

LATTICES GENERATED BY PARTIAL MAPS OF FINITE SETS

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Abstract: In this paper, the Lattices generated by partial maps for the finite set $[n] = \{1, 2, \dots, n\}$ is investigated. By using the rank function and the Möbius function, we discuss the geometricity of such lattices. Finally, their characteristic polynomials are obtained, which generalize the results of lattice generated by finite set.

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1 Introduction

Let $(P, \leq)$ be a poset. We write $a < b$ whenever $a \leq b$ and $a \neq b$. For any two elements $a, b \in P$, we say a covers b , denoted by $b \lessdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If P has the minimum (respectively maximum) element, then we denote it by 0 (respectively 1), and say that P is a poset with 0 (respectively 1). A poset P is said to be a lattice if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. For $a \in P$, if $0 \lessdot a$, then a is called an atom. A lattice P with 0 is called an atomic lattice if $a \in P \setminus \{0\}$ is the least upper bound of some atoms. Let P be a finite poset with 0. If there is a function r from P to set of all the nonnegative integers such that

- (1) $r(0) = 0$,
- (2) $r(b) = r(a) + 1$, if $a \lessdot b$.

Then r is said to be the rank function on P . Note that the rank function on P is unique if it exists.

Let P be a finite atomic lattice. P is said to be geometric lattice, if P admits a rank function r and for any two elements $a, b \in P$,

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b).$$

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Let P be a poset with 0 and 1 , and let P admit the rank function r . The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

is called the characteristic polynomial of P .

Let A be an m -subset of $[n] := \{1, 2, \dots, n\}$, and $f : A \rightarrow [n]$ be a map. Then the pair (A, f) is said to be an m -partial map of $[n]$. In particular, we write $(A, f) = 0$ if $A = \emptyset$.

Let $P = \{(A, f) \mid (A, f) \text{ be a partial map of } [n] \cup \{1\}\}$. For any two elements $(A, f) \in P \setminus \{1\}$, $(B, g) \in P \setminus \{1\}$, we define that 1 includes (A, f) , and (B, g) includes (A, f) if $A \subseteq B$ and $g|_A = f$. Partially ordered P by ordinary or reverse inclusion, two families of finite posets are obtained, denoted by P_O and P_R , respectively.

In this paper we will prove that P_O and P_R are finite atomic lattices, discuss their geometricity and compute their characteristic polynomials.

The results on the lattices generated by transitive sets of subspaces under finite classical groups may be found in Huo, Liu and Wan [3–5]. In [1], Guo discussed the lattices associated with finite vector spaces and finite affine spaces. The lattices generated by the orbits of subspaces under finite classical groups have been obtained in a series of papers by Huo and Wan [6], Wang and Feng [8], Wang and Guo [9, 10], Guo and Nan [2, 7], Wang and Li [11].

2 The Poset P_O

In this section we will prove that P_O is a finite atomic lattice and computes its characteristic polynomial. We begin with a useful lemma.

Lemma 2.1 P_O is a finite lattice.

Proof For any $(A, f) \in P_O \setminus \{1\}$, it is easy to see that

$$1 = (A, f) \vee 1 \quad \text{and} \quad (A, f) = 1 \wedge (A, f).$$

For any $(A, f), (B, g) \in P_O \setminus \{1\}$, we assert that

$$(A, f) \vee (B, g) = \begin{cases} (A \cup B, h), h|_A = f, h|_B = g, & f|_{A \cap B} = g|_{A \cap B}, \\ 1, & f|_{A \cap B} \neq g|_{A \cap B}. \end{cases}$$

Case 1 $f|_{A \cap B} = g|_{A \cap B}$. Let (C, φ) be an upper bound of (A, f) and (B, g) , then

$$A \subseteq C, B \subseteq C \text{ and } \varphi|_A = f = h|_A, \varphi|_B = g = h|_B.$$

Thus, $A \cup B \subseteq C$ and $\varphi|_{A \cup B} = h$, i.e., $(A \cup B, h) \leq (C, \varphi)$. Hence $(A, f) \vee (B, g) = (A \cup B, h)$.

Case 2 $f|_{A \cap B} \neq g|_{A \cap B}$. Assume that (C, φ) is an upper bound of (A, f) and (B, g) , i.e.,

$$(A, f) \leq (C, \varphi) \text{ and } (B, g) \leq (C, \varphi).$$

Then $\varphi|_{A \cap B} = f|_{A \cap B}, \varphi|_{A \cap B} = g|_{A \cap B}$, a contradiction.

On the other hand, for any $(A, f), (B, g) \in P_O \setminus \{1\}$, we assert that $(A, f) \wedge (B, g) = (D, h)$, where D is the maximum element of the set

$$\{C \subseteq A \cap B \mid f|_C = g|_C\}$$

and

$$h = f|_D = g|_D.$$

In fact, let (C, φ) be a lower bound of (A, f) and (B, g) . Then

$$C \subseteq A, C \subseteq B \text{ and } f|_C = g|_C.$$

Thus C belongs to $\{C \subseteq A \cap B \mid f|_C = g|_C\}$. Hence, $(A, f) \wedge (B, g) = (D, h)$.

Theorem 2.2 Let $n \geq 2$. Then P_O is a finite atomic lattice, but not a geometric lattice.

Proof Define $r_O(A, f) = |A|$ for any $(A, f) \in P_O \setminus \{1\}$ and $r_O(1) = n + 1$. Then r_O is the rank function on P_O .

Pick

$$A = \{1\}, f : A \rightarrow [n], 1 \mapsto 1; \text{ and } g : A \rightarrow [n], 1 \mapsto 2. \quad (2.1)$$

Then (A, f) and (A, g) are the atoms of P_O , and $1 = (A, f) \vee (A, g)$.

For any $(A, f) \in P_O \setminus \{1\}$ with $A = \{a_1, a_2, \dots, a_m\}$, we have

$$(A, f) = (\{a_1\}, f|_{\{a_1\}}) \vee (\{a_2\}, f|_{\{a_2\}}) \vee \dots \vee (\{a_m\}, f|_{\{a_m\}}).$$

Hence P_O is a finite atomic lattice.

Pick (A, f) and (A, g) as in (2.1). Then $(A, f) \vee (B, g) = 1$ and $(A, f) \wedge (B, g) = 0$, which implies that

$$r_O((A, f) \vee (B, g)) + r_O((A, f) \wedge (B, g)) = n + 1 > 2 = r_O(A, f) + r_O(B, g).$$

Therefore, the desired result follows.

Lemma 2.3 The Möbius function on P_O is

$$\mu_O(x, y) = \begin{cases} 0, & x \not\leq y, \\ (-1)^{|B| - |A|}, & x = (A, f) \leq (B, g) = y \neq 1, \\ -(1 - n)^{n - |A|}, & x = (A, f) < y = 1, \\ 1, & x = y = 1. \end{cases}$$

Proof In order to prove that μ_O is the Möbius function on P_O , we only need to show that

$$\sum_{x \leq z \leq y} \mu_O(x, z) = 0$$

for any $x, y \in P_O$ with $x < y$.

If $y = (B, g) \neq 1$, let $|B| - |A| = m$. Then

$$\sum_{x \leq z \leq y} \mu_O(x, z) = \sum_{k=0}^m C_m^k (-1)^k = (1-1)^m = 0.$$

If $y = 1$, let $|A| = m$. Then

$$\sum_{x \leq z \leq 1} \mu_O(x, z) = \sum_{k=0}^{n-m} C_{n-m}^k (-1)^k n^k + \mu(x, 1) = (1-n)^{n-m} + [-(1-n)^{n-|A|}] = 0.$$

Hence, the function μ_O is the Möbius function on P_O .

Theorem 2.4 The characteristic polynomial of P_O is

$$\chi(P_O, x) = x(x-n)^n - (1-n)^n.$$

Proof By Lemma 2.3 we obtain

$$\begin{aligned} \chi(P_O, x) &= \sum_{u \in P_O} \mu_O(0, u) x^{r_O(1) - r_O(u)} \\ &= \sum_{m=0}^n (-1)^m C_n^m n^m x^{r_O(1) - m} + \mu_O(0, 1) x^{r_O(1) - r_O(1)} \\ &= x \sum_{m=0}^n (-1)^m C_n^m n^m x^{n-m} + [-(1-n)^n x^0] \\ &= x(x-n)^n - (1-n)^n. \end{aligned}$$

3 The Poset P_R

In this section we will prove that P_R is a finite atomic lattice and compute its characteristic polynomials. Similar to the proof of Lemma 2.1, we obtain the following result.

Lemma 3.1 P_R is a finite lattice.

Theorem 3.2 Let $n \geq 2$. Then P_R is a finite atomic lattice, but not a geometric lattice.

Proof Define $r_R(A, f) = n + 1 - |A|$ for any $(A, f) \in P_R \setminus \{1\}$ and $r_R(1) = 0$. Then r_R is the rank function on P_R .

Pick $f : [n] \rightarrow [n], i \mapsto 1$ and $g : [n] \rightarrow [n], i \mapsto 2$. Then $0 = ([n], f) \vee ([n], g)$. For $(A, f) \in P_R \setminus \{1\}$ with $A = \{a_1, a_2, \dots, a_m\}$, pick $([n], g), ([n], h) \in P_R$, such that

$$g : [n] \rightarrow [n], a_i \mapsto f(a_i), a \mapsto 1 \text{ if } a \notin A; \quad h : [n] \rightarrow [n], a_i \mapsto f(a_i), a \mapsto 2 \text{ if } a \notin A. \quad (3.1)$$

Then $([n], g)$ and $([n], h)$ are atoms in P_R and $(A, f) = ([n], g) \vee ([n], h)$. Hence, P_R is a finite atomic lattice.

Pick f, g as in (3.1). Then

$$([n], f) \vee ([n], g) = 0, ([n], f) \wedge ([n], g) = 1,$$

which implies that

$$r_R([n], f) \vee ([n], g) + r_R([n], f) \wedge ([n], g) = n + 1 > 2 = r_R([n], f) + r_R([n], g).$$

Therefore, the desired result follows.

Lemma 3.3 The Möbius function on P_R is

$$\mu_R(x, y) = \begin{cases} 0, & x \not\leq y, \\ (-1)^{|A|-|B|}, & 1 \neq x = (A, f) \leq y = (B, g), \\ -(1-n)^{n-|B|}, & 1 = x < y = (B, g), \\ 1, & x = y = 1. \end{cases}$$

Proof If $x \neq 1$, let $|A| - |B| = m$. Then

$$\sum_{x \leq z \leq y} \mu_R(x, z) = \sum_{k=0}^m C_m^{m-k} (-1)^k = (1-1)^m = 0.$$

If $x = 1$, let $|B| = m$. Then

$$\sum_{1 \leq z \leq y} \mu_R(1, z) = \mu_R(1, 1) + \sum_{k=0}^{n-m} C_{n-m}^k [-(1-n)^{n-m-k}] n^k = 1 - [n + (1-n)]^{n-m} = 0.$$

Hence, the function μ_R is the Möbius function on P_R .

Theorem 3.4 The characteristic polynomial of P_R is

$$\chi(P_R, x) = x^{n+1} - (nx - n + 1)^n.$$

Proof By Lemma 3.3 we obtain

$$\begin{aligned} \chi(P_R, x) &= \sum_{u \in P_R} \mu_R(1, (A, f)) x^{r_R(0) - r_R(u)} \\ &= \sum_{m=0}^n C_n^m n^m [-(1-n)^{n-m}] x^m + \mu_R(1, 1) x^{r_R(0) - r_R(1)} \\ &= x^{n+1} - (nx + 1 - n)^n. \end{aligned}$$

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有限集合的部分映射生成的格

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摘要: 本文研究了有限集合 $[n] = \{1, 2, \dots, n\}$ 的部分映射生成的格. 利用秩函数和Möbius函数, 讨论了这类格的几何性, 得到了它们的特征多项式. 推广了有限集合生成格的相关性质.

关键词: 部分映射; 原子格; 特征多项式

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