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SOME RESULTS FOR CERTAIN SUBCLASS OF MULTIVALENT AND ANALYTIC FUNCTIONS

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Abstract: In this paper, we investigate functions of the class $G^*_{p,c}(a,b,\sigma)$ which are analytic and multivalent in the open unit disk $U=\{z:|z|<1\}$. By using the method of function theory, we obtain some general results concerning the quasi-Hadamard product and the extreme points and support points of $G^*_{p,c}(a,b,\sigma)$. Many interesting consequences of the main results extend related works of several earlier authors.

Keywords: analytic functions; multivalent function; quasi-Hadamard product; extreme points; support points

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1 Introduction

Let $\mathcal{A}$ denote the functions $f_p(z)$ of the form

$$f_p(z)=a_pz^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p} \quad (a_p>0; a_{n+p}\geqslant 0; p\in N^*=\{1,2,\cdots\}), \tag{1.1}$$

which are multivalent and analytic in the unit disc $U=\{z\in\mathcal{C}:|z|<1\}$.

Here, we define the general quasi-Hadamard product of the functions $f_{p,i}$ by

$$f_{p,1}*_{\chi_1}f_{p,2}*_{\chi_2}f_{p,3}*\cdots*_{\chi_{s-1}}f_{p,s}=\left\{\prod_{i=1}^{s}a_{p,i}\right\}z^p-(\prod_{i=1}^{s-1}\chi_i)\sum_{n=1}^{\infty}\left\{\prod_{i=1}^{s}a_{n+p,i}\right\}z^{n+p}, \tag{1.2}$$

where χ_i are any nonnegative real numbers and $f_{p,i}(z)\in\mathcal{A}$ are defined as (1.1), $i=1,2,\cdots,s$.

A function $f_p(z)$ defined by (1.1) is said to be in the class $G^*_p(a,b,\sigma)$ if and only if

$$\left|\frac{\frac{zf'_p(z)}{f_p(z)}-p}{\frac{bzf'_p(z)}{f_p(z)}-ap}\right|<\sigma \quad z\in U, \tag{1.3}$$

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where $-1 \leqslant a < b \leqslant 1, 0 < \sigma \leqslant 1$. Moreover, let $M_p(a,b,\sigma)$ denote the class of functions $f_p(z)$ such that $\frac{zf_p'(z)}{p}$ is in the class $G_p^*(a,b,\sigma)$.

We also have the following special cases on $G_p^*(a,b,\sigma)$ and $M_p(a,b,\sigma)$:

(I) For $a_p \equiv 1$ in (1.1) , the classes $G_p^*(a,b,\sigma) \equiv J_p^*(a,b,\sigma)$ and $M_p(a,b,\sigma) \equiv C_p(a,b,\sigma)$ were studied by Raina, Nahar [1].

(II) For $p = 1, a = -1, b = \alpha, \sigma = \beta$, the classes $G_1^*(-1,\alpha,\beta) \equiv S_0(\alpha,\beta)$ and $M_1(-1,\alpha,\beta) \equiv C_0(\alpha,\beta)$ introduced by Owa [2] are well known.

Using similar arguments as given by Raina, Nahar [1], we can easily prove the following Lemmas for functions in the classes $G_p^*(a,b,\sigma)$ and $M_p(a,b,\sigma)$:

Lemma 1.1 A function $f_p(z)$ defined by (1.1) belongs to $G_p^*(a,b,\sigma)$ if and only if

$$\sum_{n=1}^{\infty}\{(1+b\sigma)n+(b-a)p\sigma\}a_{n+p} \leqslant (b-a)p\sigma a_p, \tag{1.4}$$

where $-1 \leqslant b < a \leqslant 1, 0 < \sigma \leqslant 1, p \in N^* = \{1,2,\cdots\}$.

Lemma 1.2 A function $f_p(z)$ defined by (1.1) belongs to $M_p(a,b,\sigma)$ if and only if

$$\sum_{n=1}^{\infty}(\frac{n+p}{p})\{(1+b\sigma)n+(b-a)p\sigma\}a_{n+p} \leqslant (b-a)p\sigma a_p, \tag{1.5}$$

where $-1 \leqslant b < a \leqslant 1, 0 < \sigma \leqslant 1, p \in N^* = \{1,2,\cdots\}$.

Now, we introduce a new general class of analytic functions connected with the classes $G_p^*(a,b,\sigma)$ and $M_p(a,b,\sigma)$, which is important in the following discussion.

Definition 1.1 A function $f_p(z)$ defined by (1.1) belongs to $G_{p,c}^*(a,b,\sigma)$ if and only if

$$\sum_{n=1}^{\infty}(\frac{n+p}{p})^c\{(1+b\sigma)n+(b-a)p\sigma\}a_{n+p} \leqslant (b-a)p\sigma a_p, \tag{1.6}$$

where $-1 \leqslant b < a \leqslant 1, 0 < \sigma \leqslant 1, p \in N^* = \{1,2,\cdots\}$ and c is any fixed nonnegative real number.

In fact, for every nonnegative real number c, the class $G_{p,c}^*(a,b,\sigma)$ is nonempty as the functions of the form

$$f_p(z) = a_p z^p - \sum_{n=1}^{\infty}\frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}\lambda_{n+p}z^{n+p}, \tag{1.7}$$

where $a_p > 0, \lambda_{n+p} \geqslant 0$ and $\sum\limits_{n=1}^{\infty}\lambda_{n+p} \leqslant 1$, satisfy inequality (1.6).

We note that

(I) For $c = 0$, the class $G_{p,0}^*(a,b,\sigma) \equiv G_p^*(a,b,\sigma)$.

(II) For $c = 1$, the class $G_{p,1}^*(a,b,\sigma) \equiv M_p(a,b,\sigma)$.

(III) For $p = 1, a = -1, b = \alpha, \sigma = \beta$, the class $G_{1,c}^*(-1,\alpha,\beta) \equiv S_c(\alpha,\beta)$ was studied by Aouf [3].

(IV) For any positive integer c, we have the inclusion relation

$$G^*_{p,c}(a,b,\sigma) \subset G^*_{p,c-1}(a,b,\sigma) \subset G^*_{p,c-2}(a,b,\sigma) \subset \cdots \subset G^*_{p,2}(a,b,\sigma) \subset M_p(a,b,\sigma) \subset G^*_p(a,b,\sigma).$$

The topology of $\mathcal{A}$ is defined to be the topology of uniform convergence on compact subsets of the unit disk U. Suppose that $\mathcal{X}$ is a subset of the space $\mathcal{A}$, then $f \in \mathcal{X}$ is called an extreme point of $\mathcal{X}$ if and only if f can not be expressed as a proper convex combination of two distinct elements of $\mathcal{X}$. The set of all extreme points of $\mathcal{X}$ is denoted by $E\mathcal{X}$.

Furthermore, a function f is called a support point of a compact $\mathcal{F}$ of $\mathcal{A}$ if $f \in \mathcal{F}$ and if there is a continuous linear functional J on $\mathcal{A}$ such that $\mathcal{R}eJ$ is non-constant on $\mathcal{F}$ and

$$\mathcal{R}eJ(f) = \max\{\mathcal{R}eJ(g) : g \in \mathcal{F}\}.$$

We shall denote the set of all support points of $\mathcal{F}$ by supp$\mathcal{F}$.

Throughout this paper we use the notation $H\mathcal{F}$ for the closed convex hull of $\mathcal{F}$.

Lemma 1.3 (see [4]) Let $\mathcal{A}$ be a locally convex linear topological space and let $\mathcal{F}$ be a compact subset of $\mathcal{A}$, then

(i) If $\mathcal{F}$ is non-empty, then $E\mathcal{F}$ is non-empty.

(ii) $HE\mathcal{F} = H\mathcal{F}$.

(iii) If $H\mathcal{F}$ is compact, then $EH\mathcal{F} \subset \mathcal{F}$.

The main object of the present work is to discuss some interesting results concerning the quasi-Hadamard product of functions belonging to the class $G^*_{p,c}(a,b,\sigma)$, which extends the earlier corresponding studies in [3, 5–10]. Also, we apply this technique in Peng Zhigang [11, 12] to obtain the extreme points and support points of some important classes with $G^*_{p,c}(a,b,\sigma)$.

2 The Main Theorem

Theorem 2.1 Let the functions $f_{p,i}$ defined by (1.1) be in the class $M_p(a,b,\sigma)$ for every $i = 1,2,3,\cdots,m$; $m \in N^*$, and let the functions $g_{p,j}$ defined by (1.1) be in the class $G^*_p(a,b,\sigma)$ for every $j = 1,2,\cdots,q, q \in N^*$. If $\prod\limits_{i=1}^{m+q-1} \chi_i = 1$ or for any i, $0 < \chi_i \leqslant 1$, then the quasi-Hadamard product $f_{p,1} *_{\chi_1} f_{p,2} * \cdots *_{\chi_{m-1}} f_{p,m} *_{\chi_m} g_{p,1} *_{\chi_{m+1}} g_{p,2} * \cdots *_{\chi_{m+q-1}} g_{p,q}$ belongs to the class $G^*_{p,2m+q-1}(a,b,\sigma) \subset M_p(a,b,\sigma)$.

Proof To simplify the notation, we denote by

$$\mathcal{H}_p = f_{p,1} *_{\chi_1} f_{p,2} * \cdots *_{\chi_{m-1}} f_{p,m} *_{\chi_m} g_{p,1} *_{\chi_{m+1}} g_{p,2} * \cdots *_{\chi_{m+q-1}} g_{p,q},$$

the quasi-Hadamard product of the functions $f_{p,1}, f_{p,2}, \cdots, f_{p,m}, g_{p,1}, \cdots, g_{p,q}$.

Clearly,

$$\mathcal{H}_p = \left\{ \prod_{i=1}^{m} a_{p,i} \prod_{j=1}^{q} b_{p,j} \right\} z^p - \Big(\prod_{i=1}^{m+q-1} \chi_i \Big) \sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{m} a_{n+p,i} \prod_{j=1}^{q} b_{n+p,j} \right\} z^{n+p}. \qquad (2.1)$$

To prove the $\mathcal{H}_p \in G^*_{p,2m+q-1}$, we need to show that

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[(\frac{n+p}{p})^{2m+q-1}\{n(1+b\sigma)+(b-a)p\sigma\}\left\{\prod_{i=1}^{m+q-1}\chi_i\prod_{i=1}^{m}a_{n+p,i}\prod_{j=1}^{q}b_{n+p,j}\right\}\right]\\ \leqslant\ &(b-a)p\sigma\left\{\prod_{i=1}^{m}a_{p,i}\prod_{j=1}^{q}b_{p,j}\right\}.\end{aligned}$$

As $f_{p,i}(z) \in M_p(a,b,\sigma)$, then for every $i=1,2,\cdots,m$, we have

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)\{(1+b\sigma)n+(b-a)p\sigma\}a_{n+p,i} \leqslant (b-a)p\sigma a_{p,i}. \tag{2.2}$$

Therefore, the condition $a_{n+p,i} \geqslant 0$ can make sure that

$$\left(\frac{n+p}{p}\right)\{(1+b\sigma)n+(b-a)p\sigma\}a_{n+p,i} \leqslant (b-a)p\sigma a_{p,i}, i=1,2,\cdots,m \tag{2.3}$$

or

$$a_{n+p,i} \leqslant \frac{(b-a)p\sigma}{\frac{n+p}{p}[(1+b\sigma)n+(b-a)p\sigma]}a_{p,i} \tag{2.4}$$

for every $i=1,2,\cdots,m$. Also, since $-1\leqslant a<b\leqslant 1, 0<\sigma\leqslant 1$, it implies

$$\frac{(b-a)p\sigma}{(1+b\sigma)n+(b-a)p\sigma} \leqslant \left(\frac{n+p}{p}\right)^{-1}, \tag{2.5}$$

so the right side of the inequality (2.4) is not greater than $(\frac{n+p}{n})^{-2}a_{p,i}$, and we obtain

$$a_{n+p,i} \leqslant \left(\frac{n+p}{n}\right)^{-2}a_{p,i} \tag{2.6}$$

for $i=1,2,\cdots,m$. Similarly, for $g_{p,j}(z)\in G^*_p(a,b,\sigma)$, from Lemma 1.1 we have

$$\sum_{n=1}^{\infty}\{(1+b\sigma)n+(b-a)p\sigma\}b_{n+p,j} \leqslant (b-a)p\sigma b_{p,j} \tag{2.7}$$

for every $j=1,2,\cdots,q$. Furthermore, we can obtain

$$b_{n+p,j} \leqslant \left(\frac{n+p}{p}\right)^{-1}b_{p,j} \tag{2.8}$$

for every $j=1,2,\cdots,q$.

Using (2.6) for $i=1,2,\cdots,m$, (2.8) for $j=1,2,\cdots,q-1$, (2.7) for $j=q$ and following $\prod_{i=1}^{m+q-1}\chi_i=1$ or for any i, $0<\chi_i\leqslant 1$, we have

$$\sum_{n=1}^{\infty}\left[(\frac{n+p}{p})^{2m+q-1}\{n(1+b\sigma)+(b-a)p\sigma\}\left\{\prod_{i=1}^{m+q-1}\chi_i\prod_{i=1}^{m}a_{n+p,i}\prod_{j=1}^{q}b_{n+p,j}\right\}\right]$$

$$\leqslant \sum_{n=1}^{\infty}\left[(\frac{n+p}{p})^{2m+q-1}\{n(1+b\sigma)+(b-a)p\sigma\}b_{n+p,q}\left\{(\frac{n+p}{p})^{-2m}(\frac{n+p}{p})^{-(q-1)}\prod_{i=1}^{m}a_{p,i}\prod_{j=1}^{q-1}b_{p,j}\right\}\right]$$

$$\begin{aligned} &= \sum_{n=1}^{\infty}\{n(1+b\sigma)+(b-a)p\sigma\}b_{n+p,q}\left\{\prod_{i=1}^{m}a_{p,i}\prod_{j=1}^{q-1}b_{p,j}\right\} \leqslant (b-a)p\sigma b_{p,q}\left\{\prod_{i=1}^{m}a_{p,i}\prod_{j=1}^{q-1}b_{p,j}\right\} \\ &= (b-a)p\sigma\left\{\prod_{i=1}^{m}a_{p,i}\prod_{j=1}^{q}b_{p,j}\right\}, \end{aligned}$$

and therefore $\mathcal{H}_p \in G^*_{p,2m+q-1}(a,b,\sigma)$.

Furthermore, since $G^*_{p,2m+q-1}(a,b,\sigma) \subset G^*_{p,2m+q-2}(a,b,\sigma) \subset \cdots \subset G^*_{p,1}(a,b,\sigma) \equiv M_p(a,b,\sigma)$, which complete the proof of Theorem 2.1.

As $G^*_{p,2m-1}(a,b,\sigma) \subset G^*_{p,2m-2}(a,b,\sigma) \subset \cdots \subset G^*_{p,1}(a,b,\sigma) \equiv M_p(a,b,\sigma)$, we can obtain the following Corollary 2.1 by setting $q=0$ in Theorem 2.1.

Corollary 2.1 Let the functions $f_{p,i}$ defined by (1.1) be in the class $M_p(a,b,\sigma)$ for every $i=1,2,3,\cdots,m$; $m\in N^*$. If $\prod\limits_{i=1}^{m-1}\chi_i=1$ or for any i, $0<\chi_i\leqslant 1$, then the quasi-Hadamard product $f_{p,1} *_{\chi_1} f_{p,2} * \cdots *_{\chi_{m-1}} f_{p,m}$ belongs to the class $G^*_{p,2m-1}(a,b,\sigma) \subset M_p(a,b,\sigma)$.

As $G^*_{p,q-1}(a,b,\sigma) \subset G^*_{p,q-2}(a,b,\sigma) \subset \cdots \subset G^*_{p,1}(a,b,\sigma) \equiv M_p(a,b,\sigma) \subset G^*_p(a,b,\sigma)$, we can obtain the following Corollary 2.2 by setting $m=0$ in Theorem 2.1.

Corollary 2.2 Let the functions $g_{p,j}$ defined by (1.1) be in the class $G^*_p(a,b,\sigma)$ for every $j=1,2,\cdots,q, q\in N^*$. If $\prod\limits_{j=1}^{q-1}\chi_j=1$ or for any j, $0<\chi_j\leqslant 1$, then the quasi-Hadamard product $g_{p,1} *_{\chi_1} g_{p,2} * \cdots *_{\chi_{q-1}} g_{p,q}$ belongs to the class $G^*_{p,q-1}(a,b,\sigma) \subset M_p(a,b,\sigma)$.

Remark 2.1 (I) Putting $p=1, a=-1, b=\alpha, \sigma=\beta, \chi_i=1 (i=1,2,\cdots,m+q-1)$ in Theorem 2.1 , we obtain the Aouf [3, Theorem 1] and Owa [2, Theorem 8].

(II) Putting $p=1, a=-1, b=\alpha, \sigma=\beta, \chi_i=1 (i=1,2,\cdots,m-1)$ in Corollary 2.1 , we obtain the Aouf [3, Corollary 1] and Owa [2, Theorem 7].

(III) Putting $p=1, a=-1, b=\alpha, \sigma=\beta, \chi_i=1 (i=1,2,\cdots,q-1)$ in Corollary 2.2 , we obtain the Aouf [3, Corollary 2] and Owa [2, Theorem 6].

(IV) Obviously, $J^*_p(a,b,\sigma) \subset G^*_p(a,b,\sigma)$ and $C_p(a,b,\sigma) \subset M_p(a,b,\sigma)$, so the corresponding results in Theorem 2.1, Corollaries 2.1, 2.2 with the classes $J^*_p(a,b,\sigma)$ and $C_p(a,b,\sigma)$ defined by Raina, Nahar [1] are all right.

Theorem 2.2 The class $G^*_{p,c}(a,b,\sigma)$ is compact subset of $\mathcal{A}$.

Proof Montel's theorem implies that the $G^*_{p,c}(a,b,\sigma)$ contained in $\mathcal{A}$ is compact if and only if $G^*_{p,c}(a,b,\sigma)$ is closed and locally uniformly bounded (see [4, p.39]). We first assume

$$f_p(z)=a_pz^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\in G^*_{p,c}(a,b,\sigma),$$

then (1.6) gives that

$$a_{n+p}\leqslant\frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]},\quad n=1,2,\cdots.$$

Since $|z|=r<1$, it follows

$$|f_p(z)| \leqslant a_p|z|^p+\sum_{n=1}^{\infty} a_{n+p}|z|^{n+p} \leqslant a_p r^p+\frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}\frac{r^{1+p}}{1-r},$$

which implies that $G^*_{p,c}(a,b,\sigma)$ is locally uniformly bounded.

It remains to show that $G^*_{p,c}(a,b,\sigma)$ is sequentially closed. Suppose that a sequence $\{f_p^{(k)}(z)\}$ in $G^*_{p,c}(a,b,\sigma)$ and $\{f_p^{(k)}(z)\}\to f_p(k\to\infty)$, where

$$f_p^{(k)}(z)=a_pz^p-\sum_{n=1}^{\infty} a_{n+p}^{(k)}z^{n+p}.$$

Weierstrass' theorem asserts that $f_p\in\mathcal{A}$ (see [4, p.38]), so we can take

$$f_p=a_pz^p-\sum_{n=1}^{\infty} a_{n+p}z^{n+p},$$

moreover, $a_{n+p}^{(k)}\to a_{n+p}(k\to\infty)$. We next need to consider the $f_p\in G^*_{p,c}(a,b,\sigma)$. Since $f_p^{(k)}(z)\in G^*_{p,c}(a,b,\sigma)$, (1.6) implies that

$$\sum_{n=1}^{M}\frac{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}{(b-a)p\sigma a_p}a_{n+p}^{(k)} \leqslant \sum_{n=1}^{\infty}\frac{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}{(b-a)p\sigma a_p}a_{n+p}^{(k)} \leqslant 1$$

for any $M\in Z^+$. Thus, as $k\to\infty$, we have

$$\sum_{n=1}^{M}\frac{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}{(b-a)p\sigma a_p}a_{n+p} \leqslant 1.$$

Furthermore, taking $M\to+\infty$, it gives that

$$\sum_{n=1}^{\infty}\frac{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}{(b-a)p\sigma a_p}a_{n+p} \leqslant 1.$$

This completes the proof of Theorem 2.2.

Theorem 2.3 The extreme points of the class $G^*_{p,c}(a,b,\sigma)$ are given by

$$\begin{aligned} &EG^*_{p,c}(a,b,\sigma)\\ =&\left\{a_pz^p, a_pz^p-\frac{(b-a)p\sigma a_p}{(\frac{1+p}{p})^c[(1+b\sigma)+(b-a)p\sigma]}z^{1+p}, a_pz^p-\frac{(b-a)p\sigma a_p}{(\frac{2+p}{p})^c[2(1+b\sigma)+(b-a)p\sigma]}z^{2+p},\right.\\ &\left.\cdots, a_pz^p-\frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}z^{n+p},\cdots\right\},\end{aligned}$$

where $-1\leqslant a<b\leqslant 1, 0<\sigma\leqslant 1, n\in N^*$.

Proof Using similar arguments as given by Xiong et al. [13, Theorem 2.6], we can easily obtain the extreme points on $G^*_{p,c}(a,b,\sigma)$.

Theorem 2.4 The support points of the class $G^*_{p,c}(a,b,\sigma)$ are given by

$$
\begin{aligned}
&\mathrm{Supp}G^*_{p,c}(a,b,\sigma)\\
=&\left\{f_p(z)\in G^*_{p,c}(a,b,\sigma): f_p(z)=a_pz^p-\sum_{n=1}^{\infty}\frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}\phi_{n+p}z^{n+p}\right\},
\end{aligned}
$$

where $-1\leqslant a<b\leqslant 1, 0<\sigma\leqslant 1, \phi_{n+p}\geqslant 0, \sum\limits_{n=1}^{\infty}\phi_{n+p}\leqslant 1, n\in N^*$ and $\phi_{n+p}=0$ for some $n\geqslant 1$.

Proof First, let a function

$$f_{p,0}(z)=a_pz^p-\sum_{n=1}^{\infty}\frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}\phi_{n+p}z^{n+p},$$

where $\sum\limits_{n=1}^{\infty}\phi_{n+p}\leqslant 1, \phi_{n+p}\geqslant 0, \phi_i=0$ for some $i\geqslant 1+p$. In fact, (1.7) implies that $f_{p,0}(z)\in G^*_{p,c}(a,b,\sigma)$. Now, we need to take

$$b_{n+p}=\begin{cases}0, & n\geqslant 1, n+p\neq i,\\ 1, & n\geqslant 1, n+p=i.\end{cases}$$

Obviously, we have $\overline{\lim\limits_{n\to\infty}}(|b_{n+p}|)^{\frac{1}{n+p}}<1$. Furthermore, we define a functional J on $\mathcal{A}$ by

$$J(f_p(z))=\sum_{n=0}^{\infty}(-a_{n+p})b_{n+p}, f_p(z)=a_pz^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\subset\mathcal{A},\ g_p(z)=b_pz^p-\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\subset\mathcal{A}.$$

It is clearly that the J is a continuous linear functional on $\mathcal{A}$ (see [4, p.42]). Moreover, we note that $J(f_{p,0}(z))=-a_pb_p-\frac{(b-a)p\sigma a_p}{(\frac{i}{p})^c[(1+b\sigma)n+(b-a)p\sigma]}\phi_ib_i=-a_pb_p-0=-a_pb_p$. However, for any function

$$f_p(z)=a_pz^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\in G^*_{p,c}(a,b,\sigma), \tag{2.9}$$

we can note that

$$J(f_p(z))=-a_pb_p-a_ib_i\leqslant -a_pb_p\ (i\geqslant p+1).$$

So we have

$$\mathcal{R}eJ(f_{p,0})=\max\{\mathcal{R}eJ(f_p(z)): f_p(z)\in G^*_{p,c}(a,b,\sigma)\}$$

and $\mathcal{R}eJ(f_p(z))$ are not constant on $G^*_{p,c}(a,b,\sigma)$. Hence $f_{p,0}$ is a support point of $G^*_{p,c}(a,b,\sigma)$.

Conversely, suppose that $f_{p,0}(z)$ is a support point of $G^*_{p,c}(a,b,\sigma)$, and J is a continuous linear functional on $\mathcal{A}$. Note that $\mathcal{R}eJ$ is also a continuous linear and is non-constant on $G^*_{p,c}(a,b,\sigma)$, consequently, we have

$$\mathcal{R}eJ(f_{p,0})=\max\{\mathcal{R}eJ(f_p(z)): f_p(z)\in G^*_{p,c}(a,b,\sigma)\}.$$

Let

$$\mathcal{M} = \mathcal{R}eJ(f_{p,0})$$

and

$$\mathcal{G}_J = \{f_p(z) \in G^*_{p,c}(a,b,\sigma) : \mathcal{R}eJ(f_p(z)) = \mathcal{M}\}.$$

On the one hand, suppose that

$$\mathcal{R}eJ(f_{p,1}) = \mathcal{R}eJ(f_{p,2}) = \mathcal{M},$$

where $f_{p,1} \in G_J, f_{p,2} \in G_J, 0 < t < 1$. Then

$$\mathcal{R}eJ[tf_{p,1} + (1-t)f_{p,2}] = t\mathcal{R}eJ(f_{p,1}) + (1-t)\mathcal{R}eJ(f_{p,2}) = t\mathcal{M} + (1-t)\mathcal{M} = \mathcal{M}$$

and so $tf_{p,1} + (1-t)f_{p,2} \in \mathcal{G}_J$, which gives the convexity of $\mathcal{G}_J$.

On the other hand, suppose that $\mathcal{R}eJ(f_p^{(k)}(z)) = \mathcal{M}$ and $f_p^{(k)}(z) \to f_p(z)$, where $f_p^{(k)}(z) \in \mathcal{G}_J$. Then $\mathcal{R}eJ(f_p^{(k)}(z)) \to \mathcal{R}eJ(f_p(z))$ and so $\mathcal{R}eJ(f_p(z)) = \mathcal{M}$, which implies that the $\mathcal{G}_J$ is closed. Furthermore, Theorem 2.2 makes sure that the class $\mathcal{G}_J \subset G^*_{p,c}(a,b,\sigma)$ is locally uniformly bounded. Therefore, the class $\mathcal{G}_J$ is a convex compact subset of $G^*_{p,c}(a,b,\sigma)$. Thus, E$\mathcal{G}_J$ is not empty (see [Lemma 1.3]). Now, suppose that $g_{p,0}(z) \in$E$\mathcal{G}_J$ and $g_{p,0}(z) = tg_{p,1}(z) + (1-t)g_{p,2}(z)$, where $0 < t < 1, g_{p,1}(z) \in G^*_{p,c}(a,b,\sigma), g_{p,2}(z) \in G^*_{p,c}(a,b,\sigma)$. Then since

$$\mathcal{R}eJ(g_{p,1}) \leqslant \mathcal{M},\ \mathcal{R}eJ(g_{p,2}) \leqslant \mathcal{M},\ t\mathcal{R}eJ(g_{p,1}) + (1-t)\mathcal{R}eJ(g_{p,2}) = \mathcal{R}eJ(g_{p,0}) = \mathcal{M},$$

it follows that

$$\mathcal{R}eJ(g_{p,1}) = \mathcal{R}eJ(g_{p,2}) = \mathcal{M},$$

which implies $g_{p,1} \in \mathcal{G}_J, g_{p,2} \in \mathcal{G}_J$. Again, because $g_{p,0} \in$E$\mathcal{G}_J$, so $g_{p,1} = g_{p,2} = g_{p,0}$. Thus $g_{p,0} \in$E$G^*_{p,c}(a,b,\sigma)$. This shows that E$\mathcal{G}_J \subset$E$G^*_{p,c}(a,b,\sigma)$. Suppose

$$E\mathcal{G}_J - \{a_pz^p\} = \left\{a_pz^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n + (b-a)p\sigma]}z^{n+p} : n \in Z_1\right\},$$

where Z_1 is a subset of $Z_0 = \{1, 2, \cdots\}$. We assert that Z_1 is a proper subset of Z_0. In fact, if it is not the case, then

$$E\mathcal{G}_J - \{a_pz^p\} = \left\{a_pz^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n + (b-a)p\sigma]}z^{n+p} : n \in Z_0\right\}.$$

Since $E\mathcal{G}_J \subset \mathcal{G}_J$, it follows that

$$\mathcal{R}eJ(a_pz^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n + (b-a)p\sigma]}z^{n+p}) = \mathcal{M} \tag{2.10}$$

for all $n \in Z_0$. Hence,

$$\mathcal{R}eJ(a_pz^p) - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n + (b-a)p\sigma]}\mathcal{R}eJ(z^{n+p}) = \mathcal{M} \tag{2.11}$$

for all $n \in Z_0$. Let $n \to +\infty$. Since $z^{n+p} \to 0$ in the metric of $\mathcal{A}$ and J is a continuous linear functional on $\mathcal{A}$, it follows that $\mathcal{R}eJ(z^{n+p}) \to 0$. Thus, By (2.10) and (2.11) we have $\mathcal{R}eJ(a_p z^p) = \mathcal{M}$ and we also find that $\mathcal{R}eJ(z^{n+p}) = 0$ for all $n \in Z_0$. Furthermore, for any $f(z) = a_p z^p - \sum\limits_{n=1}^{\infty} a_{n+p} z^{n+p} \in G^*_{p,c}(a,b,\sigma)$, since J is continuous on $\mathcal{A}$ and $\mathcal{R}eJ(z^{n+p}) = 0$ for $n \in Z_0$, it follows that

$$\mathcal{R}eJ(f_p) = \mathcal{R}eJ(a_p z^p) - \sum_{n=1}^{\infty} a_{n+p} \mathcal{R}eJ(z^{n+p}) = \mathcal{R}eJ(a_p z^p) = \mathcal{M},$$

which contradicts the fact that $\mathcal{R}eJ$ is not constant on $G^*_{p,c}(a,b,\sigma)$. This shows that there is an integer $i (i \geqslant 1)$ not belonging to Z_1. In other words,

$$a_p z^p - \frac{(b-a)p\sigma a_p}{(\frac{i+p}{p})^c[(1+b\sigma)i + (b-a)p\sigma]} z^{i+p}$$

is not belonging to $E\mathcal{G}_J$. Because $\mathcal{G}_J$ is a convex compact set, so $\mathcal{G}_J = HE\mathcal{G}_J$ (see [Lemma 1.3]). Following Theorem 2.3, since $f_{p,0}(z) \in \mathcal{G}_J$, it gives that

$$f_{p,0}(z) = \phi_1 a_p z^p + \sum_{n=1}^{\infty} \phi_{n+p} f_{n+p}(z), \tag{2.12}$$

where $\phi_1 \geqslant 0, \phi_{n+p} \geqslant 0$ and $\phi_1 + \sum\limits_{n=1}^{\infty} \phi_{n+p} = 1, f_{n+p}(z) \in E\mathcal{G}_J$.

Because

$$a_p z^p - \frac{(b-a)p\sigma a_p}{(\frac{i+p}{p})^c[(1+b\sigma)i + (b-a)p\sigma]} z^{i+p}$$

is not belonging to $E\mathcal{G}_J$. So

$$\begin{aligned} f_{p,0}(z) \qquad &= \phi_1 a_p z^p - \sum_{n=1, n\neq i}^{\infty} \phi_{n+p} \Big[a_p z^p - \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n + (b-a)p\sigma]} z^{n+p} \Big] \\ &= a_p z^p - \sum_{n=1, n\neq i}^{\infty} \phi_{n+p} \frac{(b-a)p\sigma a_p}{(\frac{n+p}{p})^c[(1+b\sigma)n + (b-a)p\sigma]} z^{n+p}. \end{aligned}$$

We can obtain the following Corollary 2.3 and Corollary 2.4 by setting $c = 0$ and $c = 1$ in Theorem 2.4, respectively.

Corollary 2.3 The support points of the class $G^*_p(a,b,\sigma)$ are given by

$$\begin{aligned} &\mathrm{Supp} G^*_p(a,b,\sigma) \\ = \quad &\left\{ f(z) \in G^*_p(a,b,\sigma) : f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(b-a)p\sigma a_p}{(1+b\sigma)n + (b-a)p\sigma} \phi_{n+p} z^{n+p} \right\}, \end{aligned}$$

where $-1 \leqslant a < b \leqslant 1, 0 < \sigma \leqslant 1, \phi_{n+p} \geqslant 0, \sum\limits_{n=1}^{\infty} \phi_{n+p} \leqslant 1, n \in N^*$ and $\phi_{n+p} = 0$ for some $n \geqslant 1$.

Corollary 2.4 The support points of the class $M_p(a,b,\sigma)$ are given by

$$\begin{aligned}&\mathrm{Supp}M_p(a,b,\sigma)\\ =\ &\left\{f(z)\in M_p(a,b,\sigma): f(z)=a_pz^p-\sum_{n=1}^{\infty}\frac{(b-a)p^2\sigma a_p}{(n+p)[(1+b\sigma)n+(b-a)p\sigma]}\phi_{n+p}z^{n+p}\right\},\end{aligned}$$

where $-1\leqslant a<b\leqslant 1, 0<\sigma\leqslant 1, \phi_{n+p}\geqslant 0, \sum\limits_{n=1}^{\infty}\phi_{n+p}\leqslant 1, n\in N^*$ and $\phi_{n+p}=0$ for some $n\geqslant 1$.

Remark 2.2 (I) Putting $p=1, a=-1, b=\alpha, \sigma=\beta$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $S_c(\alpha,\beta)$ defined by Aouf [3].

(II) Putting $a_p\equiv 1, c=0$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $J_p^*(a,b,\sigma)$ defined by Raina, Nahar [1].

(III) Putting $a_p\equiv 1, c=1$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $C_p(a,b,\sigma)$ defined by Raina, Nahar [1].

(IV) Putting $p=1, a=-1, b=\alpha, \sigma=\beta, c=0$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $S_0(\alpha,\beta)$ defined by Owa [2].

(VI) Putting $p=1, a=-1, b=\alpha, \sigma=\beta, c=1$ in Theorem 2.3 and Theorem 2.4, respectively, we obtain the extreme points and support points for class $C_0(\alpha,\beta)$ defined by Owa [2].

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多叶解析函数族子类的一些结果

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摘要: 本文研究了在单位开圆盘$U=\{z:|z|<1\}$内多叶解析的函数族$G^*_{p,c}(a,b,\sigma)$的性质. 利用函数论的方法, 获得了$G^*_{p,c}(a,b,\sigma)$族相关的准哈达玛乘积的一般化结果及$G^*_{p,c}(a,b,\sigma)$的极值点与支撑点. 推广了先前相应的一些研究工作.

关键词: 解析函数; 多叶函数; 准哈达玛乘积; 极值点; 支撑点

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