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TOEPLITZ OPERATORS WITH UNBOUNDED SYMBOLS ON WEIGHTED DIRICHLET SPACE

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Abstract: In this paper, we study the properties of a class of Toeplitz operators on weighted Dirichlet space. By using the method of constructing a class of unbounded function on $\mathbb{D}$, we prove that the Toeplitz operators with these symbols are compact. Also by using the method of constructing a function ϕ in L^2_φ which is unbounded on any neighborhood of each boundary point of $\mathbb{D}$, we prove that T_ϕ is a trace class operator on weighted Dirichlet space.

Keywords: weighted Dirchlet space; unbounded symbol; trace class operator; Toeplitz operator

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1 Introduction

Let φ be a positive continuous function on $[0, 1)$, φ is called a normal function if there are two constants a and b: $0 < a < b$ such that $\frac{\varphi(t)}{(1-t^2)^a}$ decreases and $\frac{\varphi(t)}{(1-t^2)^b}$ increases on $[0, 1)$. The simplest example is $\varphi(t) = (1 - t^2)^\beta$, $\beta > 0$.

Also, let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ be its boundary, $dA(z) = \frac{r}{\pi} dr d\theta$ be the normalized Lebesgue measure on $\mathbb{D}$. We write $dA^p_\varphi(z)$ $(1 \le p < \infty)$ for the weighted Lebesgue measure: $\frac{\varphi^p(|z|)}{1-|z|^2} dA(z) = \frac{\varphi^p(r)}{1-r^2} \frac{r}{\pi} dr d\theta$, $z = re^{i\theta}$. A normal function φ and $p \in [1, \infty)$ are often used to define a Banach space $L^p(\varphi)$ with norm

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{D}} |f(z)|^p dA^p_\varphi(z) \right)^{1/p} < \infty.$$

Next for any $\alpha > b$ let $\psi(t) = \frac{(1-t^2)^\alpha}{\varphi(t)}$, then $\{\varphi, \psi\}$ is called a normal pair. Obviously ψ is also a normal function with two constants $\alpha - b$ and $\alpha - a$. Of course, every ψ can be used to give a Banach space $L^q(\psi)$ for $1 \le q < \infty$, relative to norm $\| \cdot \|_{q,\psi}$.

For $f \in L^p(\varphi)$ and $g \in L^q(\psi)$, we can define a dual form $\langle f, g\rangle$ as follows:

$$\langle f, g\rangle = \alpha \int_{\mathbb{D}} f(z)\overline{g(z)}(1 - |z|^2)^{\alpha-1} dA(z).$$

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Let $L^{2,1}_\varphi$ be the subspace of $L^2(\varphi)$ satisfying condition

$$\|u\|^2_{\frac{1}{2}} = \int_{\mathbb{D}} \Big(\Big|\frac{\partial u}{\partial z}(z)\Big|^2 + \Big|\frac{\partial u}{\partial \bar z}(z)\Big|^2\Big) dA^2_\varphi(z) < \infty.$$

Let $\mathfrak{L}^{2,1}_\varphi$ be the quotient space $L^{2,1}_\varphi/\mathbb{C}$, where $\mathbb{C}$ is complex constant functions subspace of $L^{2,1}_\varphi$, then $\mathfrak{L}^{2,1}_\varphi$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{\frac{1}{2}} = < \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} >_{L^2(\varphi)} + \langle \frac{\partial f}{\partial \bar z}, \frac{\partial g}{\partial \bar z}\rangle_{L^2(\varphi)}.$$

The weighted Dirichlet space $\mathcal{D}_\varphi$ is the subspace of all analytic function in $\mathfrak{L}^{2,1}_\varphi$. Let $K_z(w) = \int_0^{\bar z}\int_0^w \frac{d\zeta d\eta}{(1-\zeta\eta)^{\alpha+1}}$ for $z, w \in \mathbb{D}$, then the linear operator P is defined as follows:

$$(Pf)(z) = \alpha \int_{\mathbb{D}} \frac{\partial f(w)}{\partial w} \overline{\left(\frac{\partial K_z(w)}{\partial w}\right)} (1-|w|^2)^{\alpha-1} dA(w), \quad f \in \mathcal{D}_\varphi.$$

We note that the operator P is bounded from $L^{2,1}_\varphi$ onto $\mathcal{D}_\varphi$. Moreover, $(Pf)(z) = f(z)$ for $f \in \mathcal{D}_\varphi$, so K_z is called the reproducing kernel (see [1–3]). Let G be a domain in $\mathbb{C}$, and define

$$L^{\infty,1}(G) = \{u : u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar z} \in L^\infty(G)\}$$

for $u \in L^{\infty,1}(G)$, $\|u\|_{1,\infty} = \operatorname*{esssup}_{z\in G} \max\{|u(z)|, |\frac{\partial u}{\partial z}(z)|, |\frac{\partial u}{\partial \bar z}(z)|\}$.

Definition 1.1 Suppose $u \in L^{2,1}_\varphi$, the operator

$$T_u f(z) = P(uf)(z) = \alpha \int_{\mathbb{D}} \frac{\partial (u(w)f(w))}{\partial w} \overline{\left(\frac{\partial K_z(w)}{\partial w}\right)} (1-|w|^2)^{\alpha-1} dA(w) \quad (f \in \mathcal{D}_\varphi)$$

is said to be the Toeplitz operator with symbol u, this operator is densely defined.

In the case of Hardy space, it is well known that T_φ is bounded if and only if φ is essentially bounded, and T_φ is compact if and only if $\varphi = 0$ (see [4, 5]). However, there are indeed bounded and compact Toeplitz operators with unbounded symbols, in fact, Miao and Zheng [6] introduced a class of functions, called BT, which contains L^∞, for $\varphi \in BT$, T_φ is compact on Bergman space $L^2_a(\mathbb{D})$, if and only if the Berezin transform of φ vanishes on the unit circle $\mathbb{T}$. Zorboska [7] proved that if φ belongs to the hyperbolic BMO space, then T_φ is compact if and only if the Berezin transform of φ vanishes on the unit circle. Cima and Cuckovic [8] constructed a class of unbounded functions built over Cantor set, the Toeplitz operator with these functions are compact. Essentially, if the value of the function φ vanishes rapidly near the unit circle in the sense of measure dA, the T_φ will be compact. Cao [9] constructed compact Toeplitz operators on Bergman space $L^2_a(\mathbb{B}_n, d\nu)$ with unbounded symbols. Wang, Xia and Cao [10] constructed a trass class Toeplitz operator T_φ on Dirichlet space $\mathcal{D}$ with unbounded symbols.

In this paper, we construct a class of unbounded function on $\mathbb{D}$, the Toeplitz operators with these symbols are compact. We also construct a function ϕ on any countable dense subset in $\mathbb{T}$ which has nontagential limit infinity everywhere, such that T_ϕ is trace class.

2 Compact Toeplitz Operators with Unbounded Symbols

For $\delta > 0, \xi \in \mathbb{T}$, set

$$\Omega(\xi,\delta) = \{z \in \mathbb{D} : [1-(1-|z|)^{\delta}]^{\frac{1}{2}}|z-\xi| < |\mathrm{Re}(\xi\overline{(z-\xi)})|, \mathrm{Re}(z\bar{\xi}) > 0\}.$$

Then $\Omega(\xi,\delta)$ is an open subset of $\mathbb{D}$ and this domain is said to be circle cone like with vertex ξ. For any $0 < r < 1$, let $\mathbb{D}_r = \{z : |z| < r\}$ be the disc with center 0 and radius r, $\mathbb{T}_r$ its boundary. We denote the Lebesgue measure on $\mathbb{T}$ by $d\theta$. Assume b is an arbitrary positive number, it is obvious that we may choose a suitable $\delta = \delta(b) > 0$ such that for arbitrary $0 < r < 1$,

$$\theta[\Omega(\xi,\delta) \cap \mathbb{T}_r] < d(1-r^2)^b,$$

where d is a constant which is independent of ξ and r. For convenience, we write $\Omega_b(\xi) = \Omega(\xi, \delta(b))$.

Lemma 2.1 Suppose $\{f_k\} \subset \mathcal{D}_\varphi$ and $\|f_k\|_{L^{2,1}_\varphi} = 1$, $f_k \longrightarrow 0$ weakly in $\mathcal{D}_\varphi$, then $\|f_k\|_{L^2(\varphi)} \le A$ and $\|f_k\|_{L^2(\varphi)} \longrightarrow 0$, where A is a constant.

Theorem 2.2 Suppose $c > 0$, $U_c(z) = (1-|z|^2)^{-c}$, $z \in \mathbb{D}$. For any $\xi \in \mathbb{T}$, let $b \ge 2c+4$ and $\chi_{\Omega_b(\xi)}$ be the characteristic function of $\Omega_b(\xi)$, then $\phi = \chi_{\Omega_b(\xi)}U_c(z)$ induces a compact Toeplitz operator on weighted Dirichlet space $\mathcal{D}_\varphi$.

Proof Suppose $\{f_k\} \subset \mathcal{D}_\varphi$ with $\|f_k\|_{L^{2,1}_\varphi} = 1$ is a sequence which weakly converges to zero, it is enough to prove that $\|T_\phi f_k\|_{L^{2,1}_\varphi} \to 0$ when $k \to \infty$. Note

$$\begin{aligned} T_\phi f_k(z) &= \alpha \int_{\mathbb{D}} \frac{\partial(\phi(w)f_k(w))}{\partial w} \overline{\left(\frac{\partial K_z(w)}{\partial w}\right)} (1-|w|^2)^{\alpha-1} dA(w) \\ &= \alpha \int_{\mathbb{D}} \frac{\partial(\phi(w))}{\partial w} f_k(w) \overline{\left(\frac{\partial K_z(w)}{\partial w}\right)} (1-|w|^2)^{\alpha-1} dA(w) \\ &\quad + \alpha \int_{\mathbb{D}} \phi(w) f_k'(w) \overline{\left(\frac{\partial K_z(w)}{\partial w}\right)} (1-|w|^2)^{\alpha-1} dA(w). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial T_\phi f_k(z)}{\partial z} &= \alpha \int_{\mathbb{D}} \frac{\partial(\phi(w))}{\partial w} f_k(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w) \\ &\quad + \alpha \int_{\mathbb{D}} \phi(w) f_k'(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w), \end{aligned}$$

we see that

$$\begin{aligned} \|T_\phi f_k\|^2_{L^{2,1}_\varphi} &= \alpha \int_{\mathbb{D}} \Big|\frac{\partial T_\phi f_k(z)}{\partial z}\Big|^2 (1-|z|^2)^{\alpha-1} dA(z) \\ &\le C\alpha \int_{\mathbb{D}} \Big(\Big|\alpha \int_{\Omega_b(\xi)} \bar{w}(1-|w|^2)^{-c-1} f_k(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w)\Big|^2 \\ &\quad + \Big|\alpha \int_{\Omega_b(\xi)} f_k'(w)(1-|w|^2)^{-c} \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w)\Big|^2 \Big) (1-|z|^2)^{\alpha-1} dA(z). \end{aligned}$$

For $m\in(0,1)$, set $\Omega_b(\xi,m)=\{z\in\Omega_b(\xi):|z|>m\}$, then

$$\begin{aligned}\|T_\phi f_k\|^2_{L^{2,1}_\varphi} &\le 2C\alpha\int_{\mathbb{D}}\Big[\Big|\alpha\int_{\Omega_b(\xi)-\Omega_b(\xi,m)}\bar w(1-|w|^2)^{-c-1}f_k(w)\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2\\ &\quad+\Big|\alpha\int_{\Omega_b(\xi,m)}\bar w(1-|w|^2)^{-c-1}f_k(w)\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2\\ &\quad+\Big|\alpha\int_{\Omega_b(\xi)-\Omega_b(\xi,m)}f_k'(w)(1-|w|^2)^{-c}\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2\\ &\quad+\Big|\alpha\int_{\Omega_b(\xi,m)}f_k'(w)(1-|w|^2)^{-c}\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2\Big](1-|z|^2)^{\alpha-1}dA(z).\end{aligned}$$

Note

$$\begin{aligned}&\Big|\alpha\int_{\Omega_b(\xi,m)}\bar w(1-|w|^2)^{-c-1}f_k(w)\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2\\ \le\ &\alpha\int_{\Omega_b(\xi,m)}|f_k(w)|^2(1-|w|^2)^{\alpha-1}dA(w)\alpha\int_{\Omega_b(\xi,m)}\frac{(1-|w|^2)^{-2c-2}(1-|w|^2)^{\alpha-1}}{|1-z\bar w|^{2(\alpha+1)}}dA(w),\end{aligned}$$

and

$$\alpha\int_{\mathbb{D}}\frac{(1-|z|^2)^{\alpha-1}}{|1-z\bar w|^{2(\alpha+1)}}dA(z)=\frac{1}{(1-|w|^2)^{\alpha+1}}.$$

Since $\|f_k(w)\|_{L^2(\varphi)}\le A\|f_k(w)\|_{L^{2,1}_\varphi}\le A$, thus

$$\begin{aligned}&\alpha\int_{\mathbb{D}}\Big|\alpha\int_{\Omega_b(\xi,m)}\bar w(1-|w|^2)^{-c-1}f_k(w)\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2(1-|z|^2)^{\alpha-1}dA(z)\\ \le\ &A\alpha\int_{\mathbb{D}}\alpha\int_{\Omega_b(\xi,m)}\frac{(1-|w|^2)^{-2c-2}(1-|w|^2)^{\alpha-1}}{|1-z\bar w|^{2(\alpha+1)}}(1-|z|^2)^{\alpha-1}dA(w)dA(z)\\ =\ &A\alpha\int_{\Omega_b(\xi,m)}\frac{(1-|w|^2)^{-2c-2}(1-|w|^2)^{\alpha-1}}{(1-|w|^2)^{\alpha+1}}dA(w)\\ \le\ &A_0\int_m^1(1-r^2)^{b-2c-4}dr^2\\ =\ &\frac{A_0}{b-2c-3}(1-m^2)^{b-2c-3}\le A_0(1-m^2),\end{aligned}$$

where A_0 is a constant. It is obvious that for any $\varepsilon>0$, there is an $m_1\in(0,1)$ such that

$$\alpha\int_{\mathbb{D}}\Big|\alpha\int_{\Omega_b(\xi,m)}\bar w(1-|w|^2)^{-c-1}f_k(w)\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2(1-|z|^2)^{\alpha-1}dA(z)\le A_0(1-m^2)<\varepsilon$$

for $m\in[m_1,1)$. For the same reason, there exists an $m_2\in(0,1)$ such that

$$\alpha\int_{\mathbb{D}}\Big|\alpha\int_{\Omega_b(\xi,m)}f_k'(w)(1-|w|^2)^{-c}\frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar w)^{\alpha+1}}dA(w)\Big|^2(1-|z|^2)^{\alpha-1}dA(z)\le A_0(1-m^2)<\varepsilon$$

for $m\in[m_2,1)$. We write $m_0=\max\{m_1,m_2\}$, then both inequalities hold for $m\in[m_0,1)$. On the other hand, since $\Omega_b(\xi)-\Omega_b(\xi,m_0)\subset\{z\in\mathbb{D}:|z|\le m_0\}$, we know that $f_k(w)\to 0$

and $f_k'(w) \to 0$ uniformly on $\Omega_b(\xi) - \Omega_b(\xi, m_0)$. Hence for any $\varepsilon > 0$, there is a K_0, such that for $k > K_0$, $|f_k(w)| < \varepsilon$ and $|f_k'(w)| < \varepsilon$ for any $w \in \Omega_b(\xi) - \Omega_b(\xi, m_0)$. Therefore

$$\begin{aligned} & \alpha \int_{\mathbb{D}} \Big|\alpha \int_{\Omega_b(\xi)-\Omega_b(\xi,m)} \bar{w}(1-|w|^2)^{-c-1} f_k(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w)\Big|^2 (1-|z|^2)^{\alpha-1} dA(z) \\ \leq \ & \varepsilon\alpha \int_{\Omega_b(\xi)-\Omega_b(\xi,m)} \frac{1}{(1-|w|^2)^{2c+4}} dA(w) \leq \frac{\varepsilon\alpha}{(1-m_0^2)^{2c+4}}, \end{aligned}$$

and

$$\begin{aligned} & \alpha \int_{\mathbb{D}} \Big|\alpha \int_{\Omega_b(\xi)-\Omega_b(\xi,m)} (1-|w|^2)^{-c} f_k'(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w)\Big|^2 (1-|z|^2)^{\alpha-1} dA(z) \\ \leq \ & \varepsilon\alpha \int_{\Omega_b(\xi)-\Omega_b(\xi,m)} \frac{1}{(1-|w|^2)^{2c+2}} dA(w) \leq \frac{\varepsilon\alpha}{(1-m_0^2)^{2c+2}} \end{aligned}$$

consequently $\|T_\varphi f_k\|_{L^{2,1}_\varphi} \to 0$.

Theorem 2.3 There is a function $\phi \in L^{2,1}_\varphi$ which is unbounded on any neighborhood of each boundary point of $\mathbb{D}$ (i.g. for any $\xi \in \mathbb{T}$, and $r > 0$, $\operatorname*{esssup}_{z \in \mathbb{D} \cap \mathbb{D}(\xi,r)} |\phi(z)| = \infty$, where $\mathbb{D}(\xi, r) = \{z : |z-\xi| < r\}$) such that T_ϕ is a compact operator on $\mathcal{D}_\varphi$.

Proof Set $c > 0$, $b \geq 4c+5$ and let $U_c(z)$ be the function as in Theorem 2.2. Choose a countable dense subset $\{\xi_i \big| i = 1, 2, \cdots\}$ of $\mathbb{T}$. For each ξ_i, write $\phi_i = \chi_{\Omega_b(\xi_i)} U_c$ then T_{ϕ_i} is a compact operator by Theorem 2.2. For any $f \in \mathcal{D}_\varphi$,

$$\begin{aligned} & \|T_{\phi_i} f\|^2_{L^{2,1}_\varphi} \\ \leq \ & \alpha \int_{\mathbb{D}} \Big(\Big|\alpha \int_{\Omega_b(\xi_i)} \frac{\partial \phi_i(w)}{\partial w} f(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w)\Big|^2 \\ & + \Big|\alpha \int_{\Omega_b(\xi_i)} \phi_i(w) f'(w) \frac{(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}} dA(w)\Big|^2 (1-|z|^2)^{\alpha-1}\Big) dA(z) \\ \leq \ & \|f\|^2_{L^{2,1}_\varphi} \Big(\alpha \int_{\mathbb{D}} \alpha \int_{\Omega_b(\xi_i)} \frac{(1-|w|^2)^{-2c-2}(1-|w|^2)^{\alpha-1}}{|1-z\bar{w}|^{2(\alpha+1)}} dA(w)(1-|z|^2)^{\alpha-1} dA(z) \\ & + \alpha \int_{\mathbb{D}} \alpha \int_{\Omega_b(\xi_i)} \frac{(1-|w|^2)^{-2c}(1-|w|^2)^{\alpha-1}}{|1-z\bar{w}|^{2(\alpha+1)}} dA(w)(1-|z|^2)^{\alpha-1} dA(z)\Big) \\ = \ & \|f\|^2_{L^{2,1}_\varphi} \left(\alpha \int_{\Omega_b(\xi_i)} \big((1-|w|^2)^{-2c-2} + (1-|w|^2)^{-2c}\big) \frac{1}{(1-|w|^2)^2} dA(w) \right) \\ \leq \ & \|f\|^2_{L^{2,1}_\varphi} \alpha \int_0^1 \int_{\Omega_b(\xi_i) \cap \mathbb{T}_r} \big((1-r^2)^{-2c-2} + (1-r^2)^{-2c}\big) \frac{r}{(1-r^2)^2} dr d\theta \\ \leq \ & \frac{d\alpha}{2\pi} \|f\|^2_{L^{2,1}_\varphi} \int_0^1 \big((1-r^2)^{-2c-2} + (1-r^2)^{-2c}\big)(1-r^2)^b \frac{1}{(1-r^2)^2} dr^2 \\ \leq \ & K\|f\|^2_{L^{2,1}_\varphi}, \end{aligned}$$

where K is a constant, hence $\|T_{\phi_i}\| \leq K$. Write $T_N = \sum\limits_{i=1}^{N} \frac{1}{2^i} T_{\phi_i}$, then T_N is a compact, and for any M, N and $f \in \mathcal{D}_\varphi$,

$$\|\sum_{i=N}^{M}\frac{1}{2^i}T_{\phi_i}f\|_{L^{2,1}_\varphi} \le C\sum_{i=N}^{M}\frac{1}{2^i}\|f\|_{L^{2,1}_\varphi}.$$

It follows that

$$\|\sum_{i=N}^{M}\frac{1}{2^i}T_{\phi_i}\|_{L^{2,1}_\varphi} \le C\sum_{i=N}^{M}\frac{1}{2^i},$$

hence $T = \sum\limits_{i=1}^{\infty}\frac{1}{2^i}T_{\phi_i}$ converges in norm. Furthermore, T is a compact operator. It is easy to check that $\phi_i \in L^{2,1}_\varphi$ and $\|\phi_i\|_{L^{2,1}_\varphi} \le C$. Thus $\sum\limits_{i=1}^{\infty}\frac{1}{2^i}\phi_i$ converges to a function $\phi \in L^{2,1}_\varphi$. It is not difficult to see that for each polynomial $P(w)$,

$$\|(T_\phi - T_N)P\|_{L^{2,1}_\varphi} = \|T_{\sum\limits_{i=N+1}^{\infty}\frac{1}{2^i}\phi_i}P\| \le C\|P\|_{L^{2,1}_\varphi}\sum_{i=N+1}^{\infty}\frac{1}{2^i} \to 0.$$

Therefore $T = T_\phi$, namely, T is a Toeplitz operator with symbol $\phi = \sum\limits_{i=1}^{\infty}\frac{1}{2^i}\phi_i$. Since $\{\xi_i : i = 1, 2, \cdots\}$ is dense in $\mathbb{T}$, it is obvious that for any $\xi_i \in \mathbb{T}$ and $r > 0$, $\operatorname*{esssup}\limits_{z\in\mathbb{D}\cap\mathbb{D}(\xi,r)}|\phi(z)| = \infty$.

3 Trace Class Toeplitz Operators

In this section, we prove that there is a family of trace class of Toeplitz operators with unbounded symbols.

Theorem 3.1 There is a function $\phi \in L^{2,1}_\varphi$ which is unbounded on any neighborhood of each boundary point of $\mathbb{D}$ (that is, for arbitrary $\xi \in \mathbb{T}$, and $r > 0$, $\operatorname*{esssup}\limits_{z\in\mathbb{D}\cap\mathbb{D}(\xi,r)}|\phi(z)| = \infty$, where $\mathbb{D}(\xi, r)$ is same as in Theorem 2.2) such that T_ϕ is a trace class operator.

Proof Let $e_k(w) = w^k/\sqrt{C_{\alpha,k}}(k \in \mathbb{Z}^+)$ be the standard orthonormal basis of $\mathcal{D}_\varphi$, where $C_{\alpha,k} = k\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha+k)}$. Let $\{\xi_i : i = 1, 2, \cdots\}$ be a countable dense subset of $\mathbb{T}$, and $U_c = (1 - |w|^2)^{-c}(c > 0)$. We must show that

$$\sum_{k=1}^{\infty}\langle|T|e_k, e_k\rangle_{L^{2,1}_\varphi} < \infty.$$

For any $m \in (0, 1)$ set $\phi_i(w) = \chi_{\Omega_b(\xi_i,m)}U_c(w)$, where $b \ge c + 5$, then

$$\begin{aligned}
&|\langle T_{\phi_i}e_k, e_k\rangle_{L^{2,1}_\varphi}|\\
=\ & \frac{1}{C_{\alpha,k}}\Big|\alpha\int_{\mathbb{D}}\Big(\alpha\int_{\mathbb{D}}\frac{\partial\phi_i(w)}{\partial w}\frac{w^k k\bar{z}^{k-1}(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}}dA(w)\\
&+\alpha\int_{\mathbb{D}}\phi_i(w)\frac{w^{k-1}k^2\bar{z}^{k-1}(1-|w|^2)^{\alpha-1}}{(1-z\bar{w})^{\alpha+1}}dA(w)\Big)(1-|z|^2)^{\alpha-1}dA(z)\Big|\\
=\ & \frac{1}{C_{\alpha,k}}\Big|\alpha\int_{\mathbb{D}}\frac{\partial\phi_i(w)}{\partial w}w_k\bar{w}^{k-1}(1-|w|^2)^{\alpha-1}dA(w)
\end{aligned}$$

$$
\begin{aligned}
&+\alpha\int_{\mathbb{D}}\phi_i(w)k^2|w|^{k-1}(1-|w|^2)^{\alpha-1}dA(w)\big| \\
= & \ \big|\frac{c\alpha k}{C_{\alpha,k}}\int_{\Omega_b(\xi_i,m)}(1-|w|^2)^{\alpha-c-2}|w|^{2k}dA(w) \\
&+\frac{\alpha k^2}{C_{\alpha,k}}\int_{\Omega_b(\xi_i,m)}(1-|w|^2)^{\alpha-c-1}|w|^{2(k-1)}dA(w) \\
\leq & \ \frac{Ak\alpha}{C_{\alpha,k}}\big|\int_m^1(1-r^2)^{b+\alpha-c-2}r^{2k}dr^2+k\int_m^1(1-r^2)^{b+\alpha-c-1}r^{2(k-1)}dr^2\big|.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_m^1(1-r^2)^{b+\alpha-c-2}r^{2k}dr^2+k\int_m^1(1-r^2)^{b+\alpha-c-1}r^{2(k-1)}dr^2 \\
\leq & \int_{m^2}^1(1-r)^{\alpha+3}r^k dr+k\int_{m^2}^1(1-r)^{\alpha+4}r^{k-1}dr \\
\leq & \ 2\frac{\Gamma(k+1)\Gamma(\alpha+5)}{\Gamma(\alpha+k+5)},
\end{aligned}
$$

then

$$
|\langle T_{\phi_i}e_k,e_k\rangle_{L^{2,1}_\varphi}|\leq\frac{Ak\alpha}{C_{\alpha,k}}\frac{2\Gamma(k+1)\Gamma(\alpha+5)}{\Gamma(\alpha+k+5)}\leq\frac{B(\alpha)}{k^5},
$$

where $B(\alpha)$ is a constant only depending on α. Set $T=\sum\limits_{i=1}^{\infty}\frac{1}{2^i}T_{\phi_i}$, then T is a compact operator. Note T is positive, thus

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\big|\langle Te_k,e_k\rangle_{L^{2,1}_\varphi}\big| \\
= & \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2^i}\langle T_{\phi_i}e_k,e_k\rangle_{L^{2,1}_\varphi} \\
\leq & \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2^i}\frac{B(\alpha)}{k^5}\leq\sum_{k=1}^{\infty}\frac{B(\alpha)}{k^5}<\infty.
\end{aligned}
$$

Hence T is a trace class operator. In the same manner we can also construct Hilbert-Schmidt Toeplitz operator T_ϕ on $\mathcal{D}_\varphi$ with unbounded symbols $\phi\in L^{2,1}_\varphi$.

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加权Dirichlet空间上具有无界符号的Toeplitz算子

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摘要: 本文研究了加权 Dirichlet 空间上一类 Toeplitz 性质的问题. 利用构造单位圆盘 $\mathbb{D}$ 上一类无界函数的方法, 获得了以它为符号的 Toeplitz 算子是紧的结果. 同时也通过构造一类 $L^2(\varphi)$ 上的函数, 使得它们在单位圆周上每一点的任何一个邻域都无界的方法, 获得了以这些函数为符号的 Toeplitz 算子是迹类算子的结果.

关键词: 加权Dirichlit空间; 无界符号; 迹类算子; Toeplitz算子

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