d_1}}+1},r\notin E,\\
\frac{M(r,w_2)^{q_2+1}}{r}\leq \frac{K_{17}M(r,w_1)^{p_{21}/n_2}M(r,w_2)^{(p_{22}+q_2(n_2-1))/n_2}\{\sum\limits_{i=0}^{p_{21}}\sum\limits_{j=0}^{p_{22}}M(r,c_{ij})\}^{1/n_2}\{\sum\limits_{i=0}^{q_2}M(r,d_i)\}^{(n_2-1)/n_2}}{r^{\overline{d_2}}}\\
\qquad +\frac{K_{18}M(r,w_2)^{q_2}\sum\limits_{i=0}^{q_2-1}\{r[M(r,d_i)]^2+M(r,d_i)\}}{r^{\overline{d_2}+1}},r\notin E,
\end{cases}$$

where $K_{15}, K_{16}, K_{17}, K_{18}$ are positive constants.

Further, we get

$$\begin{cases} M(r,w_1)^{\min\{1,\frac{n_1+q_1-p_{11}}{n_1}\}} \le \dfrac{K_{19}[M(r,w_2)^{p_{12}/n_1}\{\sum\limits_{i=0}^{p_{11}}\sum\limits_{j=0}^{p_{12}} M(r,a_{ij})\}^{1/n_1}\{\sum\limits_{i=0}^{q_1} M(r,b_i)\}^{(n_1-1)/n_1}}{r^{\overline{d_1}}} \\ \qquad + \dfrac{\sum\limits_{i=0}^{q_1-1}\{r[M(r,b_i)]^2+M(r,b_i)\}]}{r^{\overline{d_1}}}, \\ M(r,w_2)^{\min\{1,\frac{n_2+q_2-p_{22}}{n_2}\}} \le \dfrac{K_{20}[M(r,w_1)^{p_{21}/n_2}\{\sum\limits_{i=0}^{p_{21}}\sum\limits_{j=0}^{p_{22}} M(r,c_{ij})\}^{1/n_2}\{\sum\limits_{i=0}^{q_2} M(r,d_i)\}^{(n_2-1)/n_2}}{r^{\overline{d_2}}} \\ \qquad + \dfrac{\sum\limits_{i=0}^{q_2-1}\{r[M(r,d_i)]^2+M(r,d_i)\}]}{r^{\overline{d_2}}}, \end{cases}$$

where K_{19}, K_{20} are positive constants.

By calculating $\log^+$ of the both sides of the above inequalities, we have

$$\begin{cases} \min\{n_1, n_1+q_1-p_{11}\}\log^+ M(r,w_1) \le K_{19}p_{12}\log^+ M(r,w_2) + S_1(r), \\ \min\{n_2, n_2+q_2-p_{22}\}\log^+ M(r,w_2) \le K_{20}p_{21}\log^+ M(r,w_1) + S_2(r), \end{cases}$$

where

$$S_1(r) = K_{19}[\sum_{i=0}^{p_{11}}\sum_{j=0}^{p_{12}}\log^+ M(r,a_{ij}) + \sum_{i=0}^{q_1}\log^+ M(r,b_i) + O(\log r)],$$
$$S_2(r) = K_{20}[\sum_{i=0}^{p_{21}}\sum_{j=0}^{p_{22}}\log^+ M(r,c_{ij}) + \sum_{i=0}^{q_2}\log^+ M(r,d_i) + O(\log r)].$$

Further, we get

$$(\min\{n_1, n_1+q_1-p_{11}\}\min\{n_2, n_2+q_2-p_{22}\} - K_{19}K_{20}p_{12}p_{21})\log^+ M(r,w_i) \le S(r), r \notin E,$$

where $i = 1, 2$,

$$\begin{aligned} S(r) \quad = \quad & K_{21}[\sum_{i=0}^{p_{11}}\sum_{j=0}^{p_{12}}\log^+ M(r,a_{ij}) + \sum_{i=0}^{q_1}\log^+ M(r,b_i) \\ & + \sum_{i=0}^{p_{21}}\sum_{j=0}^{p_{22}}\log^+ M(r,c_{ij}) + \sum_{i=0}^{q_2}\log^+ M(r,d_i)] + O(\log r), \end{aligned}$$

where K_{21} is a positive constant.

Since there exists a positive constant K such that $\min\{n_1, n_1+q_1-p_{11}\}\min\{n_2, n_2+q_2-p_{22}\} > Kp_{12}p_{21}$, then we obtain

$$m(r,w_1) \le S(r), \tag{3.15}$$
$$m(r,w_2) \le S(r). \tag{3.16}$$

Combining the inequalities (3.1), (3.2), (3.15) and (3.16), we have

$$T(r,w_i) = O(\log r) \quad (r \notin E), \quad i = 1, 2,$$

which shows that $(w_1(z), w_2(z))$ is an algebraic solution of (1.1).

This completes the proof of Theorem 1.1.

4 Some Examples

Example 4.1 and example 4.2 show that the conditions in Theorem 1.1 are sharp. Example 4.3 and example 4.4 show that Theorem 1.1 holds.

Example 4.1 $(w_1(z), w_2(z)) = (e^{2z}, 2e^z)$ is a nonconstant transcendental meromorphic solution of the following system of differential equations

$$\begin{cases} (w_1')^3 = \dfrac{8w_1^3 + w_1w_2^2 + 2w_1^3w_2^2 - 4w_1^2}{w_1 + 1}, \\ (w_2')^3 = \dfrac{4w_1^2 - 3w_1w_2 - 4w_1^2w_2 - w_1w_2^2 - \frac{1}{4}w_2^3 + 5w_1w_2^3}{w_2^2 - 1}. \end{cases}$$

It is easy to know that

$$n_1 = 3, q_1 = 1, p_{11} = 3, p_{12} = 2; n_2 = 3, q_2 = 2, p_{21} = 2, p_{22} = 3.$$

Thus

$$n_1 + q_1 = 4 < 5 = p_{11} + p_{12}, n_2 + q_2 = 5 = p_{21} + p_{22},$$

or

$$n_1 + q_1 = 4 < 5 = p_{11} + p_{21}, n_2 + q_2 = 5 = p_{12} + p_{22}.$$

Example 4.2 $(w_1(z), w_2(z)) = (\sin z, \cos^2 z)$ is a nonconstant transcendental meromorphic solution of the following system of differential equations

$$\begin{cases} (w_1')^2 = \dfrac{(w_2^2)}{1 - w_1^2}, \\ (w_2')^2 = \dfrac{5w_1^2w_2 + 4w_1^2w_2^2 - w_2 + w_2^2}{w_2 + 1}. \end{cases}$$

In this case,

$$n_1 = 2, q_1 = 2, p_{11} = 0, p_{12} = 2; n_2 = 2, q_2 = 1, p_{21} = 2, p_{22} = 2.$$

Thus

$$n_1 + q_1 = 4 > 2 = p_{11} + p_{12}, n_2 + q_2 = 3 < 4 = p_{21} + p_{22},$$

or

$$n_1 + q_1 = 4 > 2 = p_{11} + p_{21}, n_2 + q_2 = 3 < 4 = p_{12} + p_{22}.$$

Example 4.3 $(w_1(z), w_2(z)) = (z^2, 2z)$ is a nonconstant algebraic solution of the following system of differential equations

$$\begin{cases} (w_1')^2 = \dfrac{6zw_1 + 2zw_2 + 2z^2w_1^2 - 3w_1w_2 - 3zw_1^2w_2}{1 - w_1^2}, \\ (w_2')^3 = \dfrac{z^2 - 2z + 8 + 3w_1^2w_2^2 + (9z - 7)w_2 - 3z^3w_1w_2 + 7zw_1^2w_2 + 13w_1 - 5z^2w_1w_2^2}{w_2^2 - w_2 + 1}. \end{cases}$$

Clearly, we get

$$n_1 = 2, q_1 = 2, p_{11} = 2, p_{12} = 1; \quad n_2 = 3, q_2 = 2, p_{21} = 2, p_{22} = 2.$$

In this case, Case (i) holds, that is

$$n_1 + q_1 = 4 > 3 = p_{11} + p_{12}, n_2 + q_2 = 5 > 4 = p_{21} + p_{22},$$

but Case (ii) does not hold, that is

$$n_1 + q_1 = 4 = p_{11} + p_{21}, n_2 + q_2 = 5 > 3 = p_{12} + p_{22}.$$

There exists a positive constant $K = 2$ such that

$$\min\{n_1, n_1 + q_1 - p_{11}\} \min\{n_2, n_2 + q_2 - p_{22}\} = 2 \times 3 = 6 > 4 = Kp_{12}p_{21}.$$

Example 4.4 $(w_1(z), w_2(z)) = (\frac{1}{z}, 3z)$ is a nonconstant algebraic solution of the following system of differential equations

$$\begin{cases} (w_1')^3 = \dfrac{2w_2^3 - 11z^2w_1^2 + zw_1^2w_2 - z^2w_1^2w_2^3 + w_1^2w_2^2 - 2zw_2^2 - 3z^2w_2 - 2zw_1}{z^2w_1^2 - 2zw_1 + z^6 + 1}, \\ (w_2')^2 = \dfrac{(4z^2+1)w_1^2w_2 - 5z^2w_1^2 - w_1w_2 - 2w_2 + (5z^2 + 7z - 3)w_1 - 2z + 1}{\frac{1}{9}w_2^3 - zw_2^2 + 2z^2w_2 + z}. \end{cases}$$

Easily, we obtain

$$n_1 = 3, q_1 = 2, p_{11} = 2, p_{12} = 3; n_2 = 2, q_2 = 3, p_{21} = 2, p_{22} = 1.$$

In this case, Case (i) does not hold, that is

$$n_1 + q_1 = 5 = p_{11} + p_{12}, n_2 + q_2 = 5 > 3 = p_{21} + p_{22},$$

but Case (ii) holds, that is

$$n_1 + q_1 = 5 > 4 = p_{11} + p_{21}, n_2 + q_2 = 5 > 4 = p_{12} + p_{22}.$$

There exists a positive constant $K = 1$ such that

$$\min\{n_1, n_1 + q_1 - p_{11}\} \min\{n_2, n_2 + q_2 - p_{22}\} = 3 \times 4 = 12 > 6 = Kp_{12}p_{21}.$$

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复微分方程组的代数解

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摘要: 本文研究了一类复微分方程组的代数解的存在问题. 利用最大模原理和Nevanlinna值分布理论, 得到了一个结论, 推广和改进了一些文献的结果, 例子表明结论精确.

关键词: 微分方程; 代数解; 最大模原理

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