DIMENSIONS RELATIVE TO A COTORSION PAIR

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Abstract: In this paper, let R be a ring and $(\mathcal{F}, \mathcal{C})$ a cotorsion pair of right R-modules. $\mathcal{F}$-dimension ($\mathcal{C}$-dimension) of right R-modules and the global $\mathcal{F}$-dimension ($\mathcal{C}$-dimension) of R are introduced which unifies some known dimensions of modules and rings. By using homological methods, some new characterizations of the flat dimension of modules are given. In addition, von Neumann regular rings and perfect rings are characterized from new aspects.

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1 Introduction

Cotorsion theory, which was introduced by L. Salce in [1], plays an important role in homological algebra and tilting theory. For instance, it was used to settle the "flat cover conjecture" by Bican, Bashir and Enochs in [2]. Moreover, Trlifaj [3] related cotorsion theory to (co)tilting modules which came from the representation theory of algebras. A cotorsion theory (see definition below) is also called a cotorsion pair by many authors nowadays. We adopt the later terminology here. In the last decade, many cotorsion pairs were investigated in the literature (see e.g. [4–8]). Some dimensions relative to certain specific contorsion pairs were also introduced respectively.

In this paper, we introduce and study $\mathcal{F}$-dimension ($\mathcal{C}$-dimension) of modules and the global $\mathcal{F}$-dimension ($\mathcal{C}$-dimension) of rings with respect to any (complete) hereditary cotorsion pair $(\mathcal{F}, \mathcal{C})$ so that some known dimensions of modules and rings can be contained in this unified framework. For instance, the classical flat dimension of modules can now be characterized by the functor Ext. In addition, some new characterizations of von Neumann regular rings and perfect rings are given.

Throughout this paper, R will denote an associative ring with identity and all modules are unitary right R-modules. Recall that a pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a cotorsion theory or cotorsion pair if $\mathcal{F}^{\perp} = \mathcal{C}$ and $^{\perp}\mathcal{C} = \mathcal{F}$, where

$$\mathcal{F}^{\perp} = \{M \mid \operatorname{Ext}_R^1(F, M) = 0, \forall F \in \mathcal{F}\}$$

and

$$^{\perp}\mathcal{C} = \{M \mid \mathrm{Ext}_R^1(M, C) = 0, \forall C \in \mathcal{C}\}.$$

In [9], a cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called hereditary provided it satisfies the following equivalent conditions:

(1) If $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$, then we have $F' \in \mathcal{F}$.

(2) If $0 \to C'' \to C \to C' \to 0$ is exact with $C'', C \in \mathcal{C}$, then we have $C' \in \mathcal{C}$.

(3) $\mathrm{Ext}_R^i(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$ and $i \geq 1$.

Given a class $\mathcal{C}$ of modules, a homomorphism $\phi : M \to C$ with $C \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of M if the induced map $\mathrm{Hom}(\phi, C') \colon \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for all $C' \in \mathcal{C}$. If, in addition, $f \circ \phi = \phi$ implies $f : C \to C$ is an automorphism of C, then $\phi : M \to C$ is called a $\mathcal{C}$-envelope of M. $\mathcal{C}$-(pre)cover is defined dually. We refer the reader to [10] for more details.

According to [11], by a $\mathcal{C}$-envelope with the unique mapping property we mean a $\mathcal{C}$-envelope $\phi : M \to C$ such that $\mathrm{Hom}(\phi, C') \colon \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is injective. A similar property is defined for $\mathcal{C}$-covers.

Following [7], a monomorphism $\lambda : M \to C$ with $C \in \mathcal{C}$ is called a special $\mathcal{C}$-preenvelope of M if $\mathrm{Coker}\lambda \in {}^{\perp}\mathcal{C}$. Special $\mathcal{C}$-precover is defined dually. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called complete [7] provided that every module has a special $\mathcal{F}$-precover or, equivalently, every module has a special $\mathcal{C}$-preenvelope (see [10]).

Let us list some known cotorsion pairs as follows:

(1) $(\mathcal{P}, \mathcal{M})$ is a complete hereditary cotorsion pair, where $\mathcal{P}$ is the class of all projective modules and $\mathcal{M}$ is the class of all modules.

(2) $(\mathcal{M}, \mathcal{I})$ is a complete hereditary cotorsion pair, where $\mathcal{I}$ is the class of all injective modules.

(3) $(\mathcal{F}lat, \mathcal{C}ot)$ is a complete hereditary cotorsion pair, where $\mathcal{F}lat$ $(\mathcal{C}ot)$ is the class of all flat (cotorsion) modules.

(4) $(\mathcal{PF}, \mathcal{PC})$ is a complete cotorsion pair, where $\mathcal{PF}$ $(\mathcal{PC})$ is the class of all P-flat (P-cotorsion) modules (see [12, Theorem 2.3]). It is hereditary in case the base ring R is left generalized morphic (see [4, Proposition 2.6] or [8, Proposition 2.15(2)]).

(5) $(\mathcal{PP}, \mathcal{PI})$ is a complete cotorsion pair, where $\mathcal{PP}$ $(\mathcal{PI})$ is the class of all P-projective (P-injective) modules (see [7, Theorem 3.4]). It is hereditary if R is right generalized morphic (see [8, Proposition 2.15(1)]).

(6) $(\mathcal{FP}\text{-proj}, \mathcal{FP}\text{-inj})$ is a hereditary cotorsion pair, where $\mathcal{FP}$-proj $(\mathcal{FP}$-inj) is the class of all $\mathcal{FP}$-projective $(\mathcal{FP}$-injective) modules over a right FC ring.

(7) Every tilting (cotilting) cotorsion pair is complete and hereditary (see [3, Lemma 2.7(1), Lemma 3.9(1)]).

Now, let us present our definition.

Definition 1.1 Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair of right R-modules and M a right R-module. If there is a smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(M, C) = 0$ for all $C \in \mathcal{C}$,

we say that the $\mathcal{F}$-dimension of M is n and write $\mathcal{F}$-$\dim(M) = n$. If no such n exists, set $\mathcal{F}$-$\dim(M) = \infty$.

Dually, if there is a smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for all $F \in \mathcal{F}$, we call n the $\mathcal{C}$-dimension of M and denote it by $\mathcal{C}$-$\dim(M)$. If no such n exists, set $\mathcal{C}$-$\dim(M) = \infty$.

The right $\mathcal{F}$-dimension and right $\mathcal{C}$-dimension of R are

$$\mathrm{r}.\mathcal{F}.\mathrm{D}(R) = \sup\{\mathcal{F}\text{-}\dim(M) \mid M \text{ is a right } R\text{-module}\}$$

and

$$\mathrm{r}.\mathcal{C}.\mathrm{D}(R) = \sup\{\mathcal{C}\text{-}\dim(M) \mid M \text{ is a right } R\text{-module}\},$$

respectively.

2 $\mathcal{F}$-Dimension and $\mathcal{C}$-Dimension of Modules

Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair. In this section, we mainly discuss properties of $\mathcal{F}$-dimension and $\mathcal{C}$-dimension of modules. Our main results of this section are Theorem 2.1 and Theorem 2.3.

Theorem 2.1 Let R be a ring, $(\mathcal{F}, \mathcal{C})$ a hereditary cotorsion pair of right R-modules and $n \geq 0$. Then the following are equivalent for a right R-module M:

(1) $\mathcal{F}$-$\dim(M) \leq n$.

(2) $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for all $N \in \mathcal{C}$.

(3) $\mathrm{Ext}_R^{n+i}(M, N) = 0$ for all $N \in \mathcal{C}$ and $i \geq 1$.

(4) There exists an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with $F_0, F_1, \cdots, F_n \in \mathcal{F}$.

(5) $F_n \in \mathcal{F}$ whenever there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with $F_0, F_1, \cdots, F_{n-1} \in \mathcal{F}$.

Proof (3)$\Rightarrow$(1) is obvious by the definition of $\mathcal{F}$-$\dim(M)$.

(2)$\Rightarrow$(3) For any $N \in \mathcal{C}$, there exists an exact sequence $0 \to N \to E \to L \to 0$, where E is an injective right R-module. Then we have $L \in \mathcal{C}$ since $(\mathcal{F}, \mathcal{C})$ is hereditary and $N, E \in \mathcal{C}$. Hence $\mathrm{Ext}_R^{n+1}(M, L) = 0$ by (2). Now, in view of the following long exact sequence

$$\mathrm{Ext}_R^{n+1}(M, L) \to \mathrm{Ext}_R^{n+2}(M, N) \to \mathrm{Ext}_R^{n+2}(M, E) \to \cdots$$

$$\to \mathrm{Ext}_R^{n+i-1}(M, L) \to \mathrm{Ext}_R^{n+i}(M, N) \to \mathrm{Ext}_R^{n+i}(M, E) \to \cdots,$$

it is easy to see (3) by induction.

(1)$\Rightarrow$(2) is similar to (2)$\Rightarrow$(3).

(2)$\Rightarrow$(5) Let $0 \to F_n \to F_{n-1} \to F_{n-2} \xrightarrow{d_{n-2}} F_{n-3} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$ be an exact sequence with $F_0, F_1, \cdots, F_{n-1} \in \mathcal{F}$, then

$$\mathrm{Ext}_R^1(F_n, N) \cong \mathrm{Ext}_R^2(\mathrm{Ker}d_{n-2}, N) \cong \cdots \cong \mathrm{Ext}_R^n(\mathrm{Ker}d_0, N) \cong \mathrm{Ext}_R^{n+1}(M, N) = 0$$

for any $N \in \mathcal{C}$. Therefore (5) follows.

(5)$\Rightarrow$(4) It suffices to take a projective resolution of M

$$\cdots \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

and put $F_n = \mathrm{Ker}(P_{n-1} \to P_{n-2})$.

(4)$\Rightarrow$(2) Note that $\mathrm{Ext}_R^{n+1}(M, N) \cong \mathrm{Ext}_R^1(F_n, N) = 0$ for any $N \in \mathcal{C}$.

Let $\mathcal{P}$ be the class of all projective right R-modules and $\mathcal{M}$ the class of all right R-modules. It is well known that $(\mathcal{P}, \mathcal{M})$ is a hereditary cotorsion pair. Thus, one can obtain the characterization of the classical projective dimension of modules by substituting $(\mathcal{P}, \mathcal{M})$ for $(\mathcal{F}, \mathcal{C})$ in Theorem 2.1. Thus [5, Proposition 3.1] is also a special case of Theorem 2.1. Furthermore, the classes of flat and cotorsion right R-modules constitute a hereditary cotorsion pair. Consequently, we have

Corollary 2.2 Let M be a right R-module and $\mathcal{F}$ the class of all flat right R-modules, then the following are equivalent:

(1) $\mathcal{F}$-$\dim(M) \leq n$.

(2) $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for any cotorsion right R-module N.

(3) $\mathrm{Ext}_R^{n+i}(M, N) = 0$ for any cotorsion right R-module N and $i \geq 1$.

(4) There exists an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$, where $F_0, F_1, \cdots, F_n$ are flat right R-modules.

(5) F_n is flat whenever there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_0, F_1, \cdots, F_{n-1}$ are flat.

From Corollary 2.2 one can see that $\mathcal{F}$-$\dim(M)$ coincides with the classical flat dimension of a right R-module M in case $\mathcal{F}$ is the class of all flat right R-modules. Thus Corollary 2.2 provides a new characterization of flat dimension via functor Ext instead of Tor.

The following theorem is dual to Theorem 2.1. The proof is omitted.

Theorem 2.3 Let R be a ring, $(\mathcal{F}, \mathcal{C})$ a hereditary cotorsion pair of right R-modules and $n \geq 0$. Then the following are equivalent for a right R-module M:

(1) $\mathcal{C}$-$\dim (M) \leq n$.

(2) $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for all $N \in \mathcal{F}$.

(3) $\mathrm{Ext}_R^{n+i}(N, M) = 0$ for all $N \in \mathcal{F}$ and $i \geq 1$.

(4) There exists an exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0$ with $C^0, C^1, \cdots, C^n \in \mathcal{C}$.

(5) $C^n \in \mathcal{C}$ whenever there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

with $C^0, C^1, \cdots, C^{n-1} \in \mathcal{C}$.

Let $\mathcal{PF}$ be the class of right R-modules M such that $\mathrm{Tor}_1^R(M, R/Ra) = 0$ for all $a \in R$. Modules in $\mathcal{PF}$ are called torsion-free, $(1, 1)$-flat, or P-flat in the literature(see [8, 13, 14]).

In [4], a right R-module N is called P-cotorsion in case $\operatorname{Ext}_R^1(F, N) = 0$ for all $F \in \mathcal{PF}$. The class of P-cotorsion right R-modules is denoted by $\mathcal{PC}$. It is well known that $(\mathcal{PF}, \mathcal{PC})$ is a (perfect) cotorsion pair [12, Theorem 2.3]. Thus, P-cotorsion dimension defined in [4] coincides with $\mathcal{C}$-dimension in case $\mathcal{C} = \mathcal{PC}$. By [4, Proposition 2.6] or [8, Proposition 2.15(2)], $(\mathcal{PF}, \mathcal{PC})$ is hereditary if R is left generalized morphic [8], i.e., for each $a \in R$, there exists $b \in R$ such that the left annihilator $\mathbf{l}(a) \cong R/Rb$. So, the equivalence of (1) through (5) of [4, Proposition 3.1] is a special case of Theorem 2.3.

Moreover, $(\mathcal{PP}, \mathcal{PI})$ is a (complete) cotorsion pair [7, Theorem 3.4], where $\mathcal{PP}$ ($\mathcal{PI}$) is the class of P-projective (P-injective) right R-modules. By [8, Proposition 2.15(1)], $(\mathcal{PP}, \mathcal{PI})$ is hereditary if R is right generalized morphic. In this case, P-injective dimension defined in [8] coincides with $\mathcal{C}$-dimension when $\mathcal{C} = \mathcal{PI}$ (see [8, Lemma 3.6] which contains a special case of our Theorem 2.3).

The following propositions will be used in the next section.

Proposition 2.4 Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules. Then

(1) $\mathcal{F}\text{-dim}(B) \leq \max\{\mathcal{F}\text{-dim}(A), \mathcal{F}\text{-dim}(C)\}$.

(2) $\mathcal{F}\text{-dim}(A) \leq \max\{\mathcal{F}\text{-dim}(B), \mathcal{F}\text{-dim}(C) - 1\}$.

(3) $\mathcal{F}\text{-dim}(C) \leq \max\{\mathcal{F}\text{-dim}(A) + 1, \mathcal{F}\text{-dim}(B)\}$.

(4) $\mathcal{F}\text{-dim}(A) = \mathcal{F}\text{-dim}(C) - 1$ in case $B \in \mathcal{F}$ and $C \notin \mathcal{F}$.

(5) $\mathcal{C}\text{-dim}(B) \leq \max\{\mathcal{C}\text{-dim}(A), \mathcal{C}\text{-dim}(C)\}$.

(6) $\mathcal{C}\text{-dim}(A) \leq \max\{\mathcal{C}\text{-dim}(B), \mathcal{C}\text{-dim}(C) + 1\}$.

(7) $\mathcal{C}\text{-dim}(C) \leq \max\{\mathcal{C}\text{-dim}(A) - 1, \mathcal{C}\text{-dim}(B)\}$.

(8) $\mathcal{C}\text{-dim}(C) = \mathcal{C}\text{-dim}(A) - 1$ in case $B \in \mathcal{C}$ and $A \notin \mathcal{C}$.

Proof Note that $0 \to A \to B \to C \to 0$ induces a long exact sequence

$$\cdots \to \operatorname{Ext}_R^1(C, N) \to \operatorname{Ext}_R^1(B, N) \to \operatorname{Ext}_R^1(A, N)$$
$$\to \operatorname{Ext}_R^2(C, N) \to \operatorname{Ext}_R^2(B, N) \to \operatorname{Ext}_R^2(A, N) \to \cdots$$

for each $N \in \mathcal{C}$. Then (1) through (4) are followed easily by Theorem 2.1.

(5) Through (8) can be proved dually.

Proposition 2.5 Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair and $\{M_i \mid i \in I\}$ a family of right R-modules, then

(1) $\mathcal{F}\text{-dim}(\coprod_{i \in I} M_i) = \sup\{\mathcal{F}\text{-dim}(M_i)\}$.

(2) $\mathcal{C}\text{-dim}(\prod_{i \in I} M_i) = \sup\{\mathcal{C}\text{-dim}(M_i)\}$.

Proof (1) is an immediate consequence of Theorem 2.1 by virtue of the well known isomorphism $\operatorname{Ext}_R^n(\coprod_{i \in I} M_i, N) \cong \prod_{i \in I} \operatorname{Ext}_R^n(M_i, N)$. (2) is dual to (1).

It is easy to see that $\mathcal{F}\text{-dim}(M) \leq \operatorname{pd}(M)$ and $\mathcal{C}\text{-dim}(M) \leq \operatorname{id}(M)$, where $\operatorname{pd}(M)$ ($\operatorname{id}(M)$) stands for the projective (injective) dimension of M. The next proposition provides a condition under which the equality holds.

Proposition 2.6 Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair of right R-modules and M a right R-module.

(1) If $\mathrm{pd}(M) < \infty$ and every right R-module N is an epimorphic image of some $C \in \mathcal{C}$, then $\mathcal{F}\text{-dim}(M) = \mathrm{pd}(M)$.

(2) If $\mathrm{id}(M) < \infty$ and every right R-module N is isomorphic to a submodule of some $F \in \mathcal{F}$, then $\mathcal{C}\text{-dim}(M) = \mathrm{id}(M)$.

Proof (1) Suppose $\mathrm{pd}(M) = n < \infty$ and $\mathrm{Ext}_R^n(M, N) \neq 0$. By hypothesis, there is an exact sequence $0 \to K \to C \to N \to 0$ with $C \in \mathcal{C}$. Then we have an exact sequence $\mathrm{Ext}_R^n(M, C) \to \mathrm{Ext}_R^n(M, N) \to \mathrm{Ext}_R^{n+1}(M, K) = 0$. Hence $\mathrm{Ext}_R^n(M, C) \neq 0$. By Theorem 2.1, $\mathcal{F}\text{-dim}(M) > n - 1$ and the result follows.

(2) can be proved in a similar way.

Example 2.7 (1) If R is a right FC ring (i.e., R is right $\mathcal{FP}$-injective and right coherent) then $(\mathcal{FP}\text{-proj}, \mathcal{FP}\text{-inj})$ is a hereditary cotorsion pair and $\mathcal{P} \subseteq \mathcal{FP}\text{-inj}$, where $\mathcal{FP}\text{-proj}$, $\mathcal{FP}\text{-inj}$ and $\mathcal{P}$ are the classes of all $\mathcal{FP}$-projective, $\mathcal{FP}$-injective and projective right R-modules, respectively. By Proposition 2.6(1), the $\mathcal{FP}$-projective dimension $\mathrm{fpd}(M)$ of a right R-module (see [5]) coincides with $\mathrm{pd}(M)$ in case $\mathrm{pd}(M) < \infty$.

(2) Suppose that R is a right IF ring (i.e., every injective right R-module is flat) and $\mathrm{id}(M_R) = n < \infty$. By Proposition (2), we have $\mathrm{cd}(M) = \mathrm{id}(M)$, where $\mathrm{cd}(M)$ denotes the cotorsion dimension of M (see [6]).

3 Global $\mathcal{F}$-Dimension and $\mathcal{C}$-Dimension of Rings

In this section, we discuss the global $\mathcal{F}$-dimension and $\mathcal{C}$-dimension of rings, where $(\mathcal{F}, \mathcal{C})$ is a complete hereditary cotorsion pair. We first simplify the calculation of $\mathrm{r}.\mathcal{F}.\mathrm{D}(R)$ and $\mathrm{r}.\mathcal{C}.\mathrm{D}(R)$.

Theorem 3.1 Let $(\mathcal{F}, \mathcal{C})$ be a complete hereditary cotorsion pair of right R-modules. Then

(1) $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) = \sup\{\mathcal{F}\text{-}\dim(M) \mid M \text{ is a finitely generated right } R\text{-module}\}$
$$= \sup\{\mathcal{F}\text{-}\dim(M) \mid M \text{ is a cyclic right } R\text{-module}\}$$
$$= \sup\{\mathrm{id}(C) \mid C \in \mathcal{C}\}$$
$$= \sup\{\mathcal{F}\text{-}\dim(C) \mid C \in \mathcal{C}\}.$$

(2) If $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) < \infty$, then
$$r.\mathcal{F}.D(R) = \sup\{\mathrm{id}(M) \mid M \in \mathcal{F} \cap \mathcal{C}\}$$
$$= \sup\{\mathcal{F}\text{-}\dim(M) \mid M \text{ is an injective right } R\text{-module}\}.$$

(3) $\mathrm{r}.\mathcal{C}.\mathrm{D}(R) = \sup\{\mathrm{pd}(F) \mid F \in \mathcal{F}\} = \sup\{\mathcal{C}\text{-}\dim(F) \mid F \in \mathcal{F}\}.$

(4) If $\mathrm{r}.\mathcal{C}.\mathrm{D}(R) < \infty$, then
$\mathrm{r}.\mathcal{C}.\mathrm{D}(R) = \sup\{\mathrm{pd}(M) \mid M \in \mathcal{F} \cap \mathcal{C}\}$
$$= \sup\{\mathcal{C}\text{-}\dim(M) \mid M \text{ is a projective right } R\text{-module}\}.$$

Proof (1) It is obvious that

$$\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \geq \sup\{\mathcal{F}\text{-dim}(M) \mid M \text{ is a finitely generated right } R\text{-module}\}$$
$$\geq \sup\{\mathcal{F}\text{-dim}(M) \mid M \text{ is a cyclic right } R\text{-module}\}$$

and $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \geq \sup\{\mathcal{F}\text{-}\dim(C) \mid C \in \mathcal{C}\}$.

Now, suppose that $\sup\{\mathcal{F}\text{-}\dim(M) \mid M \text{ is a cyclic right } R\text{-module}\} = n < \infty$. Then we have $\mathrm{Ext}_R^{n+1}(R/I, C) = 0$ for all right ideal I of R and $C \in \mathcal{C}$. Hence $n \geq \sup\{\mathrm{id}(C) \mid C \in \mathcal{C}\}$.

Next, let $\sup\{\mathrm{id}(C) \mid C \in \mathcal{C}\} = n < \infty$. It follows that $\mathrm{Ext}_R^{n+1}(M, C) = 0$ for all right R-module M and $C \in \mathcal{C}$. Thus $\mathcal{F}\text{-}\dim(M) \leq n$ for all right R-module M, i.e., $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \leq n$.

Finally, suppose that $\sup\{\mathcal{F}\text{-}\dim(C) \mid C \in \mathcal{C}\} = n < \infty$. For any right R-module M, we have an exact sequence $0 \to M \to C \to F \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$ since $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair. By Proposition 2.4(2), we have $\mathcal{F}\text{-}\dim M \leq \mathcal{F}\text{-}\dim(C) \leq n$. Therefore $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \leq \sup\{\mathcal{F}\text{-}\dim(C) \mid C \in \mathcal{C}\}$.

(2) It will suffices to show $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \leq \sup\{\mathrm{id}(M) \mid M \in \mathcal{F} \cap \mathcal{C}\}$ and $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \leq \sup\{\mathcal{F}\text{-}\dim(C) \mid M \text{ is an injective right } R\text{-module}\}$.

First, let $\sup\{\mathrm{id}(M) \mid M \in \mathcal{F} \cap \mathcal{C}\} = n < \infty$. Then for any $C \in \mathcal{C}$, we may assume that $\mathcal{F}\text{-}\dim(C) = m < \infty$ since $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) < \infty$. In view of the completeness of $(\mathcal{F}, \mathcal{C})$ and Theorem 2.1, one can construct an exact sequence

$$0 \to F_m \to F_{m-1} \xrightarrow{d_{m-1}} F_{m-2} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} C \to 0$$

with each $F_i \in \mathcal{F} \cap \mathcal{C}$. Thus, $\mathrm{Ext}_R^{n+1}(-, F_i) = \mathrm{Ext}_R^{n+2}(-, F_i) = 0$ for each i by hypothesis. Let $K_i = \mathrm{Ker}d_i$ for $i = 0, 1, \cdots, m-1$. For any right R-module N and each i, we have an exact sequence

$$\mathrm{Ext}_R^{n+1}(N, F_i) \to \mathrm{Ext}_R^{n+1}(N, K_{i-1}) \to \mathrm{Ext}_R^{n+2}(N, K_i)$$

where $\mathrm{Ext}_R^{n+2}(N, K_{m-1}) = \mathrm{Ext}_R^{n+2}(N, F_m) = 0$ and $K_{-1} = C$. It is easy to check in turn that $\mathrm{Ext}_R^{n+1}(-, K_{m-2}) = 0$, $\cdots$, $\mathrm{Ext}_R^{n+1}(-, K_0) = 0$ and $\mathrm{Ext}_R^{n+1}(-, C) = 0$. Hence $\mathrm{id}(C) \leq n$. Consequently, $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) = \sup\{\mathrm{id}(C) \mid C \in \mathcal{C}\} \leq n$. This proves the first desired inequality.

Now suppose that $\sup\{\mathcal{F}\text{-}\dim(C) \mid M \text{ is an injective right } R\text{-module}\} = n < \infty$. For any $C \in \mathcal{C}$, we may assume $\mathrm{id}(C) = m < \infty$ since $\sup\{\mathrm{id}(C) \mid C \in \mathcal{C}\} = \mathrm{r}.\mathcal{F}.\mathrm{D}(R) \leq \infty$. Then there is an injective resolution

$$0 \to C \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0$$

of C. Consequently, the second desired inequality follows by a process similar to the proof of the previous one.

We omit the proof of (3) and (4) to avoid repeating.

By Theorem 2.1, Theorem 2.3 and Theorem 3.1(1) and (3), we have

Corollary 3.2 Let $(\mathcal{F}, \mathcal{C})$ be a complete hereditary cotorsion pair of right R-modules. Then

(1) $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for all $M, N \in \mathcal{C}$ if and only if $\mathrm{Ext}_R^m(M, N) = 0$ for all $M, N \in \mathcal{C}$ and $m > n$.

(2) $\mathrm{r}.\mathcal{C}.\mathrm{D}(R) \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for all $M, N \in \mathcal{F}$ if and only if $\mathrm{Ext}_R^m(M, N) = 0$ for all $M, N \in \mathcal{F}$ and $m > n$.

Whenever a kind of global dimension of rings is studied, it is of special interest to characterize a ring R with such dimension zero. As far as $\mathrm{r}.\mathcal{F}.\mathrm{D}(R)$ and $\mathrm{r}.\mathcal{C}.\mathrm{D}(R)$ are concerned, we have

Theorem 3.3 Let $(\mathcal{F}, \mathcal{C})$ be a complete hereditary cotorsion pair of right R-modules. The following are equivalent.

(1) $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) = 0$.

(2) Every module in $\mathcal{C}$ is injective.

(3) $\mathcal{C} \subseteq \mathcal{F}$.

(4) $\mathrm{Ext}_R^1(M, N) = 0$ for all $M, N \in \mathcal{C}$.

(5) $\mathrm{Ext}_R^n(M, N) = 0$ for all $M, N \in \mathcal{C}$ and $n \geq 1$.

(6) Every module in $\mathcal{C}$ has an injective envelope with the unique mapping property.

(7) Every module in $\mathcal{C}$ has an $\mathcal{F}$-cover with the unique mapping property.

Proof (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) follows from Theorem 3.1 (1).

(1)$\Leftrightarrow$(4)$\Leftrightarrow$(5) follows from Corollary 3.2.

(2)$\Rightarrow$(6) and (3)$\Rightarrow$(7) are clear.

(6)$\Rightarrow$(2) For any $C \in \mathcal{C}$, we have an exact sequence $0 \to C \xrightarrow{\varphi} E_1 \xrightarrow{\psi} E_2$, where E_1 and E_2 are injective and φ is an injective envelope of C with the unique mapping property. Note that $\psi \circ \varphi = 0 = 0 \circ \varphi$. It follows that $\psi = 0$. Hence C is isomorphic to E_1 under φ.

(7)$\Rightarrow$(3) Given $C \in \mathcal{C}$, one can construct by hypothesis an exact sequence $F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} C \to 0$, where $F_1, F_2 \in \mathcal{F}$ and φ is an $\mathcal{F}$-cover of C with the unique mapping property. Then it is easy to see $C \cong F_1 \in \mathcal{F}$.

It is well known that R is von Neumann regular if and only if its weak global dimension $\mathrm{WD}(R) = 0$. But $\mathrm{WD}(R) = \mathrm{r}.\mathcal{F}.\mathrm{D}(R)$ in case $\mathcal{F}$ is the class of flat right R-modules. So we have the following corollary as a special case of Theorem 3.3.

Corollary 3.4 The following are equivalent for a ring R.

(1) R is von Neumann regular.

(2) Every cotorsion right R-module is injective.

(3) Every cotorsion right R-module is flat.

(4) $\mathrm{Ext}_R^1(M, N) = 0$ for all cotorsion right R-modules M and N.

(5) $\mathrm{Ext}_R^n(M, N) = 0$ for all cotorsion right R-modules M, N and $n = 1, 2, \cdots$.

(6) Every cotorsion right R-module has an injective envelope with the unique mapping property.

(7) Every cotorsion right R-module has a flat cover with the unique mapping property.

Dual to Theorem 3.3, we have

Theorem 3.5 Let $(\mathcal{F}, \mathcal{C})$ be a complete hereditary cotorsion pair of right R-modules. The following are equivalent.

(1) $\mathrm{r}.\mathcal{C}.\mathrm{D}(R) = 0$.

(2) Every module in $\mathcal{F}$ is projective.

(3) $\mathcal{F} \subseteq \mathcal{C}$.

(4) $\mathrm{Ext}_R^1(M, N) = 0$ for all $M, N \in \mathcal{F}$.

(5) $\mathrm{Ext}_R^n(M, N) = 0$ for all $M, N \in \mathcal{F}$ and $n \geq 1$.

(6) Every module in $\mathcal{F}$ has a projective cover with the unique mapping property.

(7) Every module in $\mathcal{F}$ has a $\mathcal{C}$-envelope with the unique mapping property.

In Theorem 3.5, if we let $\mathcal{F}$ $(\mathcal{C})$ be the class of flat (cotorsion) right R-modules, then we have the following corollary, where (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4) is well known and (1)$\Leftrightarrow$(8) is also established in [15, Theorem 2.18].

Corollary 3.6 The following are equivalent for a ring R.

(1) R is right perfect.

(2) Every right R-module is cotorsion.

(3) Every flat right R-module is projective.

(4) Every flat right R-module is cotorsion.

(5) $\mathrm{Ext}_R^1(M, N) = 0$ for all flat right R-modules M and N.

(6) $\mathrm{Ext}_R^n(M, N) = 0$ for all flat right R-modules M, N and $n = 1, 2, \cdots$.

(7) Every flat right R-module has a projective cover with the unique mapping property.

(8) Every flat right R-module has a cotorsion envelope with the unique mapping property.

When R is right generalized morphic we may substitute $(\mathcal{PP}, \mathcal{PI})$ for $(\mathcal{F}, \mathcal{C})$ in Theorem 3.5, where $\mathcal{PP}$ $(\mathcal{PI})$ is the class of P-projective (P-injective) right R-modules. Consequently, we have the following corollary in view of the fact that a ring R is von Neumann regular if and only if every right R-module is P-injective.

Corollary 3.7 The following are equivalent for a right generalized morphic ring R.

(1) R is a von Neumann regular ring.

(2) Every P-projective right R-module is projective.

(3) Every P-projective right R-module is P-injective.

(4) $\mathrm{Ext}_R^1(M, N) = 0$ for all P-projective right R-modules M and N.

(5) $\mathrm{Ext}_R^n(M, N) = 0$ for all P-projective right R-modules M, N and $n \geq 1$.

(6) Every P-projective right R-module has a projective cover with the unique mapping property.

(7) Every P-projective right R-module has a P-injective envelope with the unique mapping property.

Finally, we estimate the classical right global dimension $\mathrm{r.D}(R)$ of a ring R with $\mathrm{r}.\mathcal{F}.\mathrm{D}(R)$ and $\mathrm{r}.\mathcal{C}.\mathrm{D}(R)$.

Theorem 3.8 Let $(\mathcal{F}, \mathcal{C})$ be a complete hereditary cotorsion pair of right R-modules. Then $\mathrm{r.D}(R) \leq \mathrm{r}.\mathcal{F}.\mathrm{D}(R) + \mathrm{r}.\mathcal{C}.\mathrm{D}(R)$.

Proof Suppose $\mathrm{r}.\mathcal{F}.\mathrm{D}(R) = m < \infty$ and $\mathrm{r}.\mathcal{C}.\mathrm{D}(R) = n < \infty$. Then for any right R-module M there is an exact sequence

$$0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with each $F_i \in \mathcal{F}$. By Theorem 3.1 (3), we have $\mathrm{pd}(F_i) \leq n$, $i = 0, 1, \cdots, m$. Now, break

the above long exact sequence in the following short exact sequences

$$0 \to F_m \to F_{m-1} \to K_{m-2} \to 0$$
$$0 \to K_{m-1} \to F_{m-2} \to K_{m-3} \to 0$$
$$\cdots$$
$$0 \to K_0 \to F_0 \to M \to 0$$

from which we can obtain

$$\mathrm{pd}(K_{m-2}) \leq \sup\{\mathrm{pd}(F_{m-1}),\ \mathrm{pd}(F_m)+1\} \leq n+1$$

and $\mathrm{pd}(K_{m-3}) \leq n+2, \cdots, \mathrm{pd}(K_0) \leq n+m-1, \mathrm{pd}(M) \leq n+m$. Therefore, the result follows.

Corollary 3.9   (1)   (see [4]) If R is left generalized morphic ring, then

$$\mathrm{r.D}(R) \leq \mathrm{WD}(R) + \mathrm{r.P\text{-}cD}(R),$$

where $\mathrm{r.P\text{-}cD}(R)$ is the right P-cotorsion dimension of R.

(2) (see [15]) $\mathrm{r.D}(R) \leq \mathrm{WD}(R) + \mathrm{r.CD}(R)$, where $\mathrm{r.CD}(R)$ is the right cotorsion dimension of R.

(3) (see [5]) If R is right coherent ring, then $\mathrm{r.D}(R) \leq \mathrm{WD}(R) + \mathrm{r.fpD}(R)$, where $\mathrm{r.fpD}(R)$ is the right $\mathcal{FP}$-projective dimension of R.

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相对于余挠对的维数

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摘要: 本文介绍了右R-模的$\mathcal{F}$-维数($\mathcal{C}$-维数)以及环R上整体$\mathcal{F}$-维数($\mathcal{C}$-维数). 利用同调方法, 给出了平坦模维数的新刻画. 另外, 得到了von Neumann正则环和完全环的新刻画.

关键词: $\mathcal{F}$-维数; $\mathcal{C}$-维数; 余挠对

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