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ROSENTHAL TYPE INEQUALITY OF B-VALUED QUASI-MARTINGALE

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Abstract: In this paper we discuss the Rosenthal type inequality of quasi-martingale. By using good λ inequality, we prove that Rosenthal type inequality of quasi-martingale and geometric properties of Banach space are equivalent. As a consequence, we prove the law of large numbers. These conclusions generalize some known results.

Keywords: Rosenthal type inequality; quasi-martingale; geometric properties

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1 Introduction

The inequalities of partial sums have been studied by a lot of authors. The moments of random variables play an important role in limit theory of random variable sequence. By this the Marcinkiewicz-Zygmund-Burkholder type inequality, Kolmogorov type inequality, Rosenthal type inequality and Bernstein inequality are discussed.

In 1970, Rosenthal [1] proved that for real valued independent random variables, the following inequality is true

$$E|\sum_{k=1}^{n} X_k|^r \le B_r \max\{\sum_{k=1}^{n} E|X_k|^r, \quad (\sum_{k=1}^{n} E|X_k|^2)^{r/2}\}, \tag{1.1}$$

where $\{X_k; 1 \le k \le n\}$ are independent random variables with zero mean, $r > 2$ and B_r is a positive constant only depending on r.

Martingale difference can be regarded as the generalization of independent random variables and some classical inequalities can also be generalized such as Rosenthal type inequality [2].

$$\begin{aligned} & c_1\{E\left[\big(\sum_{i=1}^{n} E(X_i^2|\Sigma_{i-1})\big)^{r/2}\right] + \sum_{i=1}^{n} E|X_i|^r\} \\ \le\quad & E|f_n|^r \le c_2\{E\left[\big(\sum_{i=1}^{n} E(X_i^2|\Sigma_{i-1})\big)^{r/2}\right] + \sum_{i=1}^{n} E|X_i|^r\}, \end{aligned} \tag{1.2}$$

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where $(f_i, \Sigma_i, 1 \le i \le n)$ is a real valued martingale, $X_1 = f_1, X_i = f_i - f_{i-1}, i = 2, 3, \cdots, n$, $2 \le r < \infty$, c_1 and c_2 are all positive constants only depending on r.

In 1981, De Acosta [3] proved the following inequality for Banach valued independent random variables which can be regarded as the generalization of Rosenthal type inequality:

$$E\Big|\|f_n\| - E\|f_n\|\Big|^r \le c_r\{(\sum_{i=1}^n E\|X_i\|^2)^{r/2} + \sum_{i=1}^n E\|X_i\|^r\}, \tag{1.3}$$

where $r > 2$, $(X_i; 1 \le i \le n)$ are all independent random variables in Banach space and $f_n = \sum\limits_{i=1}^n X_i, n \ge 1$.

2 Preliminaries and Notations

Let (Ω, Σ, P) be a probability space, $(X, \|\cdot\|)$ be a Banach space and $(\Sigma_n, n \ge -1)$ be a increasing σ-sub-algebra sequence in Σ, where $\Sigma = \cup_{n\ge -1}\Sigma_n,\ \Sigma_{-1} = \{\Omega, \emptyset\}$.

Definition 1 [4] Let $1 \le \alpha < \infty$. $f = (f_n, \Sigma_n, n \ge 0)$ is called a α quasi-martingale if

$$\sum_{n=1}^\infty \|E(f_{n+1}|\Sigma_n) - f_n\|_\alpha < \infty.$$

When $\alpha = 1$, $(f_n, \Sigma_n, n \ge 0)$ is called a quasi- martingale.

Definition 2 [4] Banach space X is called p smoothable, if there exists a constant $c > 0$ such that $\rho_X(\tau) \le c\tau^p, \tau > 0$, where

$$\rho_X(\tau) = \sup\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\}.$$

Definition 3 [4] Banach space X is called q convexiable, if there exists a constant $c > 0$ such that $\delta_X(\varepsilon) \ge c\varepsilon^q, \quad 0 \le \varepsilon \le 2$, where

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}.$$

In this paper the following notations will be used.

Let $0 < p < \infty$ and $f = (f_n, n \ge 0)$ be an adapted process,

$$\begin{aligned}
f_n^* &= \sup_{n\ge k\ge 0}\|f_k\|, \quad f^* = \sup_{n\ge 0}\|f_n\|;\\
df_n &= f_n - f_{n-1}, \quad n \ge 0, \quad f_{-1} = 0;\\
S^{(p)}(f) &= (\sum_{i=0}^\infty \|df_i\|^p)^{1/p}, \quad S_n^{(p)}(f) = (\sum_{i=0}^n \|df_i\|^p)^{1/p};\\
\sigma^{(p)}(f) &= \Big(\sum_{i=0}^\infty E(\|df_i\|^p\big|\Sigma_{i-1})\Big)^{1/p}, \quad \sigma_n^{(p)}(f) = \Big(\sum_{i=0}^n E(\|df_i\|^p\big|\Sigma_{i-1})\Big)^{1/p};\\
R_n(f) &= \sum_{i=0}^n \|E(df_i|\Sigma_{i-1})\|, \quad R(f) = \sup_{n\ge 0} R_n(f).
\end{aligned}$$

In this paper, the constants c may denote different constants in different contexts.

3 Main Results

Lemma 1[2] Let $r>1,\beta>1,\delta>0$, ξ and η be two non-negative random variables. If for all $\delta>0$ small enough, there exists constant ε_δ satisfying $\lim\limits_{\delta\to 0}\varepsilon_\delta=0$, such that

$$P(\xi>\beta\lambda,\eta\le\delta\lambda)\le\varepsilon_\delta P(\xi>\lambda),\quad\forall\lambda>0,$$

then there exists a constant c such that $E(\xi^r)\le cE(\eta^r)$.

Lemma 2 [5] Let X be a Banach space. Then the following statements are equivalent:

(1) X is p smoothable;

(2) There exists a constant $c>0$ such that for every X-valued quasi-martingale $f=(f_n,\Sigma_n,n\ge 0)$, $\|f^*\|_p\le c\big(\|\sigma^{(p)}(f)\|_p+\|R(f)\|_p\big)$.

Lemma 3 [5] Let X be a Banach space. Then the following statements are equivalent:

(1) X is q convexifiable;

(2) There exists a constant $c>0$ such that for every X-valued quasi-martingale $f=(f_n,\Sigma_n,n\ge 0)$, $\|S^{(q)}(f)\|_q\le c\big(\|f^*\|_q\le+\|R(f)\|_q\big)$.

Remark If $\|\cdot\|_q$ is replaced by any $\|\cdot\|_r,\ r\ge q$ in Lemma 3, the conclusion is also true.

Theorem 1 Let X be a Banach space, $1<p\le 2,\ p\le r<\infty$. Then the following statements are equivalent:

(1) X is p smoothable;

(2) There exists a constant $c>0$ only depending on p and r such that for every X-valued quasi-martingale $f=(f_n,\Sigma_n,n\ge 0)$,

$$\|f_n^*\|_r^r\le c\big(\|\sigma_n^{(p)}(f)\|_r^r+\|R_n(f)\|_r^r+\sum_{i=1}^n E(\|df_i\|^r)\big),\quad\forall n\ge 1.$$

Proof (1) $\Longrightarrow$ (2) Suppose X is p smoothable. Let $\xi=\max_{i\le n}\|f_i\|$,

$$\eta=\max\{\sigma_n^{(p)}(f),R_n(f),\max_{i\le n}\|df_i\|\}$$

and $\varepsilon=\delta^p/(\beta-\delta-1)^p$, where $\beta>1,0<\delta<\beta-1$.

Let

$$A_k=\{\omega|\lambda<\max_{i\le k-1}\|f_i(\omega)\|\le\beta\lambda,\quad\max_{i\le k-1}\|df_i(\omega)\|\le\delta\lambda,\max\{\sigma_k^{(p)}(f)(\omega),R_k(f)(\omega)\}\le\delta\lambda\}$$

and $T_i=\sum\limits_{k=1}^i\chi_{A_k}df_k, 1\le i\le n$. Then $T=(T_i,\Sigma_i,\ 1\le i\le n)$ is an X-valued quasi-martingale. Now we let

$$k_0=\inf\{k;\lambda<\max_{i\le k-1}\|f_i(\omega)\|,1\le k\le n\},n_0=\sup\{k;\max_{i\le k-1}\|f_i(\omega)\|\le\beta\lambda\}.$$

Then $1 \le k_0 < n_0 \le n$ and $\|f_{k_0-1}\| > \lambda,\ \ \|f_{n_0}\| > \beta\lambda$.

On the set $\{\xi > \beta\lambda,\ \eta \le \delta\lambda\}$, we have $T_n = \sum\limits_{i=k_0}^{n_0} df_i = f_{n_0} - df_{k_0} + f_{k_0-1}$ and

$$\|T_n\| \ge \|f_{n_0}\| - \|df_{k_0}\| - \|f_{k_0-1}\| \ge (\beta - \delta - 1)\lambda. \tag{3.1}$$

Moreover

$$P(\xi > \beta\lambda, \eta \le \delta\lambda) \le P(\|T_n\| \ge (\beta-\delta-1)\lambda) \le (\beta-\delta-1)^{-p}\lambda^{-p}E(T^*)^p. \tag{3.2}$$

Since X is p smoothable, by Lemma 2 we have

$$E(T^*)^p \le c\big(\|\sigma^{(p)}(T)\|_p + \|R(T)\|_p\big)^p \le c\big(\|\sigma^{(p)}(T)\|_p^p + \|R(T)\|_p^p\big).$$

By calculation we have

$$\begin{aligned}
\|\sigma^{(p)}(T)\|_p^p &= E[\sum_{k=1}^n \chi_{A_k} E(\|df_k\|^p|\Sigma_{k-1})] \\
&\le E[\sum_{k=1}^n \chi_{\{\omega|\lambda<\max_{i\le k-1}\|f_i(\omega)\|,\ \sigma_n^{(p)}(f)(\omega)\le\delta\lambda\}} E(\|df_k\|^p|\Sigma_{k-1})] \\
&\le c\delta^p\lambda^p P(\max_{i\le n}\|f_i\| > \lambda)
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\|R(T)\|_p^p &= \big\|\sum_{k=1}^n \|E(dT_k|\Sigma_{k-1})\|\big\|_p^p = \big\|\sum_{k=1}^n \chi_{A_k}\|E(df_k|\Sigma_{k-1})\|\big\|_p^p \\
&\le E[\sum_{k=1}^n \chi_{\{\omega|\lambda<\max_{i\le k-1}\|f_i(\omega)\|,\ R_k(f)(\omega)\le\delta\lambda\}} \|E(df_k|\Sigma_{k-1})\|]^p \\
&\le c\delta^p\lambda^p P(\max_{i\le n}\|f_i\| > \lambda).
\end{aligned} \tag{3.4}$$

By (3.2), (3.3) and (3.4) we have $P(\xi > \beta\lambda, \eta \le \delta\lambda) \le c(\beta-\delta-1)^{-p}\delta^p P(\max\limits_{i\le n}\|f_i\| > \lambda)$, where $\lim\limits_{\delta\to 0}\varepsilon = \lim\limits_{\delta\to 0}(\beta-\delta-1)^{-p}\delta^p = 0$. By Lemma 1, we have $\forall n \ge 1$,

$$\begin{aligned}
\|f_n^*\|_r^r = E(\xi^r) &\le cE(\eta^r) \le c\big(E(\sigma_n^{(p)}(f)^r) + E((\max_{i\le n}\|df_i\|)^r) + E(R_n(f)^r)\big) \\
&\le c\big(E(\sigma_n^{(p)}(f)^r) + \sum_{i=1}^n E(\|df_i\|^r) + E(R_n(f)^r)\big) \\
&= c\big(\|\sigma_n^{(p)}(f)\|_r^r + \|R_n(f)\|_r^r + \sum_{i=1}^n E(\|df_i\|^r)\big).
\end{aligned} \tag{3.5}$$

(2) $\Longrightarrow$ (1) Suppose (2) is true. Let $r=p$. For all $n\geq 1$ we have

$$
\begin{aligned}
\|f_n^*\|_p &\leq c\big(\|\sigma_n^{(p)}(f)\|_p^p+\|R_n(f)\|_p^p+\sum_{i=1}^n E(\|df_i\|^p)\big)^{1/p}\\
&\leq c\big(\|\sigma_n^{(p)}(f)\|_p+\|R_n(f)\|_p+E(\sum_{i=1}^n \|df_i\|^p)^{1/p}\big)\\
&= c\big(\|\sigma_n^{(p)}(f)\|_p+\|R_n(f)\|_p+E(\sum_{i=1}^n E(\|df_i\|^p|\Sigma_{i-1}\big)^{1/p}\big)\\
&\leq c\big(\|\sigma_n^{(p)}(f)\|_p+\|R_n(f)\|_p\big). \qquad (3.6)
\end{aligned}
$$

By Lemma 2, X is p-smoothable.

Theorem 2 Let X be a Banach space, $2\leq q<\infty, q\leq r<\infty$. Then the following statements are equivalent:

(1) X is q convexifiable;

(2) There exists a constant $c>0$ only depending on p and r such that for every X-valued quasi-martingale $f=(f_n,\Sigma_n,n\geq 0)$,

$$\|\sigma_n^{(q)}(f)\|_r^r+\sum_{i=1}^n E(\|df_i\|^r)\leq c\{\|f_n^*\|_r^r+\|R_n(f)\|_r^r\},\quad \forall n\geq 1.$$

Proof (1) $\Longrightarrow$ (2) Suppose X is q-convexifiable, by Remark there exists a constant $c>0$ such that for every X-valued quasi-martingale $f=(f_n,\Sigma_n,n\geq 0)$

$$\|S^{(q)}(f)\|_r\leq c(\|f^*\|_r+\|R(f)\|_r),\quad r\geq q. \qquad (3.7)$$

By the fact $\sum\limits_{i=1}^n E(\|df_i\|^r)=E(\sum\limits_{i=1}^n \|df_i\|^r)\leq E\big((\sum\limits_{i=1}^n \|df_i\|^q)^{r/q}=\|S_n^{(q)}(f)\|_r^r\big),\quad r\geq q$ we have

$$
\begin{aligned}
\|\sigma_n^{(q)}(f)\|_r^r+\sum_{i=1}^n E(\|df_i\|^r) &\leq c\|S_n^{(q)}(f)\|_r^r+\sum_{i=1}^n E(\|df_i\|^r)\\
&\leq c\|S_n^{(q)}(f)\|_r^r\leq cE(\max_{i\leq n}\|f_i\|^r)+\|R_n(f)\|_r^r. \qquad (3.8)
\end{aligned}
$$

(2) $\Longrightarrow$ (1) Suppose (2) is true. Let $r=q$. By the fact

$$\|S_n^{(q)}(f)\|_q^q=E(\sum_{i=0}^n \|df_i\|^q)=\|\sigma_n^{(q)}(f)\|_q^q,$$

we have

$$\|S_n^{(q)}(f)\|_q^q=1/2(\|\sigma_n^{(q)}(f)\|_q^q+\sum_{i=1}^n E(\|df_i\|^q))\leq c\{\|f_n^*\|_q^q+\|R_n(f)\|_q^q\}.$$

Thus

$$\|S^{(q)}(f)\|_q\leq c\big(\|f^*\|_q\leq +\|R(f)\|_q\big).$$

By Lemma 3 X is q-convexifiable.

Corollary 1 Let X be a Banach space, $2 \le r < \infty$, Then the following statements are equivalent:

(1) X is a Hilbert space;

(2) There exists a constant c such that

$$\begin{aligned} & c^{-1}\big(\|\sigma_n^{(2)}(f)\|_r^r + \sum_{i=1}^{n} E(\|df_i\|^r)\big) \\ \le & \quad \|f_n^*\|_r^r + \|R_n(f)\|_r^r \le c\big(\|\sigma_n^{(2)}(f)\|_r^r + \sum_{i=1}^{n} E(\|df_i\|^r) + \|R_n(f)\|_r^r\big), \forall n \ge 1. \end{aligned}$$

Theorem 3 Let X be a p-smoothable Banach space, $1 < p \le 2, p \le r < \infty$. If $f = (f_n, \Sigma_n, n \ge 0)$ is an X-valued quasi-martingale satisfying $\sum\limits_{n=1}^{\infty}(\frac{E(\|df_n\|^r|\Sigma_{n-1})}{n^r})^{1/r} < \infty$ a.s., then $\frac{1}{n}f_n = \frac{1}{n}\sum\limits_{i=1}^{n} df_i \to 0$ a.s..

Proof Since $\sum\limits_{n=1}^{\infty}(\frac{E(\|df_n\|^r|\Sigma_{n-1})}{n^r})^{1/r} < \infty$ a.s. for $0 < \varepsilon < 1$, there exists $\mathbf{N}$, such that

$$\sum_{n=\mathbf{N}+1}^{\infty}(\frac{E(\|df_n\|^r|\Sigma_{n-1})}{n^r})^{1/r} \le \varepsilon < 1 \quad \text{a.s..}$$

Now, for any $m > n \ge \mathbf{N}$, let $dg_i = \frac{df_i}{i}$, then $\sum\limits_{i=n+1}^{m} \frac{df_i}{i} = \sum\limits_{i=n+1}^{m} dg_i = g_m - g_n := g_m^{(n)}$. By the fact $dg_i^{(n)} = g_i^{(n)} - g_{i-1}^{(n)} = g_i - g_{i-1} = dg_i = \frac{df_i}{i}, (i > n)$ and

$$\begin{aligned} & \sum_{m=n+1}^{\infty} \|E(g_{m+1}^{(n)}|\Sigma_m) - g_m^{(n)}\|_1 = \sum_{m=n+1}^{\infty} \|\frac{E(df_{m+1}|\Sigma_m)}{m+1}\|_1 \\ \le & \sum_{m=n+1}^{\infty} \|E(f_{m+1}|\Sigma_m) - f_m)\|_1 < \infty, \end{aligned}$$

we know $g^{(n)} = (g_m^{(n)}, m \ge n)$ is a quasi-martingale.

By Theorem 1, we have

$$\|g_m^{(n)}\|_r^r \le c\big(\sum_{i=n+1}^{m} E(\|dg_i^{(n)}\|^r) + \|\sigma_m^{(p)}(g^{(n)})\|_r^r + \|R_m(g^{(n)})\|_r^r\big),$$

i.e.,

$$\begin{aligned} & E\|\sum_{i=n+1}^{m} \frac{df_i}{i}\|^r \\ \le & \quad c\{\sum_{i=n+1}^{m} \frac{E\|df_i\|^r}{i^r} + E\big(\sum_{i=n+1}^{m} \frac{E(\|df_i\|^p|\Sigma_{i-1})}{i^p}\big)^{r/p} + E\big(\sum_{i=n+1}^{m} \frac{\|E(df_i|\Sigma_{i-1})\|}{i}\big)^r\}. \\ = & \quad : I + II + III. \end{aligned} \tag{3.9}$$

Since

$$\lim_{n\to\infty}\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}=0 \quad \text{a.s.},$$

then

$$\sup_{n\geq 1}\frac{E(\|df_n\|^r|\Sigma_{n-1})}{n^r}<\infty \text{ a.s..} \tag{3.10}$$

Thus we have

$$\begin{aligned} I &= \sum_{i=n+1}^{\infty}\frac{E(\|df_i\|^r)}{i^r} \\ &= E\Big(\sum_{i=n+1}^{\infty}\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big) \\ &= E\Big(\sum_{i=n+1}^{\infty}\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{1/r}\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{1-1/r}\Big) \\ &\leq E\Big(\Big(\sup_{n\geq 1}\frac{E(\|df_n\|^r|\Sigma_{n-1})}{n^r}\Big)^{1-1/r}\sum_{i=n+1}^{\infty}\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{1/r}\Big) \\ &\to 0. \end{aligned} \tag{3.11}$$

Since $r\geq p$, by Jensen inequality

$$\frac{E(\|df_i\|^p|\Sigma_{i-1})}{i^p}\leq\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{p/r}\leq\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{1/r}, \quad \forall i\geq 1,$$

then

$$II=E\Big(\sum_{i=n+1}^{m}\frac{E(\|df_i\|^p|\Sigma_{i-1})}{i^p}\Big)\leq E\Big(\sum_{i=n+1}^{m}\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{1/r}\Big)\to 0 \ \ (n\to\infty). \tag{3.12}$$

Since

$$\begin{aligned} \sum_{i=n+1}^{m}\frac{\|E(df_i|\Sigma_{i-1})\|}{i} &\leq \sum_{i=n+1}^{m}\frac{E(\|df_i\|\big|\Sigma_{i-1})}{i} \\ &\leq \sum_{i=n+1}^{m}\Big(\frac{E(\|df_i\|^r|\Sigma_{i-1})}{i^r}\Big)^{1/r}\to 0 \qquad (n\to\infty). \end{aligned}$$

Then

$$III=E\Big(\sum_{i=n+1}^{m}\frac{\|E(df_i|\Sigma_{i-1})\|}{i}\Big)^r\to 0 \quad (n\to\infty). \tag{3.13}$$

By (14), (15) and (16) we have $\lim\limits_{n\to\infty}E\|\sum\limits_{i=n+1}^{m}\frac{df_i}{i}\|^r=0$ a.s. and $\sum\limits_{i=1}^{m}\frac{df_i}{i}$ is convergent almost everywhere. By Kronecher lemma $\frac{1}{n}f_n=\frac{1}{n}\sum\limits_{i=1}^{n}df_i\to 0$ a.s..

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B值拟鞅Rosenthal型不等式

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摘要: 本文研究了拟鞅Rosenthal型不等式的问题. 利用好λ 不等式得到拟鞅Rosenthal型不等式与值空间几何性质间的等价刻画, 进而得到大数定律. 这些结论丰富了已有结果.

关键词: Rosenthal 型不等式; 拟鞅; 几何性质

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