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ON STABILITY OF RANDOM FIXED POINTS WITH SET-VALUED MAPPINGS

SONG Qi-qing

(*College of Science, Guilin University of Technology, Guilin 541004, China*)

Abstract: This paper studies the stability of fixed points for random set-valued mappings. Through set-valued analyses, the existence of essentially stable sets of random fixed points is established. In the sense of Baire category, each random fixed point for most of random set-valued mappings is essentially stable. These generalize some results in the corresponding references.

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1 Introduction

Many works in relation to random fixed points have appeared for the existence and uniqueness for random single-valued and set-valued mappings, see [1–5] and references therein. These have been also applied to random generalized games, random quasi-variational inequalities, random equations, etc (see [6–8]).

It is known that the stability of fixed point theory is an important aspect in nonlinear analysis. Originally, Fort introduced a conception of essential fixed point in the sense of resisting the perturbation of functions in 1950 (see [9]). Nowadays, we can find that essential stabilities have been widely used in many fields [10–16]. In the paper [17] by Beg, essential fixed points in relation to random single-valued mappings were studied, and a sufficient and necessary condition for the continuity of fixed point mappings was obtained.

Inspired by these methods for the study of fixed points, this paper studies the essential stability of random fixed points in depth. Essentially stable sets are introduced to random fixed points with set-valued mappings, and the existence of essentially stable sets of fixed points is proved. In the sense of Baire category, we show that each random fixed point for most of random set-valued mappings is essential.

2 Preliminaries

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Biography: Song Qiqing (1980–), male, born at Dingxiang, Shanxi, associate professor, major in set-valued optimization and game theory.

Let E, Y be two topological spaces. We need recall some notions with set-valued mappings. Let $F : E \to 2^Y$ be a set-valued mapping, where 2^Y denotes the collection of all subsets of Y.

(i) F is said to be upper semi-continuous (lower semi-continous) at $x \in E$, if for each open set U with $U \supset F(x)$ $(U \cap F(x) \neq \emptyset)$, there exists an open neighborhood $O(x)$ of x such that $U \supset F(x')$ $(U \cap F(x') \neq \emptyset)$ for any $x' \in O(x)$;

(ii) F is continuous at $x \in E$ if it is both upper semi-continuous and lower semi-continuous at x;

(iii) F is said to an usco mapping if F is upper semi-continuous with compact values.

Let (X, d) be a compact convex subset of a separable metric linear space and (Ω, Σ) be a Σ-measureable space, where Σ is a σ algebra on the set Ω. $T : \Omega \times X \to 2^X$ is a random set-valued mapping with nonempty closed convex values such that

(i) for each $w \in \Omega$, $T(w, \cdot) : X \to 2^X$ is continuous;

(ii) for each $x \in X$, $T(\cdot, x) : \Omega \to 2^X$ is measurable.

Let $CB(X)$ be the collection of such random set-valued mapping T. Employ the Hausdorff metric, for any two $T, S \in CB(X)$, we can define the metric between them by

$$\rho(T, S) = \sup_{(w,x) \in \Omega \times X} H(T(w, x), S(w, x)).$$

Then $(CB(X), \rho)$ is a metric space. Let M be the set of all Σ-measurable functions from Ω to X. Define the metric between any two $\xi, \eta \in M$ as $m(\xi, \eta) = \sup_{w \in \Omega} d(\xi(w), \eta(w))$, then (M, m) is a compact metric space.

Definition 2.1 For each $T \in CB(X)$, a mapping $\xi : \Omega \to X$ is called a random fixed point of T if $\xi \in M$ and $\xi(\omega) \in (T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

To study the stability of random fixed points for any $T \in CB(X)$, denote by $F(T)$ the set of random fixed points of T, then we define a set-valued mapping F as $F : CB(X) \to 2^M$. By Theorem 3.2 in [18], $F(T) \neq \emptyset$, $\forall\, T \in CB(X)$.

Definition 2.2 For each $T \in CB(X)$, a subset $e(F)$ of $F(T)$ is said to be an essential stable set of T if

(i) $e(F)$ is closed;

(ii) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $S \in CB(X)$ with $\rho(T, S) < \delta$ it holds that $F(S) \cap B(e(F), \varepsilon) \neq \emptyset$, where $B(e(F), \varepsilon)$ is the ε neighborhood of $e(F)$.

If an essential stable set $e(F) = \{x^*\}$, then x^* is said to be an essential random fixed point of T.

Lemma 2.1 Let $\{T_n\}$ be a sequence of measurable mappings $T_n : \Omega \to 2^X$ with nonempty closed values, and $T : \Omega \to 2^X$ a mapping such that for each $w \in \Omega$, $H(T_n(w), T(w)) \to 0$ as $n \to \infty$. Then T is measurable.

Lemma 2.2 (see [19]) Let Y be a metric space, E be a Baire space, and $F : E \to 2^Y$ be an usco mapping. Then, there is a dense residual subset Q of E such that F is lower semi-continuous at each $x \in Q$.

3 Essential Stability of Random Fixed Points

Theorem 3.1 The set-valued mapping F is usco.

Proof For each $T \in CB(X)$, $F(T)$ is compact. Let $\xi_n \in F(T)$ and $\xi_n \to \xi \in M$. Then $\xi_n(w) \in T(w, \xi_n(w)), \forall\, w \in \Omega$. Sine $\xi_n \to \xi$, we have $m(\xi_n, \xi) \to 0$, hence, $d(\xi_n(w), \xi(w)) \to 0, \forall\, w \in \Omega$. Noting that $T(w, \cdot)$ is continuous for each $w \in \Omega$, the right side hand of the following inequality

$$d(\xi(w), T(w, \xi(w))) \leq d(\xi(w), \xi_n(w)) + d(\xi_n(w), T(w, \xi_n(w)) + H(T(w, \xi_n(w)), T(w, \xi(w)))$$

gets close to zero as n tends to infinity, where $d(x, A) = \inf_{y \in A} d(x, y)$. Then $\xi \in F(T)$ because $T(w, \xi(w))$ is closed, hence $F(T)$ is closed and compact also.

For each $T \in CB(X)$, F is upper semi-continuous at T. By way of contradiction, suppose that F is not upper semi-continuous at T. Then there exists a $\varepsilon > 0$ and $T_n \in CB(X)$ with $T_n \to T$, such that $F(T_n) \not\subset B(F(T), \varepsilon)$, $n = 1, 2, \cdots$. That is, there is a $\xi_n \in F(T_n)$ but $\xi_n \notin B(F(T), \varepsilon), n = 1, 2, \cdots$. Since the sequence $\{\xi_n\} \subset M$, by the compactness of M, there exists a convergent sequence as it's subsequence. Without loss of generality, we may assume that $\xi_n \to \xi^* \in M$. Clearly, we have $\xi^* \notin B(F(T), \varepsilon)$ and for each $w \in \Omega$, it holds that

$$\begin{aligned} d(\xi^*(w), T(w, \xi^*(w))) \quad &\leq \quad d(\xi^*(w), \xi_n(w)) + d(\xi_n(w), T_n(w, \xi_n(w)) \\ &\quad +H(T_n(w, \xi_n(w)), T(w, \xi_n(w))) + H(T(w, \xi_n(w)), T(w, \xi^*(w))). \end{aligned}$$

Since $\xi_n \in F(T_n)$, $T_n \to T$, $\xi_n \to \xi^*$ and the continuity of $T(w, \cdot)$, for arbitrary $\alpha > 0$, we have $d(\xi^*(w), T(w, \xi^*(w)) \leq \alpha$ as n gets close to infinity. Noting that $T(w, \xi^*(w))$ is closed, we obtain that $\xi^*(w) \in T(w, \xi^*(w))$, $\forall\, w \in \Omega$, that is, $\xi^* \in F(T)$, a contradiction with $\xi^* \notin B(F(T), \varepsilon)$.

Generally, the mapping F is not lower semi-continuous on $CB(X)$ though it is upper semi-continuous by Theorem 3.1. See the following example.

Example 3.1 Let $\Omega = X = [0, 1]$, $T : \Omega \times X \to 2^X$ such that $T(w, x) = [x, 1]$ for each $(w, x) \in \Omega \times X$. Let $y : \Omega \to X$ be a measurable function such that $y(w) = w$ for each $w \in \Omega$. Since $y(w) = w \in T(w, y(w)) = [w, 1]$, we have y is a random fixed point of T. For each $n = 1, 2, \cdots$, define a set-valued mapping $T_n : \Omega \times X \to 2^X$ such that for each $(w, x) \in \Omega \times X$,

$$T_n(w, x) = [x + \frac{1 - x}{n}, 1], \forall\, (w, x) \in \Omega \times X.$$

Clearly, we have $T_n \to T$ as n gets close to infinity. Consider a function $z : \Omega \to X$, for each $w \in \Omega$, if $z(w) \in [0, 1)$, then we have

$$z(w) \notin T_n(w, z(w)) = [z(w) + \tfrac{1 - z(w)}{n}, 1];$$

if $z(w) = 1$, then

$$z(w) \in T_n(w, z(w)) = 1, \forall\, n = 1, 2, \cdots.$$

Therefore, for each $n = 1, 2, \cdots$, there is only one measurable function z such that $z(w) \equiv 1$ as the random fixed point of T_n. For each $\varepsilon < 1$, we have $F(T) \cap B(y, \varepsilon) \neq \emptyset$, however, whatever close T_n is to T, it holds that $F(T_n) \cap B(y, \varepsilon) = \emptyset$ because $m(y, z) \equiv 1$. That is, F is not lower semi-continuous at T.

Thus, for each $T \in CB(X)$, the set $F(T)$ is closed, and from Theorem 3.1 and the concept of an essentially stable set, we can obtain the following result.

Corollary 3.1 For each $T \in CB(X)$, $F(T)$ itself is an essentially stable set of T.

Next, we show a property of essential fixed point set and a sufficient and necessary condition for the mapping F being continuous.

Theorem 3.2 For each $T \in CB(X)$, then the set $EF(T)$, consisting of all essential fixed points of T, is closed.

Proof Let $\xi_t \in EF(T)$ with $\xi_t \to \xi$. Then for each $t = 1, 2, \cdots$, $\xi_t(w) \in T(w, \xi_t(w))$, $\forall\, w \in \Omega$. Since $F(T)$ is closed, we have that ξ is a random fixed point of T. Next we show that ξ is essential. Suppose that ξ is not an essential fixed point of T. Then there exists $\varepsilon > 0$ and $T_n \in CB(X)$ with $T_n \to T$ such that $F(T_n) \cap B(\xi, \varepsilon) = \emptyset$, $\forall\, n = 1, 2, \cdots$. Since $\xi_t \to \xi$, there is a number N such that $m(\xi_t, \xi) < \varepsilon/2$ for each $t > N$. That is, $\xi_t \notin F(T_n), \forall\, n = 1, 2, \cdots$. For each t, by the essentiality of ξ_t and the fact $T_n \to T$, there exists a number $s(t)$ such that we can find a point $\eta_n \in F(T_n)$ satisfying that $m(\xi_t, \eta_n) < \varepsilon/2$ for each $n \geq s(t)$. Then, as t is large enough, we have

$$m(\xi, \eta_{s(t)}) \leq m(\xi, \xi_t) + m(\xi_t, \eta_{s(t)}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This contradicts with $F(T_{s(t)}) \cap B(\xi, \varepsilon) = \emptyset$. The proof is completed.

Remark 3.1 Theorem 3.2 generalizes the corresponding result for single-valued case in [17] into set-valued operators.

Theorem 3.3 For each $T \in CB(X)$, each random fixed point of T is essential if and only if the set-valued mapping F is continuous at T.

Proof Assume that F is continuous at T, then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $S \in M$ with $\rho(T, S) < \delta$ satisfying that $H(F(T), F(S)) < \varepsilon$. Therefore, for each $\xi \in F(T)$, for any $S \in M$ with $\rho(T, S) < \delta$ there is a point $\eta \in F(S)$ such that $m(\xi, \eta) < \varepsilon$, that is, ξ is essential.

Conversely, assume that each random fixed point of T is essential. From Theorem 3.1, we only need to show that F is lower semi-continuous at T. By way of contradiction, if F is not lower semi-continuous at T, then there is an open set V in M such that $V \cap F(T) \neq \emptyset$ and $T_n \in CB(X)$ with $T_n \to T$ but $F(T_n) \cap V = \emptyset$, $n = 1, 2, \cdots$. Take a point $\xi_0 \in V \cap F(T)$, then there is an open neighborhood $U(\xi_0)$ of ξ_0 with $U(\xi_0) \subset V$. Since ξ_0 is an essential random fixed point of T, there exists a number N such that $F(T_n) \cap U(\xi_0) \neq \emptyset$, $\forall\, n > N$, a contradiction with $F(T_n) \cap V = \emptyset$, $n = 1, 2, \cdots$.

Next, we will give some generic stability results for random fixed points.

Theorem 3.4 The metric space $(CB(X), \rho)$ is complete.

Proof Let $\{T_n\}_{n=1}^{\infty} \subset CB(X)$ be a Cauchy sequence. Then for each $\varepsilon > 0$, there is a number N such that $\rho(T_n, T_m) < \varepsilon$ for any $n, m > N$. That is, $H(T_n(w, x), T_m(w, x)) < \varepsilon$ for each point $(w, x) \in \Omega \times X$. Therefore, the sequence $\{T_n(w, x)\}_{n=1}^{\infty}$, consisting of nonempty closed convex sets, is a Cauchy sequence. By the completeness of X, for each $(w, x) \in \Omega \times X$, there exists a nonempty closed convex set in X denoted by $T(w, x)$ such that $T_n(w, x) \to T(w, x)$ as n tends to infinity. That is, there is a set-valued mapping $T : \Omega \times X \to 2^X$ with nonempty closed values. For each $w \in \Omega$, since ρ is the uniform metric, from the continuity of $T_n(w, \cdot)$, we have $T(w, \cdot)$ is also continuous. For each $x \in X$, noting that $T_n(\cdot, x) : \Omega \to 2^X$ is measurable with nonempty closed convex values and $H(T_n(w, x), T(w, x)) \to 0$, by Lemma 2.1, we have $T(\cdot, x)$ is also measurable. Therefore, $T \in CB(X)$, hence, (CB, ρ) is complete.

Theorem 3.5 Each random fixed point for most random set-valued mappings in $CB(X)$ is essentially stable.

Proof Noting that the complete metric space $CB(X)$ is a Baire space, from Theorem 3.1 and Fort's Lemma 2.2 (or Theorem 4.2 in [20]), there is a dense residual subset Q of $CB(X)$, such that F is continuous on Q. By Theorem 3.3, we have that each random fixed point for any $T \in Q$ is essentially stable.

References

[1] Bednarczuk E. A note on lower semicontinuity of minimal points [J]. Nonlinear Anal., 2002, 50: 285–297.

[2] Itoh S. A random fixed point theorem for a multivalued contraction mapping [J]. Pac. J. Math., 1977, 68: 85–90.

[3] Kumam W, Kumam P. Random fixed point theorems for multivalued subsequentially limit-contractive maps satisfying inwardness conditions [J]. J. Comput. Anal. Appl., 2012, 2: 239–251.

[4] Lin Tzu Chu. Random approximations and random fixed point theorems for non-self-maps [J]. Proc. Amer. Math. Soc., 1988, 103: 1129–1135.

[5] Shahzad N. Random fixed points of pseudo-contractive random operators [J]. J. Math. Anal. Appl., 2004, 296: 302–308.

[6] Yu Jian, Yuan Xianzhi. A random section theorem and its applications [J]. Math. Comput. Model, 1995, 21 :57–66.

[7] O'regan D. Random fixed point theory with applications [J]. Nonlinear Anal., 1997, 30: 3295–3299.

[8] Tan Kok Keong, Yuan Xianzhi. Random equilibria of random generalized games with applications to non-compact random quasi-variational inequalities [J]. Topol. Method. Nonl. An., 1995, 5: 59–82.

[9] Fort M K. Essential and nonessential fixed points [J]. Amer. J. Math., 1950, 72: 315–322.

[10] Song Qiqing, Tang Guoqiang, Wang Laisheng. On essential stable sets of solutions in set optimization problems [J]. J. Optim. Theory Appl., 2013, 156: 591–599.

[11] Khanh P Q, Quan N H. Generic stability and essential components of generalized KKM points and applications [J]. J. Optim. Theory Appl., 2011, 148: 488–504.

[12] Govindan S, Wilson R. Essential equilibria [J]. P. Natl. Acad. Sci. USA., 2005, 102: 15706–15711.

[13] Xiang Shuwen, Zhou Yonghui. On essential sets and essential components of efficient solutions for vector optimization problems [J]. J. Math. Anal. Appl., 2006, 315: 317–326.

[14] Yang Hui, Yu Jian. On essential components of the set of weakly Pareto-Nash equilibrium points [J]. Appl. Math. Lett., 2002, 15: 553–560.

[15] Yu Jian. Essential weak efficient solution in multiobjective optimization problems [J]. J. Math. Anal. Appl., 1992, 166: 230–235.

[16] Carbonell-Nicolau O. Essential equilibria in normal-form games [J]. J. Econ. Theory, 2010, 145: 421–431.

[17] Beg I. Essential random fixed point set of random operators [J]. Aust. J. Math. Anal. Appl., 2005, 2: 1–5.

[18] Benavides T D, Acedo G L, Xu Hongkun. Random fixed points of set-valued operators [J]. Proc. Amer. Math. Soc., 1996, 124: 831–838.

[19] Fort M K. Points of continuity of semicontinuous function [J]. Publ. Math. Debrecen, 1951, 2: 100–102.

[20] Xiang Shuwen, Jia Wensheng, He Jihao, Xia Shunyou, Chen Zhiyou. Some results concerning the generic continuity of set-valued mappings [J]. Nonlinear Anal., 2012, 75: 3591–3597.

关于集值映射的随机不动点的稳定性

宋奇庆

(桂林理工大学理学院, 广西 桂林 541004)

摘 要: 本文研究随机集值映射不动点的稳定性. 通过集值分析, 得到了随机集值不动点的本质稳定集的存在性. 在Baire分类意义下, 大多数的随机集值映射的随机不动点都是本质稳定的. 这些推广了现有文献中的相应结果.

关键词: 随机不动点; 稳定性; 本质

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