Vol. 34 (2014)
No. 3

数 学 杂 志
J. of Math. (PRC)

SOLUTIONS OF PARAMETRIZED SEXTIC THUE EQUATIONS

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Abstract: In this paper, we study parametric family of sextic Thue equation. By using elementary method and simpler method of approximating certain algebraic numbers, we completely solve the parametric family of sextic Thue equation, which extend the results of Alan Togbé.

Keywords: sextic Thue equation; padè approximations; continued fraction

2010 MR Subject Classification: 11D45

Document code: A **Article ID:** 0255-7797(2014)03-0478-09

1 Introduction

A Thue equation is a Diophantine equation of the form

$$F(x,y)=k,$$

where $F\in Z[x,y]$ is an irreducible binary form of degree $n\geq 3$ and k is a non-zero rational integer. In 1909, Thue [1] proved that the thue equation has only finitely many solutions, however Thue didn't give a complete method. In 1986 Baker [2] used the theory of linear form in logarithms of algebraic numbers to solve the problem, in particular, he gave an effective upper bound for the solutions of Thue equation. In recent years, various families of Thue equations were studied (see [3–14]).

In 2012, Xia, Chen, Zhang [14] presented a new and simpler method to approximate certain algebraic numbers. Applying the method, authors derived an effective upper bound for the solutions (x,y) of the two-parametric family of quartic Thue equation

$$tx^4-4sx^3y-6tx^2y^2+4sxy^3+ty^4=N$$

for $s>32t^3$. The purpose of this research is to extend the method to solve the two-parametric family of sextic Thue equation

$$\begin{aligned}F_n(x,y) &= x^6-2A_nx^5y-5(A_n+3)x^4y^2-20x^3y^3+5A_nx^2y^4+2(A_n+3)xy^5+y^6\\ &= \pm1, \qquad (1.1)\end{aligned}$$

* **Received date:** 2013-09-23 **Accepted date:** 2013-10-14

Foundation item: Supported by National Natural Science Foundation of China (61202305) and Fundamental Research funds for the Central Universities(2013PY120).

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where $A_n = 54n^3 + 81n^2 + 54n + 12,\ n > 6.02 \times 10^6$. In 1990, Alan Togbé proved that equation (1.1) has only trivial solutions, where $|n| \le 2.03 \times 10^6$. We extend the result of Alan Togbé, and consider n is any integer.

In this paper, we will show

Theorem 1 For any $n \in Z$, equation(1.1) has only trivial solutions,

$$(x, y) = (1, 0), (-1, 0), (0, 1), (0, -1), (1, -1), (-1, 1).$$

2 Preliminaries

Before the proof of the theorem, some lemmas are needed.

Lemma 2.1 (see [14]) Let $p_n(x) = \sum_{k=0}^{n} \binom{n-\alpha}{n-k}\binom{n+\alpha}{k}x^k$, $q_n(x) = \sum_{k=0}^{n} \binom{n-\alpha}{k}\binom{n+\alpha}{n-k}x^k$ and $R_n(x) = x^\alpha q_n(x) - p_n(x)$, where n is a positive integer and α is a real number, then we have

(i) $q_n(x) = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{n-k}\binom{n-\alpha}{k}(1-x)^k$;

(ii) Let $x = w = \frac{si-t}{si+t} = e^{i\varphi}, 0 < \varphi < \frac{\pi}{2}, 0 < \alpha < 1$, then $|q_n(w)| \le 4|1 + \sqrt{w}|^{2(n-1)}$;

(iii) Let $x = w = \frac{si-t}{si+t} = e^{i\varphi}, 0 < \varphi < \frac{\pi}{2}, 0 < \alpha < 1$, then $|q_n(w)| \le 4|1 + \sqrt{w}|^{2(n-1)}$.

Lemma 2.2 If $\alpha = 1/6$, then $12^k 3^{\lfloor k/2 \rfloor}\binom{n-\alpha}{k}$ is rational integer.

Proof Let $k! = 2^{s_2}3^{s_3}M$ with $(6, M) = 1$. Since

$$\binom{n-\alpha}{k} = \frac{(6n-1)(6(n-1)-1)\cdots(6(n-k+1)-1)}{6^k k!},$$

then there exists t such that $6t \equiv 1 (\mathrm{mod}\ M)$ and

$$\begin{aligned} & t^k(6n-1)(6(n-1)-1)...(6(n-k+1)-1) \\ \equiv\ & (n-t)\cdots(n-k+1-t) \\ \equiv\ & \binom{n-t}{k}k! \equiv 0\ (\mathrm{mod}\ M). \end{aligned}$$

Since $(t, M) = 1$, we obtain

$$M|(6n-1)\cdots(6(n-k+1)-1).$$

While

$$s_2 = \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{4} \rfloor + \lfloor \frac{k}{8} \rfloor + \cdots < k,$$

and

$$s_3 = \lfloor \frac{k}{3} \rfloor + \lfloor \frac{k}{3^2} \rfloor + \lfloor \frac{k}{3^3} \rfloor + \cdots < \lfloor \frac{k}{2} \rfloor.$$

The lemma therefore follows:

Lemma 2.3 (see [11]) Let θ be an algebraic number. Suppose that there exists $k_0 > 0, l_0, Q > 1, E > 1$ such that for all n there are rational integers P_n and Q_n with $|Q_n| < k_0 Q^n$

and $|Q_n\theta - P_n| \le l_0E^{-n}$ and suppose further that $P_nQ_{n+1} \neq Q_nP_{n+1}$. Then, for any rational integers x and y, $y \ge e/(2l_0)$, we have

$$|\theta y - x| > \frac{1}{cy^\lambda},\ \text{where } c = 2k_0Q(2l_0E)^\lambda,\ \lambda = \frac{\log Q}{\log E}.$$

3 Solutions of Thue Equation

Consider the equation

$$\begin{aligned} f(x,y) &= x^6 - \frac{6A}{B}x^5y + (\frac{15A}{B} - 15)x^4y^2 + 20x^3y^3 - \frac{15A}{B}x^2y^4 + (\frac{6A}{B} - 6)xy^5 + y^6 \\ &= \pm 1, \end{aligned} \tag{3.1}$$

where $A > 0$, $A, B \in Z$. If we set $B = -n, A = -A_nB/3 = 18n^4 + 27n^3 + 18n^2 + 4n > 0$, we have$f(x,y) = F_n(x, -y)$. Hence the solution of (1.1) could be deduced from the solution of (3.1).

3.1 Research of The Root

Denote by θ as the root of $f(x,1) = 0$, straightforward computation shows that θ satisfies

$$(\frac{\theta+\rho}{\theta+\bar\rho})^6 = \frac{A+B\bar\rho}{A+B\rho}, \tag{3.2}$$

where $\rho = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, $\bar\rho = -\frac{1}{2} - \frac{\sqrt{-3}}{2}$, and $-$ mean conjugate in $Z[\sqrt{-3}]$, so that $\overline{a+b\rho} = a + b\bar\rho$. Since$-3 \equiv 1(\mod 4)$, we know that $\{1, \frac{1+\sqrt{-3}}{2}\}$ are integral basis in $Z[\sqrt{-3}]$. Therefore, ρ, $\bar\rho$ are algebraic interger in $Z[\sqrt{-3}]$.

Putting $z = A + B\bar\rho, u = A + B\rho, w = z/u = e^{i\varphi}$, we have

$$\theta_k = \frac{\rho - \bar\rho w^{\frac{1}{6}}e^{\frac{2k\pi}{6}}}{w^{\frac{1}{6}}e^{\frac{2k\pi}{6}} - 1} = \frac{\rho e^{\frac{-k\pi}{6}} - \bar\rho w^{\frac{1}{6}}e^{\frac{k\pi}{6}}}{w^{\frac{1}{6}}e^{\frac{k\pi}{6}} - e^{\frac{-k\pi}{6}}},\ \ k = 0, 1, \cdots, 5.$$

Note that $e^{\frac{-\pi}{6}} = i\bar\rho$, $e^{\frac{\pi}{6}} = -i\rho$, $e^{\frac{-\pi}{3}} = -\rho$, $e^{\frac{\pi}{3}} = -\bar\rho$, one can check that θ_k satisfies $\theta_k = \frac{v_k + \bar{v_k}w^{\frac{1}{6}}}{r_kw^{\frac{1}{6}} + \bar{r_k}}$, where

$$v_k = \begin{cases} \bar\rho - 1, & k = 0; \\ 1, & k = 1; \\ \bar\rho + 2, & k = 2; \\ \rho, & k = 3; \\ 2\rho + 1, & k = 4; \\ \rho + 1, & k = 5, \end{cases} \quad r_k = \begin{cases} 2\rho + 1, & k = 0; \\ -\rho, & k = 1; \\ \bar\rho + 2, & k = 2; \\ -1, & k = 3; \\ \bar\rho - 1, & k = 4; \\ \bar\rho, & k = 5. \end{cases} \tag{3.3}$$

Actually, this property of θ make possible for future proof. Furthermore, one can give explicit estimation for the roots as the lemma below.

Lemma 3.1 Denote by θ_k $(k = 0, 1, \cdots, 5)$ as the root of $f(x, 1) = 0$, then it satisfies

$$
\begin{aligned}
&1+\frac{1}{2A_n}-\frac{5}{8A_n^2}-\frac{5}{16A_n^3}+\frac{9}{10A_n^4}\le\theta_0\le 1+\frac{1}{2A_n}-\frac{5}{8A_n^2}-\frac{5}{16A_n^3}+\frac{1}{A_n^4},\\
&\frac{1}{2A_n}-\frac{7}{8A_n^2}+\frac{7}{16A_n^3}+\frac{89}{32A_n^4}\le\theta_1\le\frac{1}{2A_n}-\frac{7}{8A_n^2}+\frac{7}{16A_n^3}+\frac{179}{64A_n^4},\\
&-1+\frac{3}{2A_n}-\frac{27}{8A_n^2}+\frac{69}{16A_n^3}+\frac{169}{8A_n^4}\le\theta_2\le -1+\frac{3}{2A_n}-\frac{27}{8A_n^2}+\frac{69}{16A_n^3}+\frac{85}{4A_n^4},\\
&\frac{1}{2}+\frac{3}{8A_n}-\frac{9}{16A_n^2}+\frac{3}{128A_n^3}+\frac{155}{64A_n^4}\le\theta_3\le\frac{1}{2}+\frac{3}{8A_n}-\frac{9}{16A_n^2}+\frac{3}{128A_n^3}+\frac{311}{128A_n^4},\\
&-2A_n-\frac{5}{2}-\frac{35}{8A_n}-\frac{105}{16A_n^2}-\frac{65}{32A_n^3}\le\theta_4\le -2A_n-\frac{5}{2}-\frac{35}{8A_n}-\frac{105}{16A_n^2}-\frac{129}{64A_n^3},\\
&2+\frac{3}{2A_n}-\frac{9}{8A_n^2}-\frac{39}{16A_n^3}+\frac{589}{64A_n^4}\le\theta_5\le 2+\frac{3}{2A_n}-\frac{9}{8A_n^2}-\frac{39}{16A_n^3}+\frac{295}{32A_n^4}.
\end{aligned}
$$

Proof One can check that $f(x, 1)$ change the sign between two formula in the inequalities.

3.2 Approximation of Roots

Now we will give an effective approximation of θ. Since

$$
\begin{aligned}
\theta_k &= \frac{v_k+\bar{v_k}w^{\frac{1}{6}}}{r_k w^{\frac{1}{6}}+\bar{r_k}}\\
&= \frac{v_k q_n(w)+\bar{v_k}w^{\frac{1}{6}}q_n(w)}{r_k w^{\frac{1}{6}}q_n(w)+\bar{r_k}q_n(w)}\\
&= \frac{v_k q_n(w)+\bar{v_k}p_n(w)+\bar{v_k}R_n(w)}{r_k p_n(w)+\bar{r_k}q_n(w)+r_k R_n(w)}\\
&= \frac{S+\bar{v_k}R_n(w)}{T+r_k R_n(w)},
\end{aligned}
$$

putting $S = v_k q_n(w) + \bar{v_k} p_n(w)$, $T = r_k p_n(w) + \bar{r_k} q_n(w)$, we have

$$|T\theta_k - S| = \bar{v_k} R_n(w) - \theta R_n(w).$$

Note that $1 - w = 1 - \frac{A+B\bar{\rho}}{A+B\rho} = \frac{B\sqrt{3}i}{u}$, from Lemma 2.1, we have

$$q_n(w)=\sum_{k=0}^{n}(-1)^k\binom{2n-k}{n-k}\binom{n-1/6}{k}\left(\frac{B\sqrt{3}i}{u}\right)^k. \tag{3.4}$$

From Lemma 2.2, we get

$$12^k 3^{\lfloor\frac{k}{2}\rfloor}\binom{n-\frac{1}{6}}{k}\in Z. \tag{3.5}$$

Hence, if denote $Q_n = 12^n u^n T$ and $P_n = 12^n u^n S$, it's easy to prove that both are algebraic integer in $Q[\sqrt{-3}]$. Since $\bar{u} = z$, $\overline{u^n p_n(w)} = u^n q_n(w)$. One can prove $\overline{Q_n} = Q_n$ and $\overline{P_n} = P_n$.

Therefore we have Q_n, $P_n \in Z$. In another word, we get two arrays of rational integers Q_n, P_n such that

$$|Q_n\theta_k - P_n| = |12^n u^n R_n(w)(\bar{v} - \theta r_k)| = R_n.$$

In the following, we give estimation of upper bound for Q_n, R_n.

From Lemma 2.1, we have

$$|p_n(w)| = |q_n(w)| \leq 4|1+\sqrt{w}|^{2n-2}$$

and $|R_n(w)| \leq \frac{\varphi}{3}|1-\sqrt{w}|^{2n}$. So, for $A_n = 54n^3 + 81n^2 + 54n + 12$, and $|n| \geq 2.03 \times 10^6$, if denote $\varepsilon = 2A - B + 2\sqrt{A^2 - AB + B^2}$, we obtain

$$\begin{aligned}
|Q_n| &= |12^n u^n (r_k p_n(w) + \overline{r_k} q_n(w))| \\
&\leq \frac{|6r_k|}{|1+\sqrt{w}|^2}|12u(1+\sqrt{w})^2|^n \\
&= \frac{|6r_k|}{|1+\sqrt{w}|^2}|12(2A - B + 2\sqrt{A^2 - AB + B^2})|^n \\
&= C_Q|12\varepsilon|^n,
\end{aligned}$$

and

$$\begin{aligned}
|R_n| &= |12^n u^n R_n(w)(\bar{v} - \theta r)| \\
&\leq \frac{|\overline{v_k} - \theta r_k| \cdot |\varphi|}{6}|12u(1-\sqrt{w})^2|^n \\
&= \frac{|\overline{v_k} - \theta r_k| \cdot |\varphi|}{6}|2A - B - 2\sqrt{A^2 - AB + B^2}|^n \\
&= C_R|\frac{\varepsilon}{36B^2}|^{-n},
\end{aligned}$$

where $C_Q = \frac{|6r_k|}{|1+\sqrt{w}|^2}$, $C_R = \frac{|\overline{v_k} - \theta r_k| \cdot |\varphi|}{6}$.

Now we will give estimation of C_Q, C_R. From(3.3) we have

$$\max_{k=0,1,\cdots,5}|r_k| = \max_{k=0,1,\cdots,5}|v_k| = 3. \tag{3.6}$$

From definition, we have $w = \frac{A+B\bar{\rho}}{A+B\rho} = \frac{A-\frac{B}{2}-\frac{\sqrt{3}}{2}Bi}{A-\frac{B}{2}+\frac{\sqrt{3}}{2}Bi} = \frac{(A-\frac{B}{2}-\frac{\sqrt{3}}{2}Bi)^2}{A^2-AB+B^2}$. Hence we get

$$|1+\sqrt{w}|^2 = |1 + \frac{A - \frac{B}{2} - \frac{\sqrt{3}}{2}Bi}{\sqrt{A^2 - AB + B^2}}|^2 = 2 + 2\frac{A - \frac{B}{2}}{\sqrt{A^2 - AB + B^2}}.$$

Note that $B = -n$, $A = 18B^4 - 27B^3 + 18B^2 - 4B$ and $|B| > 2.03 \times 10^6$, straightforward computation shows that

$$|1+\sqrt{w}|^2 > 3.999. \tag{3.7}$$

From (3.6) and (3.7), we have

$$C_Q = \frac{|6r_k|}{|1+\sqrt{w}|^2} < \frac{18}{3.999} < 4.502. \tag{3.8}$$

On the other hand, from definition we have $e^{i\varphi} = w = \frac{(A-\frac{B}{2}-\frac{\sqrt{3}}{2}Bi)^2}{A^2-AB+B^2} = \frac{A^2-AB-\frac{B^2}{2}}{A^2-AB+B^2} - \frac{\sqrt{3}B(A-\frac{B}{2})}{A^2-AB+B^2}$. So we have

$$|\varphi| \leq |\tan\varphi| = |\frac{\sqrt{3}B(A-\frac{B}{2})}{\frac{B^2}{2}}| < 0.0236. \tag{3.9}$$

If θ_k $(k = 0, 1, \cdots, 5)$ is denoted as in Lemma 3.1, one can get estimation of root θ_k:

$$\begin{cases} |\theta_k| < 3, & \text{if } k \neq 4, \\ |\theta_k| < 2.01|A_n|, & \text{if } k = 4. \end{cases}$$

Since $\varepsilon = 2A - B + 2\sqrt{A^2 - AB + B^2} < 4.1A < 1.367|A_nB|$, from (3.6) and (3.9), we can get estimate of C_R.

For θ_k $(k \neq 4)$, we have

$$C_R = \frac{|\overline{v_k} - \theta r_k| \cdot |\varphi|}{6} < \frac{3+9}{6} \cdot 0.0236 < 0.0472. \tag{3.10}$$

For θ_k $(k = 4)$, we also have

$$C_R = \frac{|\overline{v_k} - \theta r_k| \cdot |\varphi|}{6} < \frac{3+3.01A}{6} \cdot 0.0236 < 0.0357|A_n|. \tag{3.11}$$

Now putting $Q = 12\varepsilon$, $E = \frac{\varepsilon}{36B^2}$, $k_0 = 4.502$, then for θ_k, where $k \neq 4, k = 4$, we set $l_0 = 0.0472$ or $l_0 = 0.0357|A_n|$ separately. From Lemma 2.2, computing

$$2k_0Q(2l_0E)^\lambda = 2 \cdot 4.502 \cdot 12\varepsilon(2 \cdot 0.0472\frac{\varepsilon}{36B^2})^\lambda = 108.048\varepsilon(0.00263\frac{\varepsilon}{B^2})^\lambda,$$

and

$$2k_0Q(2l_0E)^\lambda = 2 \cdot 4.502 \cdot 12\varepsilon(2 \cdot 0.0357\frac{|A_n|\varepsilon}{36B^2}) = 108.048\varepsilon(0.00199\frac{|A_n|\varepsilon}{B^2}),$$

we have approximation to algebraic numbers as in followed lemma.

Lemma 3.2 If θ_k $(k = 0, 1, \cdots, 5)$ is the root of

$$x^6 - \frac{6A}{B}x^5 + (\frac{15A}{B} - 15)x^4 + 20x^3 - \frac{15A}{B}x^2 + (\frac{6A}{B} - 6)x + 1 = 0,$$

then for any $x, y \in Z$, we have

$$|x - y\theta_k| > \frac{1}{c_ky^\lambda},$$

where

$$\begin{cases} c_0 = c_1 = c_2 = c_3 = c_5 = 108.048\varepsilon(0.00263\frac{\varepsilon}{B^2})^\lambda, \\ c_4 = 108.048\varepsilon(0.00199\frac{|A_n|\varepsilon}{B^2})^\lambda, \\ \lambda = \frac{\log(12\varepsilon)}{\log 36B^2 - \log\varepsilon}. \end{cases}$$

3.3 The Proof of Theorem

Now we can prove the theorem. First, from Lemma 3.1, we have

$$\begin{aligned}
&\prod_{j\neq 0}|\theta_j-\theta_0| < 1.1\times 2.1\times 0.6\times 2.01|A_n|\times 1.1 < 3.065|A_n|,\\
&\prod_{j\neq 0}|\theta_j-\theta_1| < 1.1\times 1.1\times 0.6\times 2.01|A_n|\times 2.1 < 3.065|A_n|,\\
&\prod_{j\neq 0}|\theta_j-\theta_2| < 2.1\times 1.1\times 1.6\times 2.01|A_n|\times 3.1 < 23.03|A_n|,\\
&\prod_{j\neq 0}|\theta_j-\theta_3| < 0.6\times 0.6\times 1.6\times 2.01|A_n|\times 1.6 < 1.853|A_n|,\\
&\prod_{j\neq 0}|\theta_j-\theta_4| < (2.01|A_n|)^5 < 32.81|A_n|^5,\\
&\prod_{j\neq 0}|\theta_j-\theta_5| < 1.1\times 2.1\times 3.1\times 2.01|A_n|\times 1.6 < 23.03|A_n|. \qquad (3.12)
\end{aligned}$$

Second, computation shows that when $n=|B|>2.03\times 10^6$, we get $\lambda<2.18$. It's easy to get

$$|x-y\theta_k| < \frac{2^5}{|y|^5\prod_{j\neq k}|\theta_j-\theta_k|}.$$

Hence

$$\text{If } k\neq 4,\ \ |x-y\theta_k| < \frac{2^5}{23.03|A_n||y|^5}, \qquad (3.13)$$

$$\text{If } k=4,\ \ |x-y\theta_k| < \frac{2^5}{32.81|A_n|^5|y|^5}. \qquad (3.14)$$

So we get an upper bound of $|x-\theta y|$. From Lemma 3.2 and (3.13), for θ_k $(k\neq 4)$, we have

$$\frac{1}{108.048\varepsilon(0.00263\frac{\varepsilon}{B^2})^\lambda|y|^\lambda} < |x-y\theta_k| < \frac{2^5}{23.03|A_n||y|^5}.$$

Note that $\varepsilon<4.1A<1.367|A_nB|$, so if (x,y) is type k solution of (3.1), where $(k\neq 4)$, we have

$$|y| < 0.012^{\frac{1}{5-\lambda}}|A_n|^{\frac{\lambda}{5-\lambda}}|B|^{\frac{1-\lambda}{5-\lambda}} < 0.01|A_n|^{0.7735}|B|^{-0.4185}. \qquad (3.15)$$

For θ_4, we have

$$\frac{1}{108.048\varepsilon(0.00199\frac{|A_n|\varepsilon}{B^2})^\lambda} < |x-y\theta_k| < \frac{2^5}{32.81|A_n|^5|y|^5}.$$

So if (x,y) is type 4 solution of (3.1), we have

$$|y| < 0.012^{\frac{1}{5-\lambda}}|A_n|^{\frac{\lambda}{5-\lambda}}|B|^{\frac{1-\lambda}{5-\lambda}} < 0.0606|A_n|^{0.12766}|B|^{-1.19149}. \qquad (3.16)$$

This is an upper bound for $|y|$. From well-known result in number theory, we know that when $|y| > 1$, x, y is partial quotient of θ. In the following, we only need to verify whether (p_n, q_n) is solution of (3.1) or not.

From Lemma 3.1, computation shows the continued fraction expansion of θ_k $(k \neq 4)$, the result is listed as below:

$$\begin{aligned}
&\theta_0 = [1, 2A_n + 2, 1, 1, \lfloor \frac{2A_n}{25} \rfloor, \cdots], \ \{\frac{p_i}{q_i}\} = \{\frac{1}{1}, \frac{2A_n + 3}{2A_n + 2}, \frac{2An + 4}{2A_n + 3}, \cdots\},\\
&\theta_1 = [0, -2A_n - 3, 1, 1, \lfloor \frac{4A_n}{70} \rfloor, \cdots], \ \{\frac{p_i}{q_i}\} = \{\frac{0}{1}, \frac{1}{-2A_n + 3}, \frac{1}{-2A_n - 2}, \cdots\},\\
&\theta_2 = [-1, 2B_n + 1, 1, 1, \lfloor \frac{B_n}{2} \rfloor, \cdots], \ \{\frac{p_i}{q_i}\} = \{\frac{-1}{1}, \frac{-2B_n}{2B_n + 1}, \frac{-2B_n - 1}{2B_n + 2}, \cdots\},\\
&\theta_3 = [0, 1, 1, 2B_n, 1, 1, \lfloor \frac{B_n}{2} \rfloor, \cdots], \ \{\frac{p_i}{q_i}\} = \{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2B_n + 1}{4B_n + 1} \cdots\},\\
&\theta_5 = [2, 2B_n, 1, 1, \lfloor \frac{B_n}{2} \rfloor, \cdots], \ \{\frac{p_i}{q_i}\} = \{\frac{2}{1}, \frac{4B_n + 1}{2B_n}, \frac{4B_n + 3}{2B_n + 1}, \cdots\},
\end{aligned}$$

where $B_n = \frac{A_n}{3} = 18n^3 + 27n^2 + 18n + 4$.

One can observe that q_2 or q_4 has exceeded the upper bound of $|y|$, $0.01|A_n|^{0.7735}|B|^{-0.4185}$. Straight forward computation shows that it only exists trivial solution $\pm(x, y) = (0, 1), (1, -1)$.

One can also get the continued fraction expansion of θ_k $(k \neq 4)$ as below

$$\begin{aligned}
&\theta_4 = [-2A_n - 3, 2, \lfloor \frac{2A_n}{35} \rfloor, \cdots],\\
&\{\frac{p_i}{q_i}\} = \{\frac{-2A_n - 3}{1}, \frac{-4A_n - 5}{2}, \frac{-\lfloor \frac{2A_n}{35} \rfloor (4A_n + 5) - 2A_n - 3}{2\lfloor \frac{2A_n}{35} \rfloor + 1}, \cdots\}.
\end{aligned}$$

One can observe that q_2 has exceeded the upper bound of $|y|$, $0.0606|A_n|^{0.12766}|B|^{-1.19149}$. Computation shows that it doesn't exist type 4 solution.

Therefore, we know that (3.1) only has trivial solutions

$$\pm(x, y) = (1, 0), (0, 1)(1, -1).$$

Since $F_n(x, y) = f(x, -y)$, so we proved that when $n > 2.03 \times 10^6$, (2) only has trivial solutions. From the theorem developed by Alan Togbé [15], we prove the theorem.

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六次含参Thue方程的解

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摘要: 本文研究了含参的六次Thue方程. 利用初等方法和简单的代数数有理逼近方法彻底求解了该方程, 从而推广了Alan Togbé的结果.

关键词: 六次Thue方程; padè逼近; 连分数

MR(2010)**主题分类号**: 11D45 **中图分类号**: O156.7