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TRAVELING WAVE SOLUTIONS OF A GREEN-NAGHDI ASYMPTOTIC MODEL FOR SMALL ASPECT RATIO WAVES

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Abstract: A Green-Naghdi asymptotic model for small aspect ratio waves is investigated by qualitative analysis methods of planar autonomous systems. Under different parameter conditions, the bifurcation of its traveling wave system is discussed and the corresponding phase portraits are also given. The exact expressions of some bounded traveling wave solutions are obtained, such as smooth periodic wave solutions, kink-like wave solutions, antikink-like wave solutions, compacton-like wave solutions, periodic cusp wave solutions, solitary wave solutions and cusp solitary wave solutions. Furthermore, these solutions are simulated by applying the software Maple.

Keywords: Green-Naghdi asymptotic model; traveling wave solutions; phase portraits; bifurcations

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1 Introduction

The Green-Naghdi system was first derived by Green and Naghdi [1] in 1976 for describing a fully nonlinear shallow water gravity wave with a free surface or an interfacial surface. The system also appears in different physical contexts such as bubbly fluid dynamics and magneto-hydro dynamics. Its various aspects have been studied. For instance, Li [2] showed that the system has no eigenvalues with a positive real part and solitary waves with a small amplitude are linearly stable. Deng, Guo and Wang [3] obtained the exact expressions of its smooth soliton wave solutions, cusp soliton wave solutions, smooth periodic wave solutions and periodic cusp wave solutions and gave some numerical simulations of these solutions.

The Green-Naghdi asymptotic model for small aspect ratio waves

$$S_t - \frac{1}{k^2}S_{xxt} - \frac{1}{2k^2}\sqrt{\frac{g}{k}}S_{xxx} + \frac{3}{2}\sqrt{\frac{g}{k}}S_x + \sqrt{gk}SS_x = \frac{1}{3k}\sqrt{\frac{g}{k}}(5S_xS_{xx} + SS_{xxx}), \tag{1.1}$$

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which was first derived by Kraenkelb, Leona and Mannaa [4] in 2005, is a k-dependent equation that describes the nonlinear deformations of the wave with a given wave number k. Here $S(x,t)$ is the free surface deformation, g is the gravitation, t is the time variable, and x is the propagation direction. Thus the parameters g, k are both positive. Taking $\alpha = \sqrt{\frac{g}{k}}$, (1.1) becomes

$$k^2 S_t - S_{xxt} - \frac{\alpha}{2} S_{xxx} + \frac{3}{2}\alpha k^2 S_x + k^3 \alpha S S_x - \frac{5}{3} k\alpha S_x S_{xx} - \frac{1}{3} k\alpha S S_{xxx} = 0. \tag{1.2}$$

Recently, many mathematicians are very interested in traveling wave solutions of many mathematical and physical models. For instance, Huang and Liu investigated new exact traveling wave solutions of Fisher equation and Burgers-Fisher equation by using the method of an auxiliary ODE method in [5]; Rong, Tang and Huang [6] considered bifurcations of traveling solutions for the K$(n, 2n, -n)$ equations; Tang et al. [7] discussed traveling wave solutions for the generalized special type of the Tzitzeica-Dodd-Bullough equation; Zhong and Gu [8] studied the bifurcation of traveling wave solutions for symmetric regularized wave equations and so on.

To the best of our knowledge, there is no result about the traveling wave solutions of (1.2). In this paper, we discuss the bifurcation of the traveling wave system of the equation (1.2) under the different parameter conditions by qualitative analysis methods of planar systems (See, e.g., [9]), give the corresponding phase portraits by using the software Maple and show the exact expressions of smooth periodic wave solutions, kink-like wave solutions, antikink-like wave solutions, compacton-like wave solutions, periodic cusp wave solutions, solitary wave solutions and cusp solitary wave solutions. Furthermore, we simulate them.

This paper is organized as follows. In Section 2, (1.2) is changed into a traveling wave system. Phase portraits are given in Section 3. Section 4 shows the exact expressions of bounded wave solutions and the numerical simulations of these solutions.

2 Traveling Wave System

Let $\xi = x - ct$, where $c \neq 0$ is the wave speed. Substituting $S(x,t) = u(x-ct) = u(\xi)$ into (1.2) we get the following ordinary differential equation

$$(\frac{3}{2}\alpha - c)k^2 u_\xi + (c - \frac{\alpha}{2}) u_{\xi\xi\xi} + k^3 \alpha u u_\xi - \frac{5}{3} k\alpha u_\xi u_{\xi\xi} - \frac{1}{3} k\alpha u u_{\xi\xi\xi} = 0. \tag{2.1}$$

Integrating (2.1) once with respect to ξ yields the traveling wave equation

$$\beta + (\frac{3}{2}\alpha - c)k^2 u + \frac{1}{2} k^3 \alpha u^2 + (c - \frac{\alpha}{2} - \frac{1}{3} k\alpha u) u_{\xi\xi} - \frac{2}{3} k\alpha (u_\xi)^2 = 0, \tag{2.2}$$

where β is the constant of integration. Let $v = u_\xi$ and $b = \frac{\alpha}{2} - c$, and we have the following traveling wave system from (2.2)

$$\begin{cases} \frac{du}{d\xi} = v, \\ \frac{dv}{d\xi} = \frac{1}{\frac{1}{3}k\alpha u + b} \left(-\frac{2}{3} k\alpha v^2 + \frac{1}{2} k^3 \alpha u^2 + (\alpha + b) k^2 u + \beta \right). \end{cases} \tag{2.3}$$

It is not convenient to study the phase portraits of system (2.3) because it has a singular line $u = -\frac{3b}{k\alpha}$. Thus, we introduce a transformation

$$d\xi = (\frac{1}{3}k\alpha u + b)d\tau. \tag{2.4}$$

Then system (2.3) is changed to

$$\begin{cases} \frac{du}{d\tau} = (\frac{1}{3}k\alpha u + b)v, \\ \frac{dv}{d\tau} = -\frac{2}{3}k\alpha v^2 + \frac{1}{2}k^3\alpha u^2 + (\alpha + b)k^2 u + \beta. \end{cases} \tag{2.5}$$

The first integral of (2.3) and (2.5) is

$$\begin{aligned} H(u, v) = &-\frac{1}{270}k^5\alpha^3(2\alpha + 11b)u^5 - \frac{1}{2}kb^2(\alpha kb + kb^2 - \alpha\beta)u^2 \\ &+ \frac{1}{2}(\frac{1}{3}k\alpha u + b)^4 v^2 - \frac{1}{18}k^2\alpha b(9kb^2 + 6k\alpha b + 2\alpha\beta)u^3 \\ &- b^3\beta u - \frac{1}{216}k^3\alpha^2(2\alpha\beta + 18k\alpha b + 45kb^2)u^4 - \frac{1}{324}k^6\alpha^4 u^6. \end{aligned} \tag{2.6}$$

Obviously, (2.3) and (2.5) have the same topological phase portraits except the singular line l: $u = -\frac{3b}{k\alpha}$. In the following, we focus on system (2.5). Let

$$\begin{aligned} v_1 &= \sqrt{\frac{3}{2k\alpha}\left[\beta - \frac{k}{2\alpha}3b(2\alpha - b)\right]}, v_2 = -\sqrt{\frac{3}{2k\alpha}\left[\beta - \frac{k}{2\alpha}3b(2\alpha - b)\right]}, \\ u_1 &= \frac{1}{k^2\alpha}\left[-k(\alpha + b) - \sqrt{k^2(\alpha + b)^2 - 2\alpha k\beta}\right], \\ u_2 &= \frac{1}{k^2\alpha}\left[-k(\alpha + b) + \sqrt{k^2(\alpha + b)^2 - 2\alpha k\beta}\right], \end{aligned}$$

which will be used later.

3 Bifurcations and Phase Portraits of the Traveling Wave System

Let

$$f(u) = \frac{1}{2}k^3\alpha u^2 + (\alpha + b)k^2 u + \beta. \tag{3.1}$$

It is easy to see that

$$f'(u) = k^3\alpha u + (\alpha + b)k^2. \tag{3.2}$$

The symmetric axis and the discriminant of f are

$$u = -\frac{\alpha + b}{k\alpha} \tag{3.3}$$

and

$$\Delta = 2\alpha k^3[\frac{k}{2\alpha}(\alpha + b)^2 - \beta]. \tag{3.4}$$

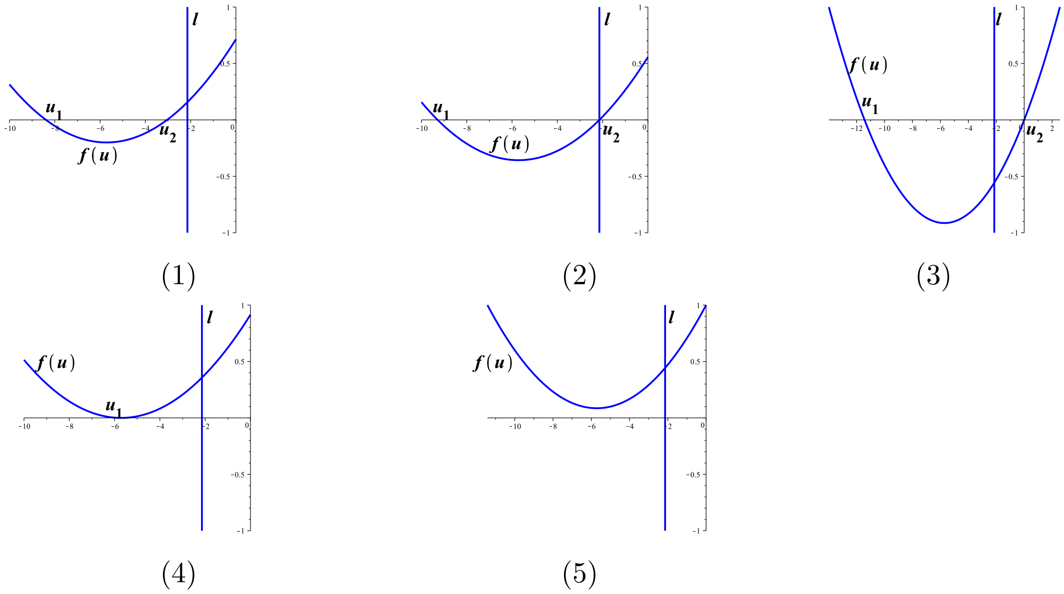

Figure 1: The graphs of the function $f(u)$ for $2b<\alpha$. (1) $\frac{k}{2\alpha}3b(2\alpha-b)<\beta<\frac{k}{2\alpha}(\alpha+b)^2$. (2) $\beta=\frac{k}{2\alpha}3b(2\alpha-b)$. (3) $\beta=\frac{k}{2\alpha}(\alpha+b)^2$. (4) $\beta<\frac{k}{2\alpha}3b(2\alpha-b)$. (5) $\beta>\frac{k}{2\alpha}(\alpha+b)^2$.

The coefficient matrix of the linearized system of (2.5) at an equilibrium $E(u_e,v_e)$ is

$$\begin{pmatrix} \frac{1}{3}k\alpha v_e & \frac{1}{3}k\alpha u_e+b \\ f'(u_e) & -\frac{4}{3}k\alpha v_e \end{pmatrix}, \tag{3.5}$$

whose determinant and trace are

$$D(E)=-\frac{4}{9}k^2\alpha^2v_e^2-(\frac{1}{3}k\alpha u_e+b)f'(u_e),\quad T(E)=-k\alpha v_e. \tag{3.6}$$

By the qualitative theory of differential equations for an equilibrium of a planar dynamical system [9], we know that the equilibrium $E(u_e,v_e)$ is a saddle point if $D<0$; It is a node if $D>0$ and $T\neq 0$; it is a center if $D>0$ and $T=0$; It is degenerate if $D=0$. Using these, we can obtain the phase portraits of (2.5) under different parameter conditions.

Note that from $c\neq 0$ and $b=\frac{\alpha}{2}-c$ we have

$$\frac{k}{2\alpha}(\alpha+b)^2-\frac{3}{2\alpha}kb(2\alpha-b)=\frac{k}{2\alpha}(\alpha-2b)^2>0,$$

which means $\frac{k}{2\alpha}(\alpha+b)^2>\frac{3}{2\alpha}kb(2\alpha-b)$. Using these, we discuss the bifurcation of system (2.5) by the relative position of the function $f(u)$ to the singular line l (see Figure).

Theorem 1 If $2b<\alpha$ and $\frac{3}{2\alpha}kb(2\alpha-b)<\beta<\frac{k}{2\alpha}(\alpha+b)^2$, then system (2.5) has two equilibriums $E_1(-\frac{3b}{k\alpha},v_1)$ and $E_2(-\frac{3b}{k\alpha},v_2)$ on the singular line l and two equilibriums $E_3(u_1,0)$ and $E_4(u_2,0)$ at the left side of l. E_1, E_2 and E_3 are saddle points, and E_4 is a center (see (1)–(3) of Figure).

Proof The definition of f yields

$$f(-\frac{3b}{k\alpha}) = \beta - \frac{3kb}{2\alpha}(2\alpha - b). \tag{3.7}$$

From

$$-\frac{2}{3}k\alpha v^2 + f(-\frac{3b}{k\alpha}) = 0,$$

we have

$$v^2 = \frac{3}{2k\alpha}[\beta - \frac{3}{2\alpha}kb(2\alpha - b)]. \tag{3.8}$$

By $\frac{3}{2\alpha}kb(2\alpha - b) < \beta$, we get that system (2.5) has two equilibrium points $E_1(-\frac{3b}{k\alpha}, v_1)$ and $E_2(-\frac{3b}{k\alpha}, v_2)$. By (3.6) we have $D(E_1) < 0$ and $D(E_2) < 0$. Thus, $E_1(-\frac{3b}{k\alpha}, v_1)$ and $E_2(-\frac{3b}{k\alpha}, v_2)$ are saddle points.

From $\beta < \frac{k}{2\alpha}(\alpha+b)^2$ and (3.4), we get that $\Delta > 0$. Thus, $f(u)$ has two real roots u_1 and u_2, and system (2.5) has two equilibriums $E_3(u_1, 0)$ and $E_4(u_2, 0)$. Next, we discriminate the relative position of E_3, E_4 and l. From $2b < \alpha$ we have

$$-\frac{3b}{k\alpha} - (-\frac{\alpha + b}{k\alpha}) = \frac{\alpha - 2b}{k\alpha} > 0. \tag{3.9}$$

By (3.3) we obtain that the symmetric axis of f is at the left side of the singular line l. By (3.7) and $\frac{3}{2\alpha}kb(2\alpha - b) < \beta$, we have $f(-\frac{3b}{k\alpha}) > 0$. Thus, we get that E_3 and E_4 are at the left side of l (see (a) of Figure 1). Furthermore, it is easy to see that $f'(u_1) < 0$, $f'(u_2) > 0$, $\frac{1}{3}k\alpha u_i + b < 0$ $(i = 1, 2)$. By (3.6) we have that $D(E_3) < 0$ and $D(E_4) > 0$. Thus, E_3 is a saddle point and E_4 a center.

Theorem 2 If $2b < \alpha$ and $\beta = \frac{3}{2\alpha}kb(2\alpha - b)$, then system (2.5) has two equilibriums $E_3(u_1, 0)$ and $E_4(u_2, 0)$. E_3 is a saddle point, E_4 is a degenerate saddle point (see (4) of Figure).

Proof The proof that E_3 is a saddle point is similar as one of Theorem 1. We only prove that E_4 is a degenerate saddle point.

By $\beta = \frac{k}{2\alpha}3b(2\alpha - b)$, we have $u_2 = -\frac{3b}{k\alpha}$. Under the transformation $u = y - \frac{3b}{k\alpha}$ and $v = z$, system (2.5) can be written as

$$\begin{cases} \frac{dy}{d\tau} = \frac{1}{3}k\alpha yz, \\ \frac{dz}{d\tau} = k^2(\alpha - 2b)y - \frac{2}{3}k\alpha z^2 + \frac{1}{2}k^3\alpha y^2. \end{cases} \tag{3.10}$$

Thus, we focus on $O(0, 0)$ of system (3.10). Let $Y = k^2(\alpha - 2b)y - \frac{2}{3}k\alpha z^2 + \frac{1}{2}k^3\alpha y^2, X = z$, and we have

$$\begin{cases} \frac{dX}{d\tau} = Y, \\ \frac{dY}{d\tau} = \frac{2}{9}k^2\alpha^2 X^3 - k\alpha XY + (|X|^3 + |XY|)O(|(X, Y)|). \end{cases} \tag{3.11}$$

By Theorem 7.2 in Chapter 2 of [9], we get that $O(0, 0)$ is a degenerate saddle point of (3.11). which means that $O(0, 0)$ is a degenerate saddle point of (3.10). Thus, E_4 is a degenerate saddle point of (2.5).

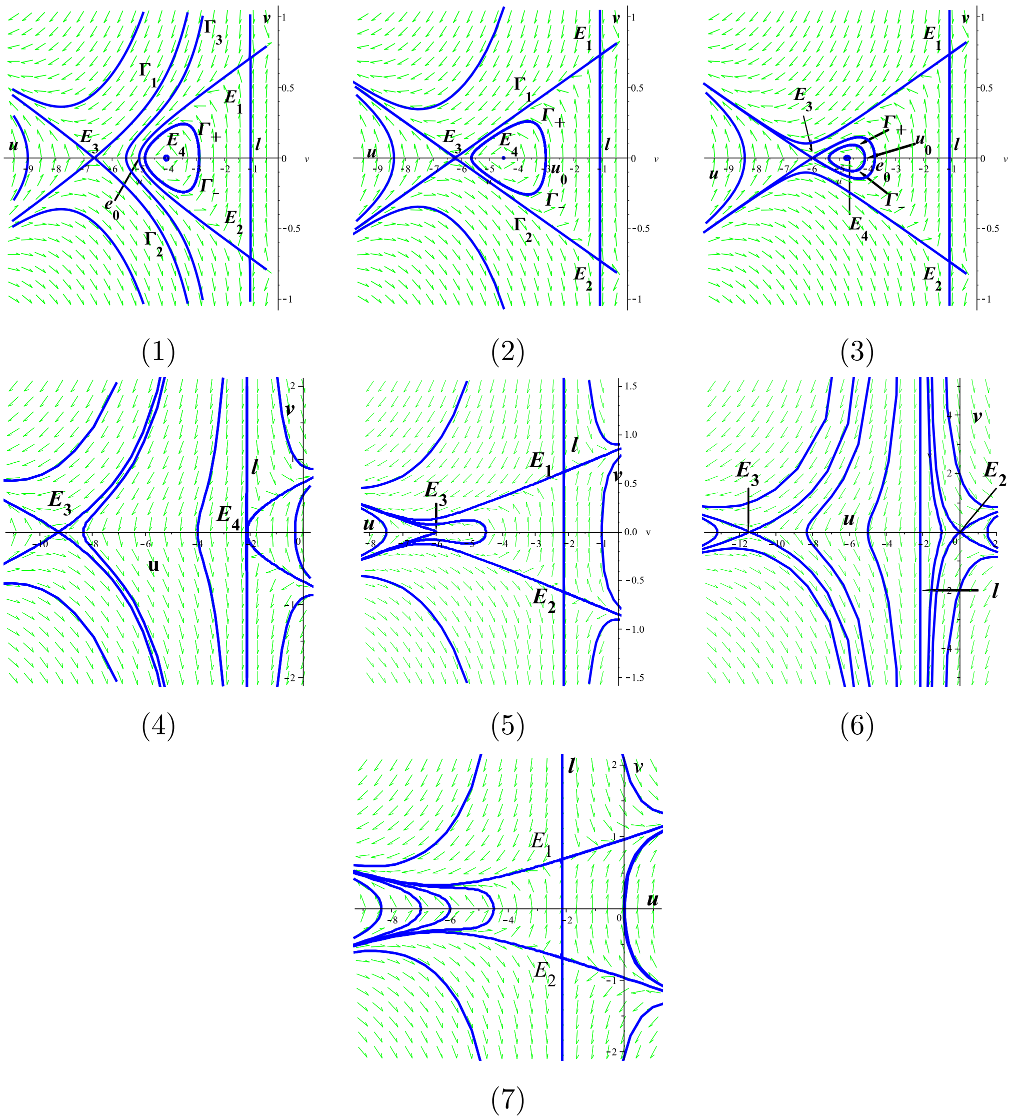

Figure 2: The phase portraits of system (2.5) for $2b < \alpha$. (1) $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_1, 0) > H(-\frac{3b}{k\alpha}, v_1)$. (2) $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_1, 0) = H(-\frac{3b}{k\alpha}, v_1)$. (3) $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_1, 0) < H(-\frac{3b}{k\alpha}, v_1)$. (4) $\beta = \frac{3}{2\alpha}kb(2\alpha - b)$. (5) $\beta = \frac{k}{2\alpha}(\alpha + b)^2$. (6) $\beta < \frac{3}{2\alpha}kb(2\alpha - b)$. (7) $\beta > \frac{k}{2\alpha}(\alpha + b)^2$.

Theorem 3 If $2b < \alpha$ and $\beta = \frac{k}{2\alpha}(\alpha+b)^2$, then system (2.5) has three equilibriums $E_1(-\frac{3b}{k\alpha}, v_1)$, $E_2(-\frac{3b}{k\alpha}, v_2)$ and $E_3(u_1, 0)$. E_1 and E_2 are saddle points, and E_3 is a cusp (see (5) of Figure).

Proof The proof that E_1 and E_2 are saddle points is similar as one of Theorem 1. Next we only prove that E_3 is a cusp. With the transformation $u = m - \frac{\alpha+b}{k\alpha}$ and $v = y$, system (2.5) becomes

$$\begin{cases} \frac{dm}{d\tau} = \frac{2b-\alpha}{3}y + \frac{1}{3}k\alpha m y, \\ \frac{dy}{d\tau} = \frac{1}{2}k^3\alpha m^2 - \frac{2}{3}k\alpha y^2. \end{cases} \tag{3.12}$$

Obviously, the topological structure of system (2.5) near the equilibrium E_3 is homeomorphic to the one of system (3.12) near O. Let $w = m$ and $z = \frac{2b-\alpha}{3}y + \frac{1}{3}k\alpha m y$. System (3.12) is changed into

$$\begin{cases} \frac{dw}{d\tau} = z, \\ \frac{dz}{d\tau} = \frac{1}{2}k^3\alpha w^2 + z^2\frac{k\alpha}{\alpha-2b+k\alpha w}. \end{cases} \tag{3.13}$$

By Theorem 7.3 in Chapter 2 of [9], we get that the equilibrium E_3 is a cusp point.

Theorem 4 If $2b < \alpha$ and $\beta < \frac{3}{2\alpha}kb(2\alpha - b)$, then system (2.5) has two equilibriums $E_3(u_1, 0)$ and $E_4(u_2, 0)$ which are both saddles (see (4) of Figure).

The proof is same as one of Theorem 1.

Theorem 5 If $2b < \alpha$ and $\beta > \frac{k}{2\alpha}(\alpha+b)^2$, then system (2.5) has two equilibriums $E_1(-\frac{3b}{k\alpha}, v_1)$ and $E_2(-\frac{3b}{k\alpha}, v_2)$ which are both saddle points (see (5) of Figure).

If $2b > a$ we have similar results as follows.

Theorem 6 For $2b > \alpha$,

(1) If $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha+b)^2$, then (2.5) has four equilibriums $E_1(-\frac{3b}{k\alpha}, v_1)$, $E_2(-\frac{3b}{k\alpha}, v_2)$, $E_3(u_1, 0)$ and $E_4(u_2, 0)$. E_1, E_2 and E_4 are saddle points, and E_3 is a center (see (1)–(3) of Figure).

(2) If $\beta = \frac{k}{2\alpha}3b(2\alpha-b)$, then system (2.5) has two equilibriums $E_3(-\frac{3b}{k\alpha}, 0)$ and $E_4(u_2, 0)$. $E_4(u_2, 0)$ is a saddle point and E_3 is degenerate saddle point (see (4) of Figure).

(3) If $\beta = \frac{k}{2\alpha}(\alpha+b)^2$. then system (2.5) has three equilibriums $E_1(-\frac{3b}{k\alpha}, v_1)$, $E_2(-\frac{3b}{k\alpha}, v_2)$ and $E_3(-\frac{\alpha+b}{k\alpha}, 0)$. E_1 and E_2 are saddle points and E_3 is a cusp (see (5) of Figure).

(4) If $\beta < \frac{k}{2\alpha}3b(2\alpha - b)$. then system (2.5) has two equilibriums $E_3(u_1, 0)$ and $E_4(u_2, 0)$. E_3 and E_4 are both saddle points (see (6) of Figure).

(5) If $\beta > \frac{k}{2\alpha}(\alpha+b)^2$. then system (2.5) has two equilibriums $E_1(-\frac{3b}{k\alpha}, v_1)$ and $E_2(-\frac{3b}{k\alpha}, v_2)$ which are both saddle points (see (7) of Figure).

4 Traveling Wave Solutions and Numerical Simulations

In this section, we give smooth periodic wave solutions, kink-like wave solutions, antikink-like wave solutions, compacton-like wave solutions, periodic cusp wave solutions, solitary

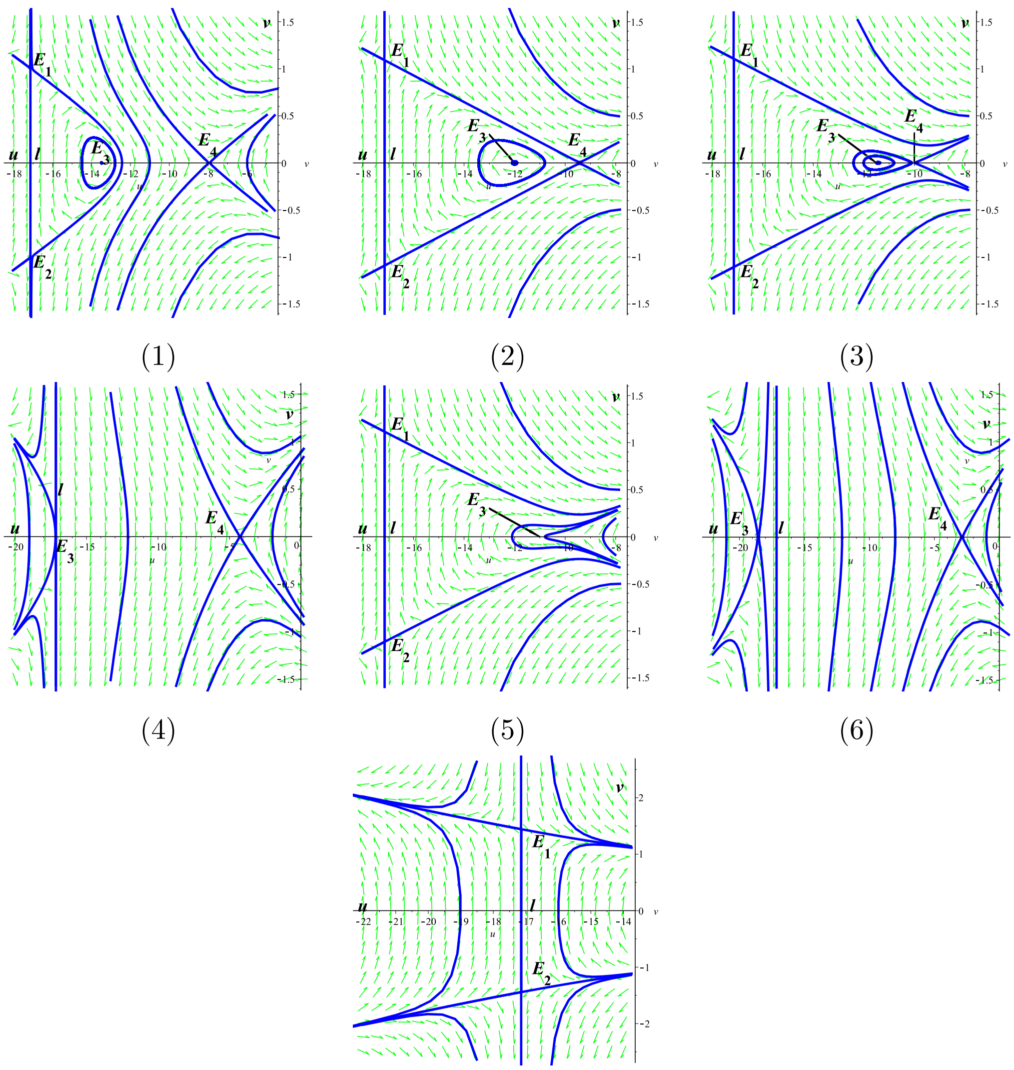

Figure 3: The phase portraits of system (2.5) for $2b > \alpha$. (1) $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_2, 0) > H(-\frac{3b}{k\alpha}, v_1)$. (2) $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_2, 0) = H(-\frac{3b}{k\alpha}, v_1)$. (3) $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_2, 0) < H(-\frac{3b}{k\alpha}, v_1)$. (4) $\beta = \frac{3}{2\alpha}kb(2\alpha - b)$. (5) $\beta = \frac{k}{2\alpha}(\alpha + b)^2$. (6) $\beta < \frac{3}{2\alpha}kb(2\alpha - b)$. (7) $\beta > \frac{k}{2\alpha}(\alpha + b)^2$.

wave solutions and cusp solitary wave solutions and their numerical simulations. Let $u(\xi)$ be a traveling wave solution of (1.2) for $\xi \in (-\infty, +\infty)$ and

$$\lim_{\xi \to -\infty} u(\xi) = A, \quad \lim_{\xi \to +\infty} u(\xi) = B,$$

where $\xi = x - ct$ and A, B are constants. $u(\xi)$ is called a solitary wave solution of (1.2) if $A = B$ and a kink (or antikink) wave solution if $A \neq B$. Usually, a solitary wave solution of (1.2) corresponds to a homoclinic orbit of system (2.5), a kink (or antikink) wave solution of (1.2) corresponds to a heteroclinic orbit of system (2.5), and a periodic traveling wave solution of (1.2) corresponds to a periodic orbit of system (2.5). In the following, we just consider the cases that $2b < \alpha$ and $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ (see (1)–(3) of Figure). Using the same way, we can discuss other cases. Note that, from (2.4), we have $\xi \to -\infty$ if $\tau \to +\infty$ and $u < -\frac{3b}{k\alpha}$.

Theorem 7 (Periodic wave solutions) Suppose that $2b < \alpha$, $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$. Consider the following conditions:

(1) $H(u_1, 0) \geq H(-\frac{3b}{k\alpha}, v_1)$ and $u_2 < u_0 < -\frac{3b}{k\alpha}$ (see (1) and (2) of Figure);

(2) $H(u_1, 0) < H(-\frac{3b}{k\alpha}, v_1)$ and $u_2 < u_0 < e_0$, where $e_0 \in (u_2, -\frac{3b}{k\alpha})$ is the solution of equation $H(u, 0) = H(u_1, 0)$ (see (3) of Figure).

If one of the two conditions holds, then (1.2) has a periodic wave solution

$$u(\xi) = \begin{cases} \phi_1(\xi - 2nT_0) & \text{for } \xi \in [2nT_0, (2n+1)T_0), \\ \phi_1(-\xi + 2(n+1)T_0) & \text{for } \xi \in ((2n+1)T_0, 2(n+1)T_0), \end{cases}$$

$n = 0, \pm 1, \pm 2, \cdots$, with $u(0) = u_0$ and $v(0) = 0$ satisfying

$$\int_{u_0}^{\phi_1} -\frac{1}{\sqrt{F(s)}} \mathrm{d}s = \xi, \tag{4.1}$$

where

$$\begin{aligned} F(u) = & \frac{2}{(\frac{1}{3}k\alpha u + b)^4} \left(\frac{1}{324}k^6\alpha^4 u^6 + \frac{1}{270}k^5\alpha^3(2\alpha + 11b)u^5 + b^3\beta u \right. \\ & + \frac{1}{2}kb^2(\alpha kb + kb^2 + \alpha\beta)u^2 + \frac{1}{18}k^2\alpha b(9kb^2 + 6k\alpha b + 2\alpha\beta)u^3 \\ & \left. + \frac{1}{216}k^3\alpha^2(2\alpha\beta + 18k\alpha b + 45kb^2)u^4 - H(u_0, 0) \right), \end{aligned} \tag{4.2}$$

T_0 is given by

$$T_0 = \int_{u_0}^{u_0^-} -\frac{1}{\sqrt{F(s)}} \mathrm{d}s,$$

and $u_0^- \in (u_1, u_2)$ is a solution of $H(u, 0) = H(u_0, 0)$.

Proof By (1) of Theorem 1 system (2.5) has a periodic orbit $\Gamma = \Gamma_+ \cup \Gamma_-$ since the equilibrium E_4 is a center. Take $(u_0, 0) \in \Gamma$. From the definition of $H(u, v)$ in (2.6), Γ

lies on the curve given by $H(u,v)=H(u_0,0)$. Note that $H(u,0)=H(u_0,0)$ has a solution $u_0^- \in (u_1,u_2)$, i.e., Γ intersects the $u-$axis at the point $(u_0^-,0)$.

$H(u,v)=H(u_0,0)$ yields $v=\pm\sqrt{F(u)}$. By (2.3) we have

$$\mathrm{d}\xi=\frac{\mathrm{d}u}{v}=\frac{\mathrm{d}u}{\pm\sqrt{F(u)}}.$$

Integrating the above along Γ in clockwise gives (4.1).

Theorem 8 (Kink-like or antikink-like wave solutions) Suppose that $2b<\alpha$, $\frac{3}{2\alpha}kb(2\alpha-b)<\beta<\frac{k}{2\alpha}(\alpha+b)^2$ and $H(u_1,0)>H(-\frac{3b}{k\alpha},v_1)$. Let Γ_1 and Γ_2 denote the orbit connecting $E_3(u_1,0)$ (see (1) of Figure). Take an initial value (u_0,v_0) with $u_1<u_0<u_2$ and $v_0>0$ (or $v_0<0$) on the Γ_1 (or Γ_2). Then (1.2) has a kink-like (or antikink-like) wave solution $u=u(\xi)$ satisfying

$$\int_{u_0}^{u}\frac{1}{\sqrt{F(s)}}\mathrm{d}s=\xi \qquad \left(\text{or } \int_{u_0}^{u}-\frac{1}{\sqrt{F(s)}}\mathrm{d}s=\xi\right)$$

for $\xi\in(-\infty,T_1)$ (or $\xi\in(\tilde{T}_1,+\infty)$), where $F(u)$ is defined in (4.2) by replacing $H(u_0,0)$ with $H(u_1,0)$ and T_1 (or $\tilde{T}_1$) is given by

$$T_1=\int_{u_0}^{-\frac{3b}{k\alpha}}\frac{1}{\sqrt{F(s)}}\mathrm{d}s \qquad \left(\text{or } \tilde{T}_1=\int_{u_0}^{-\frac{3b}{k\alpha}}-\frac{1}{\sqrt{F(s)}}\mathrm{d}s\right).$$

The proof is similar to the one of Theorem 3.3.

Theorem 9 (Compacton-like wave solutions) Suppose that $2b<\alpha$, $\frac{3}{2\alpha}kb(2\alpha-b)<\beta<\frac{k}{2\alpha}(\alpha+b)^2$ and $H(u_1,0)>H(-\frac{3b}{k\alpha},v_1)$. If $u_1<e_1<e_0$, where $e_0\in(u_1,u_2)$ is the solution of equation $H(u,0)=H(-\frac{3b}{k\alpha},v_1)$ (see (1) of Figure), then (1.2) has a compacton-like wave solution

$$u(\xi)=\begin{cases}\phi_2(\xi) & \text{for } \xi\in[0,T_2),\\ \phi_2(-\xi) & \text{for } \xi\in(-T_2,0),\end{cases}$$

with $(u(0),v(0))=(e_1,0)$ satisfying

$$\int_{e_1}^{\phi_2}\frac{1}{\sqrt{F(s)}}\mathrm{d}s=\xi,$$

where $F(u)$ is defined in (4.2) by replacing $H(u_0,0)$ with $H(e_1,0)$ and T_2 is given by

$$T_2=\int_{e_1}^{-\frac{3b}{k\alpha}}\frac{1}{\sqrt{F(s)}}\mathrm{d}s.$$

Theorem 10 (Periodic cusp wave solutions) Suppose that $2b<\alpha$, $\frac{3k}{2\alpha}b(2\alpha-b)<\beta<\frac{k}{2\alpha}(\alpha+b)^2$ and $H(u_1,0)>H(-\frac{3b}{k\alpha},v_1)$. If $u_1<e_0<u_2$, where $e_0\in(u_1,u_2)$ is the solution of the equation $H(u,0)=H(-\frac{3b}{k\alpha},v_1)$ (see (1) of Figure), then (1.2) has a periodic cusp wave solution

$$u(\xi)=\begin{cases}\phi_3(\xi-2nT_3) & \text{for } \xi\in[2nT_3,(2n+1)T_3),\\ \phi_3(-\xi+2nT_3) & \text{for } \xi\in[(2n-1)T_3,2nT_3),\end{cases}$$

$n = 0, \pm 1, \pm 2, \cdots$, with $u(0) = e_0$ and $v(0) = 0$ satisfying

$$\int_{e_0}^{\phi_3} \frac{1}{\sqrt{F(s)}} \mathrm{d}s = \xi,$$

where $F(u)$ is defined in (4.2) by replacing $H(u_0, 0)$ with $H(e_0, 0)$, and T_3 is given by

$$T_3 = \int_{e_0}^{-\frac{3b}{k\alpha}} \frac{1}{\sqrt{F(s)}} \mathrm{d}s.$$

Theorem 11 (Solitary wave solutions]) Suppose $2b < \alpha$, $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_1, 0) < H(-\frac{3b}{k\alpha}, v_1)$ (see (3) Figure). Then (1.2) has a solitary wave solution

$$u(\xi) = \begin{cases} \phi_4(\xi) & \text{for } \xi \in [0, +\infty), \\ \phi_4(-\xi) & \text{for } \xi \in (-\infty, 0) \end{cases}$$

with $u(0) = e_0$ and $v(0) = 0$ satisfying

$$\int_{e_0}^{\phi_4} -\frac{1}{\sqrt{F(s)}} \mathrm{d}s = \xi, \tag{4.3}$$

where $e_0 \in (u_2, -\frac{3b}{k\alpha})$ is the solution of equation $H(u, 0) = H(u_1, 0)$ and $F(u)$ is defined in (4.2) by replacing $H(u_0, 0)$ with $H(u_1, 0)$.

Theorem 12 (Cusp solitary wave solutions) Suppose that $2b < \alpha$, $\frac{3}{2\alpha}kb(2\alpha - b) < \beta < \frac{k}{2\alpha}(\alpha + b)^2$ and $H(u_1, 0) = H(-\frac{3b}{k\alpha}, v_1)$. Let Γ_1 (or Γ_2) denote the orbits connecting $(u_1, 0)$ and $(-\frac{3b}{k\alpha}, v_1)$ (or $(-\frac{3b}{k\alpha}, -v_1)$). Take an initial value $(u(0), v(0)) = (u_0, v_0) \in \Gamma_1$. Then (1.2) has a cusp solitary wave solution

$$u(\xi) = \begin{cases} \phi_5(\xi) & \text{for } \xi \in (-\infty, T_4); \\ \phi_5(-\xi + 2T_4) & \text{for } \xi \in (T_4, +\infty), \end{cases}$$

where ϕ_5 satisfies

$$\int_{u_0}^{\phi_5} \frac{1}{\sqrt{F(s)}} \mathrm{d}s = \xi,$$

$F(u)$ is defined in (4.2) by replacing $H(u_0, 0)$ with $H(u_1, 0)$ and

$$T_4 = \int_{u_0}^{-\frac{3b}{k\alpha}} \frac{1}{\sqrt{F(s)}} \mathrm{d}s = \xi.$$

First, in order to simulate some bounded wave solutions, we take $b = \frac{1}{2}$, $\beta = \frac{5.3}{7}$ and $k = 0.2$ which imply $\alpha = 7$ for $g = 9.8$. After simple calculations, we obtain u_1=-6.64483, u_2=-4.06944, $v_1 = 0.70801$, the singular line l: u=-1.07143, $H(u_1, 0) = 0.57234$ and $H(-1.07143, 0.70801) = 0.02327$ Thus, the conditions in Theorems 4.1–4.4 are satisfied. Solving equation $H(u, 0) = H(-1.07143, 0.70801)$ yields $e_0 = -5.03733$

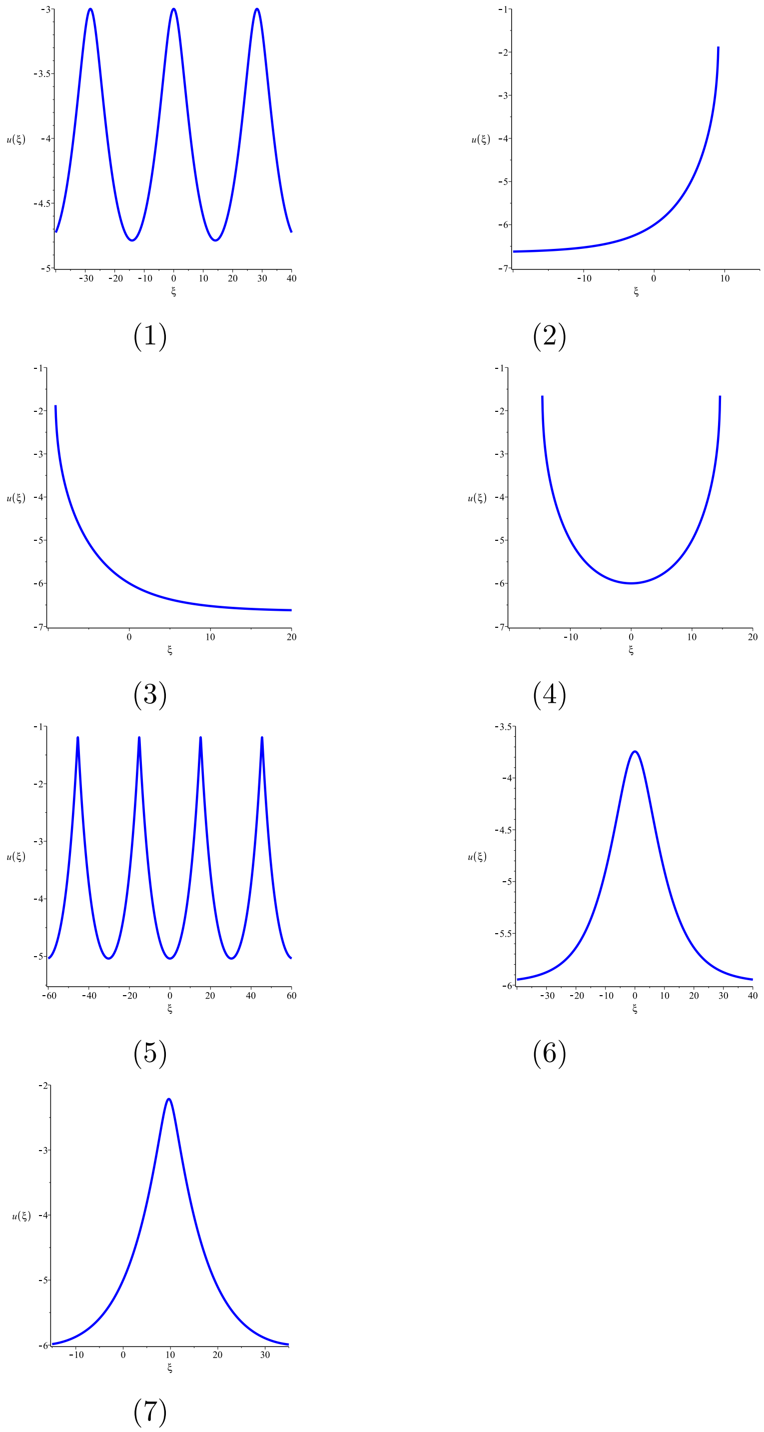

Figure 4: The simulation of the equation (1.2) for $b = \frac{1}{2}$, $k = 0.2$ and $\alpha = 7$. (1) $(u(0), v(0)) = (-3, 0)$ and $\beta = \frac{5}{7}$. (2) $(u(0), v(0)) = (-6, 0.11160)$ and $\beta = \frac{5}{7}$. (3) $(u(0), v(0)) = (-6, -0.11160)$ and $\beta = \frac{5}{7}$. (4) $(u(0), v(0)) = (-6, 0)$ and $\beta = \frac{5}{7}$. (5) $(u(0), v(0)) = (-5.03733, 0)$ and $\beta = \frac{5}{7}$. (6) $(u(0), v(0))$=$(-3.74457, 0)$ and $\beta = \frac{5.55}{7}$. (7) $(u(0), v(0)) = (-5, 0.17172)$ and $\beta = \frac{5.481}{7}$.

The graph of the periodic wave solution $u(\xi)$ of (1.2) with an initial value $(u(0),v(0)) = (-3,0)$ is shown in (1) of Figure . The graph of the kink-like (or antikink-like) wave solution $u(\xi)$ of (1.2) with an initial value

$$(u(0),v(0)) = (-6,0.11160)$$

(or $(u(0),v(0)) = (-6,-0.11160)$) is shown in (2) (or (3)) of Figure . The graph of the compacton-like wave solution $u(\xi)$ of (1.2) with an initial value $(u(0),v(0)) = (-6,0)$ is shown in (4) of Figure . The graph of the periodic cusp wave solution $u(\xi)$ of (1.2) with an initial value $(u(0),v(0)) = (-5.03733,0)$ is shown in (5) of Figure .

Second, in order to simulate solitary wave solutions, we take $b=\frac{1}{2}$, $\beta=\frac{5.55}{7}$ and $k=0.2$ which imply $\alpha=7$ for $g=9.8$. After simple calculations, we obtain

$$u_1=-5.97573, u_2=-4.73855, v_1=0.73453,$$

the singular line l: $u{=}-1.07143$, $H(u_1,0)=-0.13603$ and $H(-1.07143,0.73453)=0.02446$. Thus, the conditions in Theorem 4.5 are satisfied. Solving equation $H(u,0)=H(-5.97573,0)$ yields $e_0=-3.74457$. The graph of the solitary wave solution $u(\xi)$ of (1.2) with an initial value $(u(0),v(0)){=}(-3.74457,0)$ is shown in (6) of Figure .

Last, in order to simulate cusp solitary wave solutions, we take $b=\frac{1}{2}$, $\beta=\frac{5.481}{7}$ and $k=0.2$ which imply $\alpha=7$ for $g=9.8$. After simple calculations, we obtain

$$u_1=-6.21428, u_2=-4.49999, v_1=0.72730,$$

the singular line l: $u{=}-1.07143$, $H(u_1,0)=0.02413$ and $H(-1.07143,0.72730)=0.02413$. Thus, the conditions in Theorem 4.6 are satisfied. Solving equation

$$H(-5,v)=H(-6.21428,0)$$

yields $v=0.17172$ (or $v=-0.17172$). The graph of the cusp solitary wave solution $u(\xi)$ of (1.2) with an initial value $(u(0),v(0))=(-5,0.17172)$ is shown in (7) of Figure .

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References

[1] Green A E, Naghdi P M. A derivation of equations for wave propagation in water of variable depth[J]. J. Fluid. Mech., 1976, 78: 237–246.

[2] Li Y A. Hamiltonian structure and linear stability of solitary waves of the Green-Naghdi equations[J]. J. Nonlin. Math. Phys., 2002, 9: 99–105.

[3] Deng S, Guo B, Wang T. Some traveling wave solutions of the Green-Naghdi System[J]. Int. J. Bif. Chaos., 2011, 21: 575–585.

[4] Kraenkelb R A, Leona J, Mannaa M A. Theory of small aspect ratio waves in deep water[J]. Physica D, 2005, 211: 377–390.

[5] Huang J, Liu H. New exact traveling wave solutions for Fisher equation and Burgers-Fisher equation[J]. J. Math. (PRC), 2011, 31 (4): 631–637.

[6] Rong J, Tang S, Huang W. Bifurcations of traveling wave solutions for the k$(n, 2n, -n)$ equations[J]. J. Math. (PRC), 2010, 30 (4): 603–612.

[7] Tang S, Tang Q. Traveling wave solutions for the generalized special type of the Tzitzeica-Dodd-Bullough equation[J]. J. of Math. (PRC), 2009, 29 (1): 27–37.

[8] Zhong J, Gu X. Bifurcation of traveling wave solutions for symmetric regularized wave equations[J]. Math. Appl., 2011, 24(3): 488–492.

[9] Zhang Z, Ding T, Huang W, Dong Z. Qualitative theory of differential equations[M]. Providence: American Mathematical Society, 1991.

小展弦比波的Green-Naghdi渐进模型的行波解

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摘要: 本文研究了小展弦比波的Green-Naghdi渐进模型. 利用平面自治系统的稳定性分析方法, 在不同的参数条件下, 讨论了它的行波系统的分岔并且给出了对应的相图, 得到了光滑周期波解, 广义扭波解, 广义反扭波解, 广义紧波解, 周期尖波解, 孤波解和孤立尖波解的精确表达式. 进一步, 通过数学软件Maple模拟了这些解.

关键词: Green-Naghdi渐进模型; 行波解; 相图; 分岔

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