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NORMAL FAMILIES OF HOLOMORPHIC MAPPINGS FROM $\mathbb{C}^M$ INTO $P^n(\mathbb{C})$ WITH SOME FIXED HYPERSURFACES

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Abstract: In this paper, we study the normal families of meromorphic mappings. Applying the heuristic principle in several complex variables obtained by Aladro and Krantz [1] and some Picard theorems given by M. Shirosahi, we shall prove some normality criterias for families of holomorphic mappings of several complex variables into $P^n(\mathbb{C})$, the n-dimensional complex projective space, related to some special hypersurfaces.

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1 Introduction

Let $f(z)$ be a meromorphic function on the complex plane. By the second main theorem of value distribution theory [6], we have the following classical Picard's theorem.

Theorem A If there exist three mutually distinct points a_1, a_1 and a_3 on the Riemann sphere such that $f(z)-a_i$ $(i=1,2,3)$ has no zero on the complex plane, then f is a constant.

Let F be a family of meromorphic functions defined on a domain D of the complex plane. F is said to be normal on D if every sequence of functions of F has a subsequence which converges uniformly on every compact subset of D with respect to the spherical metric to a meromorphic function or identically ∞ on D. Perhaps the most celebrated criteria for normality in one complex variable is the following Montel-type theorem, see ref. [2].

Theorem B Let F be a family of meromorphic functions on a domain D of the complex plane. Suppose that there exist three mutually distinct points a_1, a_1 and a_3 on the Riemann sphere such that $f(z)-a_i$ $(i=1,2,3)$ has no zero on D for each $f\in F$. Then F is a normal family on D.

The fact that Picard's theorem and normality criteria were so intimately related led Bloch to hypothesise that a family of meromorphic functions which have a property P in

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common on a domain D is normal on D if the property P forces a meromorphic function on the complex plane to be a constant. This hypothesize is called Bloch's heuristic principle in complex function theory [1, 13]. Although the principle is false in general, many authors proved normality criteria for families of meromorphic functions by starting from Picard-type theorems. Hence an interesting topic is to make the principle rigorous and to find its applications. There are many investigations in this field for one complex variable (see, e.g., [13, 14]).

In the case of higher dimension, the notion of normal family has proved its importance in geometric function theory in several complex variables(see, e.g., [8, 12]). Many Picard-type theorems have established for holomorphic mappings of $\mathbb{C}^n$ into $P^N(\mathbb{C})$(e.g. [4, 5]). Afterwards, the related normality criteria in several complex variables have proved by using various methods (see, e.g., [3, 9]). In particular, Aladro and Krantz [1] proved a criterion for normality in several complex variables and for the first time implemented a Zalcman type heuristic principle in this more general content. Using the heuristic principle obtained by Aladro and Krantz [1], Tu [11]extended Theorem B to the case of families of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $P^N(\mathbb{C})$ related to Fujimoto-Green's and Nochka's Picard-type theorems.

In this paper, inspired by the idea in Tu [11], we shall prove some normality criteria for families of holomorphic mappings of several complex variables into $P^n(\mathbb{C})$, related to Shirosahi's [10] Picard-type theorems.

2 Preliminaries and Main Results

For the general reference of this paper, see references [10, 11].

Let $P^n(\mathbb{C})$ be the complex projective space of dimension n, and $\rho : \mathbb{C}^{n+1} \setminus \{0\} \to P^n(C)$ be the standard projective mapping. Let f be a holomorphic mapping of a domain D in $\mathbb{C}^m$ into $P^n(\mathbb{C})$. Then for any $z \in D$, f always has a reduced representation

$$\tilde{f}(z) = (f_0(z), \cdots, f_n(z))$$

on some neighborhood U of z in D, that is, $f_0(z), \cdots, f_n(z)$ are holomorphic functions on U without common zeroes, such that $f(z) = \rho(\tilde{f}(z))$ on U.

Definition 2.1 A sequence $\{f^{(p)}(z)\}$ of holomorphic mappings from a domain D in $\mathbb{C}^m$ into $P^n(\mathbb{C})$ is said to converge uniformly on compact subsets of D to a holomorphic mapping $f(z)$ of D into $P^n(\mathbb{C})$ if and only if, for any $z \in D$, each $f^{(p)}(z)$ has a reduced representation

$$\tilde{f}^{(p)}(z) = (f_0^{(p)}(z), \cdots, f_n^{(p)}(z))$$

on some fixed neighborhood U of z such that $\{f_i^{(p)}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function $f_i(z)(i = 0, \cdots, n)$ on U with the property that

$$\tilde{f}(z) := (f_0(z), \cdots, f_n(z))$$

is a representation of $f(z)$ on U, where $f_{i_0}(z)$ is not identically equal to zero on U for some i_0.

Let F be a family of holomorphic mappings of a domain D in $\mathbb{C}^m$ into $P^n(\mathbb{C})$. F is said to be a normal family on D if any sequence in F contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into $P^n(\mathbb{C})$.

For the detailed discussion about convergence, see reference [3].

A hypersurface S in $P^n(\mathbb{C})$ is the projection of the set of zeros of a nonconstant homogeneous form P in $n+1$ variables. Let

$$P(w_0, w_1, \cdots, w_n) := \sum a_{i_0,\cdots,i_n} w_0^{i_0} \cdots w_n^{i_n} \quad (a_{i_0,\cdots,i_n} \in \mathbb{C})$$

be a homogeneous polynomial in $w_0, \cdots, w_n$. Then

$$S := \rho(\{(w_0, w_1, \cdots, w_n) \in \mathbb{C}^{n+1} \setminus \{0\} : P(w_0, w_1, \cdots, w_n) = 0\})$$

is a hypersurface in $P^n(\mathbb{C})$ given by the homogeneous polynomial P.

Let d and p be two positive integers with $d > 2p + 8$ and $p > 2$ which have no common factors. Define homogeneous polynomials $H(w_0, w_1) = w_0^d + w_0^p w_1^{d-p} + w_1^d$ with degree d.

We consider hypersurfaces given by homogeneous polynomials P and P_n, where

$$\begin{aligned} P(w_0, w_1) &= P_1(w_0, w_1) = H(w_0, w_1), \\ P_n(w_0, \cdots, w_n) &= P_{n-1}(P(w_0, w_1), \cdots, P(w_{n-1}, w_n)) \end{aligned}$$

with degree d^n for $n \geq 2$. In [10], M. Shirosahi gave some interesting Picard-type theorems for the hypersurfaces given by homogeneous polynomials P and P_n as follows.

Theorem C Let f and g be entire functions at least one of which are not identically equal to zero. If $P(f, g) = \alpha^d$ for some entire function α, then $(f : g)$ is constant.

Theorem D Let $f_0, f_1$and f_2 be entire functions at least two of which are not identically equal to zero, and let C be a nonzero constant. If $P(f_0, f_1) = CP(f_1, f_2)$,then $(f_0 : f_1 : f_2)$ is constant.

For more $f_j's$, Shirosahi proved the following results in [10].

Theorem E Let $n \geq 2$ be an integer and $f_0, \cdots, f_n$ entire functions. If at least two of $P(f_0, f_1)$, $P(f_1, f_2)$, $\cdots, P(f_{n-1}, f_n)$ are not identically equal to zero and $(P(f_0, f_1) : P(f_1, f_2) : \cdots : P(f_{n-1}, f_n))$ is constant, then $(f_0 : f_1 : \cdots : f_n)$ is constant.

Theorem F Let n be a positive integer and f a holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$ with a reduced representation $(f_0, \cdots, f_n)$. If $P_n(f_0, \cdots, f_n) = 0$, then f is constant.

Theorem G Let f be a holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$ with a reduced representation $(f_0, \cdots, f_n)$. If $P_n(f_0, \cdots, f_n) = \alpha^{d^n}$, for some entire function α not identically equal to zero, then f is constant.

Using the heuristic principle obtained by Aladro and Krantz [1], we will prove the normality criterions for family of holomorphic mappings of several complex variables into the complex projective space, related to Shirosahi's Picard-type theorems as follows:

Theorem 2.1 Let F be a family of holomorphic mappings from a domain D in $\mathbb{C}^m$ into $P^n(\mathbb{C})$. For any $z \in D$, each $f \in F$ has a reduced representation

$$\tilde{f}(z) = (f_0(z), f_1(z), \cdots, f_n(z))$$

on some neighborhood U of z in D. If $(P(f_0, f_1) : P(f_1, f_2) : \cdots : P(f_{n-1}, f_n))$ is constant on U, then F is normal on D.

For $n = 1$, We immediately obtain normality criteria related to Theorem D.

Corollary 2.1.1 Let F be a family of holomorphic mappings from a domain D in $\mathbb{C}^m$ into $P^2(C)$. For any $z \in D$, each $f \in F$ has a reduced representation

$$\tilde{f}(z) = (f_0(z), f_1(z), f_2(z))$$

on some neighborhood U of z in D. Let C be a nonzero constant, if $P(f_0, f_1) = CP(f_1, f_2)$ on U, then F is normal on D.

Theorem 2.2 Let F be a family of holomorphic mappings from a domain D in $\mathbb{C}^m$ into $P^n(\mathbb{C})$. For any $z \in D$, each $f \in F$ has a reduced representation

$$\tilde{f}(z) = (f_0(z), f_1(z), \cdots, f_n(z))$$

on some neighborhood U of z in D. If $P_n(f_0, f_1, \cdots, f_n) = \alpha^{d^n}$ on U, for some entire function α, then F is normal on D.

When $n = 1$ in Theorem 2.2, we get the normality criteria related to Theorem C.

Corollary 2.2.1 Let F be a family of holomorphic mappings from a domain D in $\mathbb{C}^m$ into $P^1(C)$. For any $z \in D$, each $f \in F$ has a reduced representation

$$\tilde{f}(z) = (f_0(z), f_1(z))$$

on some fixed neighborhood U of z, such that $P(f_0, f_1) = \alpha^d$ on U for some entire function α, then F is normal on D.

3 Proofs of Theorems 2.1 and 2.2

In order to prove Theorem 2.1 and 2.2, we need the following Lemma.

Lemma 3.1 Let F be a family of holomorphic mappings of a domain D in $\mathbb{C}^m$ into $P^n(\mathbb{C})$. The family F is not normal on D if and only if there exists a compact set $K \subset D$ and sequences $\{f_i\} \subset F$, $\{p_i\} \subset K$, $\{r_i\}$ with $r_i > 0$ and $r_i \to 0^+$ and $\{u_i\} \subset \mathbb{C}^m$ Euclidean unit vectors such that $g_i(\xi) := f_i(p_i + r_i u_i \xi)$, where $\xi \in \mathbb{C}$ satisfies $p_i + r_i u_i \xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping g of $\mathbb{C}$ into $P^n(\mathbb{C})$.

For the proof of Lemma 3.1, see Theorem 3.1 in reference [1], Theorem 6.5 in ref. [7] (Cf. reference [14]).

Proof of Theorem 2.1 Suppose that F is not a normal family on D. Then, by Lemma 3.1, there exist a compact set $K \subset D$ and sequences $\{f_k\} \subset F$, $\{p_k\} \subset K$, $\{r_k\}$ with $r_k > 0$ and $r_k \to 0^+$ and $\{u_k\} \subset \mathbb{C}^m$ Euclidean unit vectors such that

$$g_k(\xi) := f_k(p_k + r_k u_k \xi),$$

where $\xi \in \mathbb{C}$ satisfying $p_k + r_k u_k \xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping g of $\mathbb{C}$ into $P^n(\mathbb{C})$.

On the other hand, we will prove that g must be a constant holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$.

In fact, since K is a compact subset of D, without loss of generality, we might assume that $\{p_k\}(\subset K)$ converges to $p_0(\in K)$.

Now, let k_0 be a positive integer, such that for $k \geq k_0$, $d(p_k + r_k u_k \xi, p_0) \leq \min\{\frac{1}{2}d(\mathbb{C}^m - D, p_0), 1\}$ for all $\xi \in \{\xi \in \mathbb{C}; |\ \xi\ | \leq 1\}$, where $d(p,q)$ is the Euclidean distance between p and q in $\mathbb{C}^m$.

Let g have a reduced representation $\tilde{g}(\xi) = (g_0(\xi), g_1(\xi), \cdots, g_n(\xi))$ on $\mathbb{C}$.

Since $g_k(\xi) := f_k(p_k + r_k u_k \xi)$, where $\xi \in \mathbb{C}$ satisfies $p_k + r_k u_k \xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping g, we have

$$g_k(\xi) := f_k(p_k + r_k u_k \xi)(k \geq k_0)$$

that converges to g uniformly on $\{\xi \in \mathbb{C}; |\ \xi\ | \leq 1\}$.

Therefore, there exists a compact subset $K_1 \subset \{\xi \in \mathbb{C}; |\ \xi\ | \leq 1\}$, such that every $g_k(\xi) := f_k(p_k + r_k u_k \xi)(k \geq k_0)$ has a reduced representation

$$\tilde{g}_k(\xi) = (g_{0k}(\xi), g_{1k}(\xi), \cdots, g_{nk}(\xi))$$

on K_1 and $g_k(\xi)$ converges to $g(\xi)$ uniformly on K_1 as $k \to \infty$.

By the assumption of Theorem 2.1, we have $(P(g_{0k}, g_{1k}) : P(g_{1k}, g_{2k}) : \cdots : P(g_{n-1k}, g_{nk}))$ $(k \geq k_0)$ is constant on K_1. Hence $(P(g_0, g_1) : P(g_1, g_2) : \cdots : P(g_{n-1}, g_n))$ is constant on K_1, and thus $(P(g_0, g_1) : P(g_1, g_2) : \cdots : P(g_{n-1}, g_n))$ is constant on $\mathbb{C}$.

For the case at least two of $P(g_0, g_1)$, $P(g_1, g_2)$, $\cdots$, $P(g_{n-1}, g_n)$ is not identically equal to zero, by Theorem E, g must be a constant holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$. For the other case, it is easy to see that g also is a constant mapping.

Therefore, this is a contradiction. The proof of Theorem 2.1 is completed.

Proof of Theorem 2.2 Suppose that F is not a normal family on D. According to the proof of Theorem 2.1, $g_k(\xi) := f_k(p_k + r_k u_k \xi)(k \geq k_0)$ has a reduced representation

$$\tilde{g}_k(\xi) = (g_{0k}(\xi), g_{1k}(\xi), \cdots, g_{nk}(\xi))$$

on K_1 and $g_k(\xi)$ converges to $g(\xi)$ uniformly on K_1 as $k \to \infty$.

On the other hand, $P_n(g_{0k}, g_{1k}, \cdots, g_{nk}) = \alpha^{d^n}$ on K_1, where α is an entire function.

Thus $P_n(g_0, g_1, \cdots, g_n) = \alpha^{d^n}$ on K_1, hence $P_n(g_0, g_1, \cdots, g_n) = \alpha^{d^n}$ on $\mathbb{C}$.

Case 1 α is identically equal to zero. Using Theorem F, g must be a constant holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$, a contradiction.

Case 2 α is not identically equal to zero. By Theorem G, g must be a constant holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$, a contradiction.

Thus, g is a constant. This is a contradiction. Theorem 2.2 is proved.

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涉及复代数超曲面的全纯映射正规族

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摘要: 本文研究了涉及固定超曲面的全纯映照的正规性问题. 利用Aladro 和Krantz对全纯映射族正规性的刻画和Shirosahi建立的一系列涉及一些特殊复代数超曲面的Picard 型定理, 得到了全纯映射族的一些正规定则.

关键词: 正规族; 全纯映射; 超曲面; 值分布理论

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