Vol. 34 (2014)
No. 6

数 学 杂 志
J. of Math. (PRC)

THE HELTON CLASS OPERATORS AND THE SINGLE VALUED EXTENSION PROPERTY

DONG Zhi-qing [1], CAO Xiao-hong [2], ZHAO Hai-yan [2]

(*1.School of Science, Chongqing Jiaotong University, Chongqing 400074, China*)

(*2.College of Math. and Information Science, Shaanxi Normal University, Xi'an 710062, China*)

Abstract: In this paper, the stability of single valued extension property under compact perturbations for the Helton class operator is investigated. Also, 2×2 upper triangular operator matrices for which the single valued extension property is stable under compact perturbations are characterized. Using the characteristic of semi-Fredholm domain, we give the necessary and sufficient condition for 2×2 upper triangular operator matrices for which the single valued extension property hold.

Keywords: single valued extension property; compact perturbation; Helton class

2010 MR Subject Classification: 47A53; 47A55; 47A15

Document code: A **Article ID:** 0255-7797(2014)06-1033-11

1 Introduction

Throughout this paper, H will denote a complex separable infinite dimensional Hilbert space. Let $B(H)$ denote the algebra of all bounded linear operators on H and $\mathcal{K}(H)$ the ideal of compact operators in $B(H)$. We recall that, for $T \in B(H)$, the spectrum $\sigma(T)$ collects the complex numbers λ for which $T - \lambda I$ fails to be invertible, equivalently is either not one to one or not onto. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range $R(T)$ with finite dimensional null space $N(T)$, and if $R(T)$ has finite co-dimension, $T \in B(H)$ is called a lower semi-Fredholm operator. We call $T \in B(H)$ Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension. For a semi-Fredholm operator, let $n(T) = \dim N(T)$ and $d(T) = \dim H/R(T) = \operatorname{codim} R(T)$. The index of a semi-Fredholm operator $T \in B(H)$ is given by $\operatorname{ind}(T) = n(T) - d(T)$. The ascent of T , $\operatorname{asc}(T)$, is the least non-negative integer n such that $N(T^n) = N(T^{n+1})$ and the descent, $\operatorname{des}(T)$, is the least non-negative integer n such that $R(T^n) = R(T^{n+1})$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. And $T \in B(H)$ is called Browder if it is Fredholm "of finite ascent and descent": equivalently [4, Theorem 7.9.3] if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, the Browder spectrum $\sigma_b(T)$, the Wolf spectrum

* **Received date:** 2013-01-04 **Accepted date:** 2013-07-11

Foundation item: Supported by National Natural Science Foundation of China(10726043).

Biography: Dong Zhiqing(1972–), female, born at Taiyuan, Shanxi, lecturer, major in operator theory, statistics.

$\sigma_{SF}(T)$ of $T \in B(H)$ are defined by (see [4, 5]): $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Fredholm$\}$, $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Weyl$\}$, $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder$\}$,

$$\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi} - \text{Fredholm}\}.$$

Let $\sigma_0(T) = \sigma(T)\backslash\sigma_b(T)$, $\rho_w(T) = \mathbb{C}\backslash\sigma_w(T)$, $\rho_b(T) = \mathbb{C}\backslash\sigma_b(T)$, $\rho_{SF}(T) = \mathbb{C}\backslash\sigma_{SF}(T)$ and

$$\sigma_{SF_+}(T)(\sigma_{SF_-}(T)) = \{\lambda \in \mathbb{C} : T - \lambda I\text{is not upper (lower) semi} - \text{Fredholm}\}.$$

We call $T \in B(H)$ is bounded from below if $N(T) = \{0\}$ and $R(T)$ is closed, $\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not bounded from below$\}$ denotes the approximate point spectrum and $\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not surjective$\}$.

In this note, we investigate the stability of single valued extension property under compact perturbations for the Helton class operators. Also, we characterize 2×2 upper triangular operator matrices for which the single valued extension property is stable under compact perturbations.

2 SVEP and Its Perturbations

In [6], Helton initiated the study of operators which satisfy an identity of the form

$$T^{*m} - \begin{pmatrix} m \\ 1 \end{pmatrix} T^{*m-1}T + \cdots + (-1)^m T^m = 0. \tag{1}$$

We need further study for this class of operators based on (1). Let R and S be in $B(H)$ and let $C(R,S) : B(H) \to B(H)$ be defined by $C(R,S)(A) = RA - AS$. Then

$$C(R,S)^k(I) = \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j}.$$

If there is an integer $k \geq 1$ such that an operator S satisfies $C(R,S)^k(I) = 0$, we say that S belongs to the Helton class of R with order k. We denote this by $S \in \text{Helton}_k(R)$. Let's begin with a lemma.

Lemma 2.1 Let $S \in \text{Helton}_k(R)$, then:

(1) $\sigma_a(S) \subseteq \sigma_a(R)$, $\sigma_{SF_+}(S) \subseteq \sigma_{SF_+}(R)$;

(2) $\sigma_s(R) \subseteq \sigma_s(S)$, $\sigma_{SF_-}(R) \subseteq \sigma_{SF_-}(S)$;

(3) For any $\lambda \in \mathbb{C}$, $N(S - \lambda I) \subseteq N[(R - \lambda I)^k]$;

(4) $\sigma_p(S) \subseteq \sigma_p(R)$.

Proof For any $\lambda \in \mathbb{C}$, we have the following equation:

$$
\begin{aligned}
&\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}(R-\lambda I)^j(\lambda I-S)^{k-j} \\
= &\sum_{j=0}^{k}\sum_{r=0}^{j}\sum_{s=0}^{k-j}(-1)^{k-(s+r)}\begin{pmatrix} k \\ j \end{pmatrix}\begin{pmatrix} j \\ r \end{pmatrix}\begin{pmatrix} k-j \\ s \end{pmatrix}R^r\lambda^{j+s-r}S^{k-(j+s)} \\
= &\sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}R^jS^{k-j}=0.
\end{aligned}
$$

Then

$$
\left\{
\begin{aligned}
&[(\lambda I-S)^{k-1}+\begin{pmatrix} k \\ 1 \end{pmatrix}(R-\lambda I)(\lambda I-S)^{k-2} \\
&+\cdots+\begin{pmatrix} k \\ k-1 \end{pmatrix}(R-\lambda I)^{k-1}](\lambda I-S)=-(R-\lambda I)^k, \\
&(R-\lambda I)[\begin{pmatrix} k \\ 1 \end{pmatrix}(\lambda I-S)^{k-1} \\
&+\cdots+\begin{pmatrix} k \\ k-1 \end{pmatrix}(R-\lambda I)^{k-2}(\lambda I-S)+(R-\lambda I)^{k-1}]=-(\lambda I-S)^k.
\end{aligned}
\right.
$$

From the first equation, we can prove that $\sigma_a(S)\subseteq\sigma_a(R)$, $\sigma_{SF_+}(S)\subseteq\sigma_{SF_+}(R)$, $\sigma_p(S)\subseteq\sigma_p(R)$ and for any $\lambda\in\mathbb{C}$, $N(S-\lambda I)\subseteq N[(R-\lambda I)^k]$. Using the second equation, we get that $\sigma_s(R)\subseteq\sigma_s(S)$ and $\sigma_{SF_-}(R)\subseteq\sigma_{SF_-}(S)$.

An operator T on a complex Hilbert space H is said to have the single valued extension property(SVEP for short), denoted by $T\in$(SVEP), if for every open set $U\subseteq\mathbb{C}$, the only analytic solution $f(\cdot):U\to H$ of the equation $(T-\lambda I)f(\lambda)=0$ for all $\lambda\in U$ is the zero function on U. Clearly, T has the SVEP if $\mathrm{int}\sigma_p(T)=\emptyset$, where $\sigma_p(T)$ denotes the point spectrum of T. The single valued extension property is possessed by many important classes of operators such as hyponormal operators and decomposable operators. The interested reader is referred to (see [1, 3, 9]) for more details. Next we study the Helton class of an operator which has the single valued extension property.

Theorem 2.1 Let $S\in\mathrm{Helton}_k(R)$. If $R\in B(H)$ has the single valued extension property, then S has the single valued extension property.

Proof Let $f:D\to H$ be an analytic function such that $(\lambda I-S)f(\lambda)\equiv 0$, where $D\subseteq\mathbb{C}$ is open. By (4) in Lemma 2.1, we know that $(R-\lambda I)^kf(\lambda)\equiv 0$. Since $R\in B(H)$ has the single valued extension property, it follows that $(R-\lambda I)^{k-1}f(\lambda)\equiv 0$. By induction, we have $f(\lambda)\equiv 0$. So we conclude that S has the single valued extension property.

In order to study the stability of the single valued extension property, we first give a lemma (see [10], Theorem 1.3).

Lemma 2.2 Let $T\in B(H)$, then $T+K\in$(SVEP) for all $K\in\mathcal{K}(H)$ if and only if

(1) $\mathrm{int}\sigma_{SF}(T)=\emptyset$;

(2) $\rho_{SF}(T)$ is connected.

If $R\in\mathcal{K}(H)$ and $S\in\mathrm{Helton}_k(R)$, from Lemma 2.1, we know that $\sigma_{SF_+}(S)=\{0\}$. Then $\sigma_{SF}(S)=\{0\}$ since S has the single valued extension property. For any polynomial p, $\sigma_{SF}(p(S))=\{p(0)\}$. Thus int $\sigma_{SF}(p(S))=\emptyset$ and $\rho_{SF}(p(S))$ is connected. So we have that

Corollary 2.1 If $R\in\mathcal{K}(H)$ and $S\in\mathrm{Helton}_k(R)$, then $p(S)+K$ has the single valued extension property for any polynomial p and any $K\in\mathcal{K}(H)$.

In [10], if $\rho_{SF}(T)$ is connected, then $\sigma(T+K)=\sigma_{SF}(T+K)\cup\sigma_0(T+K)$ for all $K\in\mathcal{K}(H)$.

Theorem 2.2 Let $S\in\mathrm{Helton}_k(R)$. If $R+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$, then $S+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$ if and only if $\sigma_{SF_+}(S)=\sigma_{SF_-}(S)$.

Proof We know that $\sigma(R)=\sigma_{SF}(R)\cup\sigma_0(R)$ since $\rho_{SF}(R)$ is connected (Lemma 2.2). It can induce that $\sigma_{SF_+}(R)=\sigma_{SF_-}(R)$.

If $S+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$, we know that $\rho_{SF}(S)$ is connected. Then $\sigma(S)=\sigma_{SF}(S)\cup\sigma_0(S)$. This implies that $\sigma_{SF_+}(S)=\sigma_{SF_-}(S)$.

For the converse, if $\sigma_{SF_+}(S)=\sigma_{SF_-}(S)$, we know from Lemma 2.1 that

$$\sigma_{SF_+}(S)=\sigma_{SF_+}(R)=\sigma_{SF_-}(R)=\sigma_{SF_-}(S).$$

Then

$$\sigma_{SF}(S)=\sigma_{SF_+}(S)\cap\sigma_{SF_-}(S)=\sigma_{SF_+}(S)=\sigma_{SF_+}(R)\cap\sigma_{SF_-}(R)=\sigma_{SF}(R).$$

So $\rho_{SF}(S)=\rho_{SF}(R)$ and $\mathrm{int}\sigma_{SF}(S)=\mathrm{int}\sigma_{SF}(R)$. Since $R+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$, by Lemma 2.2, $S+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$.

From the proof of Theorem 2.2, we can get: Let $S\in\mathrm{Helton}_k(R)$, if R has the single valued extension property, then $S+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$ if and only if $\sigma_{SF_+}(S)=\sigma_{SF_-}(S)$ and $R+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$.

If $S\in\mathrm{Helton}_k(R)$ and $R\in\mathrm{Helton}_k(S)$, then

$$\sigma_\tau(S)=\sigma_\tau(R),$$

where $\sigma_\tau\in\{\sigma_a,\sigma_{SF_+},\sigma_{SF_-},\sigma_p,\sigma_s\}$. By Theorem 2.1 and Theorem 2.2, we have

Corollary 2.2 Let $S\in\mathrm{Helton}_k(R)$ and $R\in\mathrm{Helton}_k(S)$, then

(1) R has the single valued extension property if and only if S has the single valued extension property;

(2) $R+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$ if and only if $S+K$ has the single valued extension property for all $K\in\mathcal{K}(H)$.

If S and R have the single valued extension property, does $S+R$ have the single valued extension property? So far we don't know the answer about this question. But we consider the special cases of this question. We start with the case of Helton class. If $S \in \mathrm{Helton}_k(R)$ and $SR = RS$, then $R \in \mathrm{Helton}_k(S)$. By Corollary 2.2, there is the result

Corollary 2.3 Let $S \in \mathrm{Helton}_k(R)$ and $SR = RS$, then

(1) R has the single valued extension property if and only if S has the single valued extension property;

(2) $R + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $S + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$;

(3) $S + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $S + R + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$.

Proof We only need to prove (3). It is easy to calculate that $C(2R, R + S)^k(I) = 0$, that is $R + S \in \mathrm{Helton}_k(2R)$. It is clear that $2R \cdot (R + S) = (R + S) \cdot 2R$, then from (2), $S + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $R + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $2R + K = 2(R + \frac{K}{2})$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $S + R + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$.

If $SN = NS$ and $N^k = 0$ for some $k \in \mathbb{N}$, then $S \in \mathrm{Helton}_k(S + N)$ and $S(S + N) = (N + S)S$; Also, we can prove that $tS \in \mathrm{Helton}_k(tS + N)$ for any $t \in \mathbb{N}$. Then

Corollary 2.4 Let $SN = NS$. If $N^k = 0$ for some $k \in \mathbb{N}$, then

(1) S has the single valued extension property if and only if $S + N$ has the single valued extension property;

(2) $S + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $S + N + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$;

(3) $S + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $tS + N + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ and for any $t \in \mathbb{N}$.

In Corollary 2.4, if we let $N \in B(H)$ be a quasi-nilpotent operator, we can get that $S + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ if and only if $S + N + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$. In fact, we know that N is a Riesz operator. Then $\rho_{SF}(S + N) = \rho_{SF}(S)$. From Lemma 2.2, we can prove the claim.

Example 2.1 Let $T_1,\ T_2 \in B(\ell^2)$ be defined by

$$T_1(x_1, x_2, \cdots) = (x_1, 0, x_3, x_4, \cdots); \qquad T_2(x_1, x_2, \cdots) = (0, x_1, 0, 0, \cdots),$$

and let $S = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix}$, then $SN = NS$, $N^2 = 0$ and $S + K$ has the single valued extension property for all $K \in \mathcal{K}(\ell^2 \oplus \ell^2)$. Thus $tS + N + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$ and for any $t \in \mathbb{N}$.

In [2], the single valued extension property of upper triangular operator matrices has been studied. We continue this work. In the following, we characterize 2×2 upper triangular operator matrices for which the single valued extension property is stable under compact perturbations. Let us give some lemmas which will be used in the main result (Lemma 2.10 in [8] and Theorem 3.1 in [7] or Corollary 4.3 in [10]).

Lemma 2.3 Let $T \in B(H)$ and suppose that $\emptyset \neq \mathcal{T} \subseteq \sigma_{SF}(T)$, then given $\epsilon > 0$, there exists a compact operator K with $\|K\| < \epsilon$ such that $T + K = \begin{pmatrix} N & C \\ 0 & A \end{pmatrix}$, where N is a normal operator and $\sigma(N) = \sigma_{SF}(N) = \overline{\mathcal{T}}$.

Lemma 2.4 Let $T \in B(H)$. If $\sigma(T) = \partial\Omega$, where Ω is a bounded connected open subset of $\mathbb{C}$, then, given $\epsilon > 0$, there exists $K \in \mathcal{K}(H)$ with $\|K\| < \sqrt{\frac{m(\Omega)}{\pi}} + \epsilon$ such that $\sigma(T + K) = \overline{\Omega}$. Here $m(\cdot)$ denotes the planar Lebesgue measure.

Theorem 2.3 Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$, then $T + K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$ if and only if

(1) $R + K_1$ and $N + K_2$ have the single valued extension property for all $K_i \in \mathcal{K}(H)$ $(i = 1, 2)$;

(2) $\sigma_w(T + K) = \sigma_b(T + K)$ for all $K \in \mathcal{K}(H \oplus H)$.

Proof Suppose $T + K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$. Then $\mathrm{int}\sigma_{SF}(T) = \emptyset$ and $\rho_{SF}(T)$ is connected, also $\sigma_w(T + K) = \sigma_b(T + K)$ for all $K \in \mathcal{K}(H \oplus H)$.

First, we will prove that $\mathrm{int}\sigma_{SF}(R) = \emptyset$ and $\rho_{SF}(R)$ is connected. If $\mathrm{int}\sigma_{SF}(R) \neq \emptyset$, then there exists $\lambda_0 \in \sigma_{SF}(R)$ and $\delta > 0$ such that $B_\delta(\lambda_0) \subseteq \sigma_{SF}(R)$. Since $\mathrm{int}\sigma_{SF}(T) = \emptyset$ and T has the single valued extension property, there must exist $\lambda_1 \in B_\delta(\lambda_0)$ such that $T - \lambda_1 I$ is an upper semi-Fredholm operator. Using the equation

$$T - \lambda_1 I = \begin{pmatrix} R - \lambda_1 I & S \\ 0 & N - \lambda_1 I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & N - \lambda_1 I \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \begin{pmatrix} R - \lambda_1 I & 0 \\ 0 & I \end{pmatrix},$$

we know that $R - \lambda_1 I$ is upper semi-Fredholm. It is in contradiction to the fact that $\lambda_1 \in B_\delta(\lambda_0) \subseteq \sigma_{SF}(R)$. For the connected property of $\rho_{SF}(R)$, if $\rho_{SF}(R)$ is not connected, then we can choose a bounded component Ω of $\rho_{SF}(R)$. Since $\partial\Omega \subseteq \sigma_{SF}(R)$, by Lemma 2.3, there exists $K_{11} \in \mathcal{K}(H)$ such that $R + K_{11} = \begin{pmatrix} N_1 & B \\ 0 & A \end{pmatrix}$, where N_1 is normal and $\sigma(N_1) = \sigma_{SF}(N_1) = \partial\Omega$. By Lemma 2.4, we can choose a compact operator K' such that $\sigma(N_1 + K') = \overline{\Omega}$. We have the fact that $N_1 + K' - \lambda I$ is Weyl for any $\lambda \in \Omega$. Let $K_{12} = \begin{pmatrix} K' & 0 \\ 0 & 0 \end{pmatrix}$, then $K_{12} \in \mathcal{K}(H)$ and $R + K_{11} + K_{12} = \begin{pmatrix} N_1 + K' & B \\ 0 & A \end{pmatrix}$. Let $K_1 = K_{11} + K_{12}$ and $K = \begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}$, we get that $T + K = \begin{pmatrix} R + K_1 & S \\ 0 & N \end{pmatrix}$. Since $\mathrm{int}\sigma_{SF}(T) = \emptyset$, there exists $\lambda_1 \in \Omega$ such that $T + K - \lambda_1 I$ is upper semi-Fredholm with $\mathrm{ind}(T + K - \lambda_1 I) \leq 0$. Thus $T + K - \lambda_2 I$ is bounded from below for some $\lambda_2 \in \Omega$ since $T + K$ has the single valued extension property. This induces that $R + K_1 - \lambda_2 I$ is bounded from below. Also $N_1 + K' - \lambda_2 I$ is bounded from below. But since $N_1 + K' - \lambda_2 I$ is Weyl, we know that $N_1 + K' - \lambda_2 I$ is invertible. It is in contradiction to the fact that $\sigma(N_1 + K') = \overline{\Omega}$.

Now we prove that $\mathrm{int}\sigma_{SF}(R)=\emptyset$ and $\rho_{SF}(R)$ is connected, then $R+K_1$ has the single valued extension property for all $K_1\in\mathcal{K}(H)$.

Second, using the same way we will prove that $\mathrm{int}\sigma_{SF}(N)=\emptyset$ and $\rho_{SF}(N)$ is connected. If $\mathrm{int}\sigma_{SF}(N)\neq\emptyset$, then there exists $\lambda_0\in\sigma_{SF}(N)$ and $\delta>0$ such that $B_\delta(\lambda_0)\subseteq\sigma_{SF}(N)$. Since $\mathrm{int}\sigma_{SF}(T)=\mathrm{int}\sigma_{SF}(R)=\emptyset$ and both T and R have the single valued extension property, there must exist $\lambda_1\in B_\delta(\lambda_0)$ such that $T-\lambda_1 I$ and $R-\lambda_1 I$ are upper semi-Fredholm operators. From the fact that $\rho_{SF}(T)$ and $\rho_{SF}(R)$ are connected, we know that $T-\lambda_1 I$ and $R-\lambda_1 I$ are Browder operators (see [10], Corollary 2.5). Then there exists $\lambda_2\in B_\delta(\lambda_0)$ such that $T-\lambda_2 I$ and $R-\lambda_2 I$ are invertible. This induces that $N-\lambda_2 I$ is invertible. It is a contradiction. If $\rho_{SF}(N)$ is not connected, then we can choose a bounded component Ω of $\rho_{SF}(N)$. Similar to the preceding proof, we can choose $K_2\in\mathcal{K}(H)$ such that $N+K_2=\begin{pmatrix} N_2+K' & B\\ 0 & A\end{pmatrix}$, where N_2 is normal, K' is compact and $\sigma(N_2+K')=\overline{\Omega}$. Also, $N_2+K'-\lambda I$ is Weyl for any $\lambda\in\Omega$. Let $K=\begin{pmatrix} 0 & 0\\ 0 & K_2\end{pmatrix}$ and $T+K=\begin{pmatrix} R & S\\ 0 & N+K_2\end{pmatrix}$. Since $\mathrm{int}\sigma_{SF}(T)=\mathrm{int}\sigma_{SF}(T+K)=\emptyset$, there exists $\lambda_1\in\Omega$ such that $T+K-\lambda_1 I$ is upper semi-Fredholm. Then $R-\lambda_1 I$ is upper semi-Fredholm. But since $\rho_{SF}(T)$ and $\rho_{SF}(R)$ are connected, we know that both $T+K-\lambda_1 I$ and $R-\lambda_1 I$ are Browder operators. Thus there is $\lambda_2\in\Omega$ such that both $T+K-\lambda_2 I$ and $R-\lambda_2 I$ are invertible. We get that $N+K_2-\lambda_2 I$ is invertible, which implies that $N_2+K'-\lambda_2 I$ is bounded from below. Then $N_2+K'-\lambda_2 I$ is invertible since $N_2+K'-\lambda_2 I$ is Weyl. It is in contradiction to the fact that $\sigma(N_2+K')=\overline{\Omega}$. We now get that $\mathrm{int}\sigma_{SF}(N)=\emptyset$ and $\rho_{SF}(N)$ is connected, so $N+K_2$ has the single valued extension property for all $K_2\in\mathcal{K}(H)$.

For the converse, suppose $R+K_1$ and $N+K_2$ have the single valued extension property for all $K_i\in\mathcal{K}(H)$ $(i=1,2)$. First we need to prove that T has the single valued extension property if R and N have the single valued extension property. Let $f=f_1\oplus f_2: D\to H\oplus H$ be an analytic function such that $(\lambda I-T)f(\lambda)\equiv 0$, then we have that

$$\begin{cases} (\lambda I-R)f_1(\lambda)-Sf_2(\lambda)=0,\\ (\lambda I-N)f_2(\lambda)=0.\end{cases}$$

Since N has the single valued extension property, it follows that $f_2(\lambda)=0$. Then

$$(\lambda I-R)f_1(\lambda)=0.$$

Thus $f_1(\lambda)=0$ since R has the single valued extension property. We get that $f(\lambda)\equiv 0$, which means that T has the single valued extension property. Second we prove that $\mathrm{int}\sigma_{SF}(T)=\emptyset$ and $\rho_{SF}(T)$ is connected. Since $\rho_{SF}(R)$ and $\rho_{SF}(N)$ are connected, it follows that $\sigma_{SF_+}(R)=\sigma_{SF}(R)$ and $\sigma_{SF_+}(N)=\sigma_{SF}(N)$. Using the fact that $\sigma_{SF}(T)=\sigma_{SF_+}(T)\subseteq\sigma_{SF_+}(R)\cup\sigma_{SF_+}(N)=\sigma_{SF}(R)\cup\sigma_{SF}(N)$ and $\mathrm{int}\sigma_{SF}(R)=\mathrm{int}\sigma_{SF}(N)=\emptyset$, we get that $\mathrm{int}\sigma_{SF}(T)\subseteq\mathrm{int}\sigma_{SF}(R)\cup\mathrm{int}\sigma_{SF}(N)=\emptyset$, that is $\mathrm{int}\sigma_{SF}(T)=\emptyset$. If $\rho_{SF}(T)$ is not connected, then we can choose a bounded component Ω of $\rho_{SF}(T)$. Thus we can choose $K \in \mathcal{K}(H \oplus H)$ such that

$$T + K = \begin{pmatrix} N_3 + K_1 & B \\ 0 & A \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2 = H \oplus H,$$

where N_3 is normal, K_1 is compact and $\sigma(N_3 + K_1) = \overline{\Omega}$. Let $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$, where K_{ij} is compact $(i, j = 1, 2)$. Then

$$\begin{aligned} T + K &= \begin{pmatrix} N_3 + K_1 & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} R + K_{11} & S + K_{12} \\ K_{21} & N + K_{22} \end{pmatrix} \\ &= \begin{pmatrix} R + K_{11} & S + K_{12} \\ 0 & N + K_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ K_{21} & 0 \end{pmatrix}. \end{aligned}$$

Since $\mathrm{int}\sigma_{SF}(R) = \mathrm{int}\sigma_{SF}(N) = \emptyset$, there exists $\lambda_1 \in \Omega$ such that $R - \lambda_1 I$ and $N - \lambda_1 I$ are semi-Fredholm. Using the fact that both $\rho_{SF}(R)$ and $\rho_{SF}(N)$ are connected, $R + K_{11} - \lambda_1 I$ and $N + K_{22} - \lambda_1 I$ are Browder (see [10], Corollary 3.5). Then $T + K - \lambda_1 I$ is Weyl. By $\sigma_w(T + K) = \sigma_b(T + K)$, there exists $\lambda_2 \in \Omega$ such that $T + K - \lambda_2 I$ is invertible, this means that $N_3 + K_1 - \lambda_2 I$ is invertible. It is a contradiction again. So $\mathrm{int}\sigma_{SF}(T) = \emptyset$ and $\rho_{SF}(T)$ is connected, which means that $T + K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$.

Similar to the proof of Theorem 2.3, let $T = (T_{ij}) \in B(\oplus_{k=1}^{n} H)$ be an $n \times n$ upper triangular operator matrix, then $T + K$ has the single valued extension property for all $K \in \mathcal{K}(\oplus_{k=1}^{n} H)$ if and only if $T_{ii} + K_i$ has the single valued extension property for all $K_i \in \mathcal{K}(H)$ $(i = 1, 2, \cdots, n)$ and $\sigma_w(T + K) = \sigma_b(T + K)$ for all $K \in \mathcal{K}(\oplus_{k=1}^{n} H)$.

If $N \in B(H)$ is a Riesz operator, then $\mathrm{int}\sigma_{SF}(N) = \emptyset$ and $\rho_{SF}(N)$ is connected. This means that $N + K$ has the single valued extension property for all $K \in \mathcal{K}(H)$. Then

Corollary 2.5 Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$. If $N \in B(H)$ is a Riesz operator, then $T+K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$ if and only if $R+K_1$ has the single valued extension property for all $K_1 \in \mathcal{K}(H)$ and $\sigma_w(T+K) = \sigma_b(T+K)$ for all $K \in \mathcal{K}(H \oplus H)$.

Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$. If $RS = SN$, we claim that $\sigma_a(T) = \sigma_a(R) \cup \sigma_a(N)$. In fact, we only need to prove that $\sigma_a(R) \cup \sigma_a(N) \subseteq \sigma_a(T)$. Let $\lambda_0 \notin \sigma_a(T)$, then $\lambda_0 \notin \sigma_a(R)$. First we will prove that $N(N - \lambda_0 I) = \{0\}$. If $x_0 \in N(N - \lambda_0 I)$, by $(R - \lambda_0 I)S = S(N - \lambda_0 I)$, then $(R - \lambda_0 I)Sx_0 = 0$. This induces that $Sx_0 = 0$ since $R - \lambda_0 I$ is bounded from below. We can find that $\begin{pmatrix} 0 \\ x_0 \end{pmatrix} \in N(T - \lambda_0 I)$. But since $N(T - \lambda_0 I) = \{0\}$, it follows that $x_0 = 0$. So $N(N - \lambda_0 I) = \{0\}$. Second we will prove that $R(N - \lambda_0 I)$ is closed. Let $(N - \lambda_0 I)y_n \to y_0 (n \to \infty)$, then $S(N - \lambda_0 I)y_n \to Sy_0 (n \to \infty)$. By $(R - \lambda_0 I)S = S(N - \lambda_0 I)$,

$(R-\lambda_0 I)Sy_n \to Sy_0 (n \to \infty)$. Since $R-\lambda_0 I$ is bounded from below, there is $k>0$ such that $\|(R-\lambda_0 I)x\| \geq k\|x\|$ for all $x \in H$. Then $\{Sy_n\}$ is a Cauchy sequence. Suppose $Sy_n \to y_1 (n \to \infty)$. Then

$$(T-\lambda_0 I)\begin{pmatrix} 0 \\ y_n \end{pmatrix} = \begin{pmatrix} Sy_n \\ (N-\lambda_0 I)y_n \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} (n \to \infty)).$$

From the fact that $R(T-\lambda_0 I)$ is closed, there exists $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$(T-\lambda_0 I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}.$$

Then $(R-\lambda_0 I)x_1 + Sx_2 = y_1$ and $(N-\lambda_0 I)x_2 = y_0$. This implies that $y_0 \in R(N-\lambda_0 I)$, which means that $R(N-\lambda_0 I)$ is closed. So $\lambda_0 \notin \sigma_a(N)$ and hence $\sigma_a(T) = \sigma_a(R) \cup \sigma_a(N)$.

Corollary 2.6 Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$ and $RS = SN$, then $T+K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$ if and only if

(1) Both R and N have the single valued extension property;

(2) $\mathrm{int}\sigma_{SF}(T) = \emptyset$;

(3) For any $K = \begin{pmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{pmatrix} \in \mathcal{K}(H \oplus H)$, $\sigma_b(T+K) = \sigma_{SF_+}(R+K_1) \cup \sigma_{SF_+}(N+K_2)$.

Proof Suppose $T+K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$. By Theorem 2.3, $\rho_{SF}(T)$, $\rho_{SF}(R)$ and $\rho_{SF}(N)$ are connected. Then $\sigma_{SF_+}(R+K_1) = \sigma_b(R+K_1)$ and $\sigma_{SF_+}(N+K_2) = \sigma_b(N+K_2)$ for any $K_1, K_2 \in \mathcal{K}(H)$. Since

$$T+K = \begin{pmatrix} R+K_1 & K_{12} \\ 0 & N+K_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ K_{21} & 0 \end{pmatrix}$$

and $\begin{pmatrix} 0 & 0 \\ K_{21} & 0 \end{pmatrix}$ is compact for any $K = \begin{pmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{pmatrix} \in \mathcal{K}(H \oplus H)$, it follows that $\sigma_b(T+K) = \sigma_{SF_+}(T+K) \subseteq \sigma_{SF_+}(R+K_1) \cup \sigma_{SF_+}(N+K_2)$. Let $\lambda_0 \notin \sigma_b(T+K)$, then $\lambda_0 \notin \sigma_{SF_+}(R+K_1)$. But since $\rho_{SF}(R)$ is connected, we know that $R+K_1-\lambda_0 I$ is a Weyl operator. Then $N+K_2-\lambda_0 I$ is Weyl, which means that $\lambda_0 \notin \sigma_{SF_+}(N+K_2)$. So $\sigma_b(T+K) = \sigma_{SF_+}(R+K_1) \cup \sigma_{SF_+}(N+K_2)$ for any $K = \begin{pmatrix} K_1 & K_{12} \\ K_{21} & K_2 \end{pmatrix} \in \mathcal{K}(H \oplus H)$.

For the converse, we only need to prove that $\rho_{SF}(T)$ is connected. If $\rho_{SF}(T)$ is not connected, then we can choose a bounded component Ω of $\rho_{SF}(T)$. Then we can choose $K \in \mathcal{K}(H \oplus H)$ such that $T+K = \begin{pmatrix} N_3+K_1' & B \\ 0 & A \end{pmatrix}$, where N_3 is normal, K_1' is compact and $\sigma(N_3 + K_1') = \overline{\Omega}$. Similar to the proof in Theorem 2.3, we know that T has the single valued extension property and

$$T + K = \begin{pmatrix} N_3 + K_1' & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} R + K_1 & S + K_{12} \\ K_{21} & N + K_2 \end{pmatrix}.$$

Since $\mathrm{int}\sigma_{SF}(T) = \emptyset$ and T has the single valued extension property, there exists $\lambda_0 \in \Omega$ such that $T - \lambda_0 I$ is bounded from below. Then $R - \lambda_0 I$ and $N - \lambda_0 I$ are bounded from below because $\sigma_a(T) = \sigma_a(R) \cup \sigma_a(N)$. This implies that $\lambda_0 \notin \sigma_{SF_+}(R+K_1) \cup \sigma_{SF_+}(N+K_2)$, thus $\lambda_0 \notin \sigma_b(T + K)$. Then there exists $\lambda_1 \in \Omega$ such that $T + K - \lambda_1 I$ is invertible. It follows that $N_3 + K_1' - \lambda_1 I$ is invertible, a contradiction.

Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$ and $N \in \mathrm{Helton}_k(R)$. If $\sigma_{SF_+}(N) = \sigma_{SF_-}(N)$ and $R + K_1$ has the single valued extension property for all $K_1 \in \mathcal{K}(H)$, then $\rho_{SF}(R) = \rho_{SF}(N)$. Thus $\rho_{SF}(T) = \rho_{SF}(R) \cap \rho_{SF}(N) = \rho_{SF}(R)$ is connected. If $R \in \mathrm{Helton}_k(N)$, $\rho_{SF}(R)$ is connected and $N + K_2$ has the single valued extension property for all $K_2 \in \mathcal{K}(H)$, also we have that $\rho_{SF}(T) = \rho_{SF}(R) \cap \rho_{SF}(N)$ is connected. By Lemma 2.1, Theorem 2.2 and Theorem 2.3, we have

Corollary 2.7 (1) Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$ and $N \in \mathrm{Helton}_k(R)$, then $T + K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$ if and only if $\sigma_{SF_+}(N) = \sigma_{SF_-}(N)$ and $R + K_1$ has the single valued extension property for all $K_1 \in \mathcal{K}(H)$;

(2) Let $T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix} \in B(H \oplus H)$ and $R \in \mathrm{Helton}_k(N)$, then $T + K$ has the single valued extension property for all $K \in \mathcal{K}(H \oplus H)$ if and only if $\rho_{SF}(R)$ is connected and $N + K_2$ has the single valued extension property for all $K_2 \in \mathcal{K}(H)$.

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Helton类算子及单值扩张性质

董志清[1], 曹小红[2], 赵海燕[2]

(1. 重庆交通大学理学部, 重庆 400074)

(2. 陕西师范大学数学与信息科学学院, 陕西 西安 710062)

摘 要: 本文研究了Helton类算子在紧摄动下单值扩张性质的稳定性, 同时研究了2×2上三角算子矩阵在紧摄动下单值扩张性质的稳定性. 利用半Fredholm域的特点, 获得了2×2上三角算子矩阵具有单值扩张性质的稳定性的充分必要条件.

关 键 词: 单值扩张性质; 紧摄动; Helton类

MR(2010)**主题分类号**: 47A53; 47A55; 47A15 **中图分类号**: O177.2