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THE SECOND VARIATION OF ARC-LENGTH AND SUBMANIFOLDS IN FINSLER MANIFOLD

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Abstract: In this paper, we study submanifolds in Finsler manifold M. By using tangent curvature and normal curvature, which are introduced in [23, 24], we derive a new second variation formula for a geodesic γ in Finsler manifold, and then obtain many relation between geometric invariants and topological invariants of Finsler submanifolds, which are generalizations of the results described in [4].

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1 Introduction

As is known to all, the theory of hypersurfaces in a Finsler space was first considered by E. Cartan (see [10]). Henceforth, many geometrists studied different properties of submanifolds of Finsler (see [6, 12, 17, 20, 25]). Recently, by using the Holmes-Thompson volume form, which appeared once in [3] and [11], analogues such as the mean curvature and the second fundamental form for Finsler submanifolds were introduced in [13] and coincide with the usual notions for the Riemannian case. Q. He and Y. B. Shen in [14] also studied the properties of Finsler minimal submanifolds and established the Bernstein type theorems for Finsler minimal graphs in the Minkowski space and the Randers space. Later, some geometrists proved that totally geodesic submanifolds of Finsler manifolds are minimal for the Holmes-Thompson volume form (see [7, 16]). On the other hand, Busemann in [9] argued strongly that the volume of a Finsler manifold should be its Hausdorff measure. His argument was based on a number of axioms that any natural definition of volume on Finsler spaces must satisfy. In 1998, Shen in [24] introduced the notions of the mean curvature and the normal curvature for Finsler submanifolds with Busemann-Hausdorff volume form. Based on this volume form, minimal surfaces and a Bernstein type theorem on a special Randers space were considered in [26] and [27]. The main purpose of this paper is to continue the investigation in this direction. By using tangent curvature and normal curvature, which

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were introduced in [23, 24], we attempt to study the relation between geometric invariants and topological invariants of Finsler submanifolds. The article is organized as follows.

In Section 2, we review some basic facts of the Finsler manifold and its submanifolds.

In Section 3, we derive the following second variation formula for a geodesic γ.

$$\begin{aligned} L''(0) &= \mathrm{g}_{\dot\gamma(t)}(D_V V, \dot\gamma(t))|_a^b + \mathbf{T}_{\dot\gamma(t)}(V)|_a^b \\ &\quad + \int_a^b \{\mathrm{g}_{\dot\gamma(t)}(D_{\dot\gamma(t)}V^\perp, D_{\dot\gamma(t)}V^\perp) - \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(V^\perp), V^\perp)\}dt. \end{aligned} \tag{1.1}$$

In Section 4, we study hypersurfaces in Finsler manifold with positive Ricci curvature, and prove that under suitable hypotheses on the curvatures, the fundamental group of the manifold is homomorphic with the fundamental group of a component of the complement of hypersurface.

In Section 5, we study submanifolds in Finsler manifold M with positive flag curvature, and prove that two compact submanifolds with vanished normal and sur-tangent curvatures must necessarily intersect if their dimension sum is at least that of M.

In Section 6, we study r -dimensional submanifold N with nonpositive mean normal and sur-tangent curvatures in complete Finsler manifold. We find that if k -th Ricci curvature $\mathrm{Ric}_{(k)}(M) \geq kc(k \leq r)$, then $d(p, N) \leq \frac{\pi}{2\sqrt{c}}$, for any $p \in M$. Furthermore, suppose that N is compact, then M is compact.

2 Preliminaries

Let M be an n-dimensional complete connected Finsler manifold with Finsler metric F, and $\pi : TM \to M$ the natural projection. For each $p \in M$, let

$$\mathrm{g}_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2(y)}{\partial y^i \partial y^j}, \mathrm{C}_{ijk}(y) = \frac{1}{4}\frac{\partial^3 F^2(y)}{\partial y^i \partial y^j \partial y^k}, \tag{2.1}$$

where $y = y^i \frac{\partial}{\partial x^i}|_p \in T_pM$. The fundamental tensors g and the Cartan tensors C on π^*TM of F are defined by

$$\mathrm{g}(X, Y) := \mathrm{g}_{ij}(y)X^iY^j, \;\; \mathrm{C}(X, Y, Z) := \mathrm{C}_{ijk}(y)X^iY^jZ^k, \tag{2.2}$$

respectively. Define a map

$$D : T_pM \times C^\infty(TM) \to T_pM, \tag{2.3}$$

$$D_yV := \{dV^i(y) + V^j(x)\frac{\partial G^i}{\partial y^j}(y)\}\frac{\partial}{\partial x^i}|_p, \tag{2.4}$$

where $G^i(y)$ are geodesic coefficients, given by

$$G^i(y) := \frac{1}{4}\mathrm{g}^{il}(y)\{2\frac{\partial \mathrm{g}_{jl}}{\partial x^k}(y) - \frac{\partial \mathrm{g}_{jk}}{\partial x^l}(y)\}y^jy^k. \tag{2.5}$$

$D_yV(p)$ is called the covariant derivative of V at p in the direction y, and the family $D := \{D_y\}_{y \in TM}$ is called the connection of F.

Extend y to a geodesic field Y in a neighborhood $\mathbf{U}$ of $p \in M$. Let $\hat{D}$ be the Levi-Civita connection of the induced Riemannian metric g_Y on $\mathbf{U}$. Define

$$\mathbf{T}_y(v) := \mathrm{g}_y(\hat{D}_v V, y) - \mathrm{g}_y(D_v V, y), v \in T_pM, \tag{2.6}$$

where V is a vector field with $V_p = v$. $\mathbf{T} = \{\mathbf{T}_y\}_{y \in TM\backslash\{0\}}$ is called the tangent curvature (see [23]).

Proposition 2.1 (see [4]) There is a family of transformations $R_y : T_pM \to T_pM, y \in T_pM\backslash\{0\}$, such that for any geodesic variation Γ of a geodesic γ, the variation vector field $J(t) := \frac{\partial \Gamma}{\partial u}(0, t)$ along γ satisfies the following equation

$$D_{\dot{\gamma}} D_{\dot{\gamma}} J + R_{\dot{\gamma}}(J) = 0. \tag{2.7}$$

The Riemann curvature, denoted R, is defined by

$$R := \{R_y : T_pM \to T_pM \mid y \in T_pM\backslash\{0\}, p \in M\}. \tag{2.8}$$

For a flag (P, y) (or (u, y)), the flag curvature $K(P, y)$ is defined by

$$K(P, y) = K(u, y) := \frac{\mathrm{g}_y(R_y(u), u)}{\mathrm{g}_y(y, y)\mathrm{g}_y(u, u) - \mathrm{g}_y(y, u)\mathrm{g}_y(y, u)}, \tag{2.9}$$

where u is a tangent vector such that $P = \mathrm{span}\{u, y\}$. The Ricci curvature and the k-th Ricci curvature of y are defined as

$$\mathrm{Ric}(y) := \sum_{i=1}^{n} K(u_i, y), \tag{2.10}$$

$$\mathrm{Ric}_{(k)}(y) := \sum_{j=1}^{k} K(u_{i_j}, y), \tag{2.11}$$

respectively, where the indices $i_1, i_2, \cdots, i_k \in \{1, \cdots, n\}, i_l \neq i_j \ \ (l \neq j), k \leq n$, and $\{u_i\}_{i=1}^n$ is the local g_Y-orthonormal frame.

Proposition 2.2 (see [4]) There is a unique set of local 1-forms ω_j^i on $TM\backslash\{0\}$ such that

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{2.12}$$

$$d\mathrm{g}_{ij} = \mathrm{g}_{jk}\omega_i^k + \mathrm{g}_{kj}\omega_i^k + 2\mathrm{C}_{ijk}\omega^{n+k}, \tag{2.13}$$

$$\omega^{n+i} = dy^i + y^j \omega_j^i. \tag{2.14}$$

Let T be a non-vanishing vector field on an open subset $\mathbf{U} \subset M$, then the above 1-forms admit a linear connection ∇^T, which is called Chern connection. For vector fields X, Y, Z on $\mathbf{U}$, the Chern curvature $\Omega^T(X, Y)Z$ is given by

$$\Omega^T(X, Y)Z = \nabla_X^T \nabla_Y^T Z - \nabla_Y^T \nabla_X^T Z - \nabla_{[X,Y]}^T Z. \tag{2.15}$$

Now let N be a smoothly embedded submanifold in Finsler manifold M, and $\mathbf{n}$ denote a unitary normal vector of N at $p \in N$. The normal curvature (see [23, 24]) and sur-tangent curvature of N at $\mathbf{n}$ is given by

$$S_{\mathbf{n}}(X,Y) = -\mathrm{g}_{\mathbf{n}}(D_X\mathbf{n}, Y), \tag{2.16}$$

$$\mathbf{T}_{\mathbf{n}}(X) = \mathbf{T}_y(v)|_{y=\mathbf{n}, v=X}, \tag{2.17}$$

respectively, for $X, Y \in T_pN$.

3 Second Variation Formula

Proposition 3.1 (see [28]) Let $\gamma : [a,b] \to M$ be a unit speed geodesic in (M,F). Consider a piecewise C^∞ variation of γ

$$\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M, \tag{3.1}$$

and let $L(s) = L_F(\gamma_s)$ be the arc-length of γ_s, then

$$L''(s) = \int_a^b \{\frac{\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde V}\nabla^{\tilde T}_{\tilde T}\tilde V, \tilde T) + \mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\tilde V, \nabla^{\tilde T}_{\tilde T}\tilde V)}{(\mathrm{g}_{\tilde T}(\tilde T, \tilde T))^{\frac12}} - \frac{(\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\tilde V, \tilde T))^2}{(\mathrm{g}_{\tilde T}(\tilde T, \tilde T))^{\frac32}}\}dt, \tag{3.2}$$

where $\tilde T = d\Gamma_{(s,t)}(\frac{\partial}{\partial t}), \tilde U = d\Gamma_{(s,t)}(\frac{\partial}{\partial s})$.

By (2.15) and (3.2), we have

$$\begin{aligned} &L''(s) \\ = \; & \int_a^b \mathrm{g}_{\tilde T}(\tilde T, \tilde T)^{-\frac12}\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\nabla^{\tilde T}_{\tilde V}\tilde V, \tilde T)dt + \int_a^b \mathrm{g}_{\tilde T}(\tilde T, \tilde T)^{-\frac12}\mathrm{g}_{\tilde T}(\Omega^{\tilde T}(\tilde V, \tilde T)\tilde V, \tilde T)dt \\ & + \int_a^b \mathrm{g}_{\tilde T}(\tilde T, \tilde T)^{-\frac12}\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\tilde V, \nabla^{\tilde T}_{\tilde T}\tilde V)dt - \int_a^b \mathrm{g}_{\tilde T}(\tilde T, \tilde T)^{-\frac32}\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\tilde V, \tilde T)^2 dt. \end{aligned} \tag{3.3}$$

Observe that

$$\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\nabla^{\tilde T}_{\tilde V}\tilde V, \tilde T) = \tilde T[\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde V}\tilde T, \tilde T)] - \mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde V}\tilde V, \nabla^{\tilde T}_{\tilde T}\tilde T), \tag{3.4}$$

$$\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde V}\tilde T, \tilde T)|_{s=0} = \mathrm{g}_{\dot\gamma(t)}(D_V V, \dot\gamma(t)) + \mathbf{T}_{\dot\gamma(t)}(V), \tag{3.5}$$

$$\nabla^{\tilde T}_{\tilde T}\tilde T|_{s=0} = 0. \tag{3.6}$$

We obtain

$$\int_a^b \mathrm{g}_{\tilde T}(\tilde T, \tilde T)^{-\frac12}\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\nabla^{\tilde T}_{\tilde V}\tilde V, \tilde T)|_{s=0}dt = \mathrm{g}_{\dot\gamma(t)}(D_V V, \dot\gamma(t))|_a^b + \mathbf{T}_{\dot\gamma(t)}(V)|_a^b. \tag{3.7}$$

Note that

$$\mathrm{g}_{\tilde T}(\Omega^{\tilde T}(\tilde V, \tilde T)\tilde V, \tilde T)|_{s=0} = -\mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(V(t)), V(t)), \tag{3.8}$$

$$\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\tilde V, \nabla^{\tilde T}_{\tilde T}\tilde V)|_{s=0} = \mathrm{g}_{\dot\gamma(t)}(D_{\dot\gamma(t)}V(t), D_{\dot\gamma(t)}V(t)), \tag{3.9}$$

$$\mathrm{g}_{\tilde T}(\nabla^{\tilde T}_{\tilde T}\tilde V, \tilde T)|_{s=0} = \mathrm{g}_{\dot\gamma(t)}(D_{\dot\gamma(t)}V(t), \dot\gamma(t)), \tag{3.10}$$

and by (3.3) and (3.7), we obtain

$$\begin{aligned} L''(0) &= \mathrm{g}_{\dot\gamma(t)}(D_V V, \dot\gamma(t))|_a^b + \mathbf{T}_{\dot\gamma(t)}(V)|_a^b \\ &\quad + \int_a^b \{\mathrm{g}_{\dot\gamma(t)}(D_{\dot\gamma(t)}V^\perp, D_{\dot\gamma(t)}V^\perp) - \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(V^\perp), V^\perp)\}dt, \end{aligned} \tag{3.11}$$

where $V^\perp = V - \mathrm{g}_{\dot\gamma(t)}(V, \dot\gamma(t))\dot\gamma(t)$.

Remark 3.1 The second variation formula for variations with fixed endpoints is derived in [4] in a different way.

Next, the index form, denoted $I_{\gamma(t)}(X)$, is defined by

$$\begin{aligned} I_{\gamma(t)}(X) &= \mathrm{g}_{\dot\gamma(t)}(D_X X, \dot\gamma(t))|_a^b + \mathbf{T}_{\dot\gamma(t)}(X)|_a^b \\ &\quad + \int_a^b \{\mathrm{g}_{\dot\gamma(t)}(D_{\dot\gamma(t)}X, D_{\dot\gamma(t)}X) - \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(X), X)\}dt, \end{aligned} \tag{3.12}$$

where X is a vector field along $\gamma(t)$ and normal to it.

4 The Hypersurface and Fundament Group

Suppose the hypersurface N separates the Finsler manifold M. The mean normal (sur-tangent) curvatures of N is said to be semidefinite if they are nonnegative on one orientation of N and nonpositive on the other. Call a component $\mathbb{U}$ of $M - N$ exterior to N if the mean normal and sur-tangent curvatures of N are nonpositive on the orientation of N corresponding to $\mathbb{U}$. $\gamma : [a, b] \to M$ is said to be an N-curve if $\dot\gamma(a), \dot\gamma(b) \in N^\perp$ and an N-geodesic if it is a geodesic parametrized by arc-length. A variation $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ is said to be an N-variation if each Γ_s given by $\Gamma_s(t) = \Gamma(s, t)$ is an N-curve.

Theorem 4.1 Let M be a Finsler manifold with positive Ricci curvature. Let the compact connected hypersurface N separate M, and the mean normal and sur-tangent curvatures of N are semidefinite. If $\mathbb{U}$ is the component of $M - N$ exterior to N, then the natural homomorphism $\pi_1(\mathbb{U}) \to \pi_1(M)$ is surjective.

Proof Since the exactness of the sequence $\pi_1(\mathbb{U}) \to \pi_1(M) \to \pi_1(M, \mathbb{U})$, it remains only to show that $\pi_1(\mathbb{U}) \to \pi_1(M, \bar{\mathbb{U}})$ is trivial.

Let $\gamma : [a, b] \to M$ be a minimal $N-$geodesic such that $\dot\gamma(a)$ points out of $\mathbb{U}$ and $\dot\gamma(b)$ points into $\mathbb{U}$. Let Q_γ be the set of vector fields along $\gamma(t)$, normal to it, autoparallel, and unitary. If $X_1, X_2, \cdots, X_{n-1} \in \mathrm{Q}_\gamma$ are mutually orthogonal, by (3.12), we have

$$\sum_{m=1}^{n-1} I_{\gamma(t)}(X_m) = \sum_{m=1}^{n-1} S_{\dot\gamma(t)}(X, X)|_a^b + \sum_{m=1}^{n-1} \mathbf{T}_{\dot\gamma(t)}(X)|_a^b - \int_a^b Ric(\dot\gamma(t))dt. \tag{4.1}$$

Because of the assumed orientations of $\dot\gamma(a)$ and $\dot\gamma(b)$ and the hypothesis on Ricci curvature, $\sum\limits_{m=1}^{n-1} I_{\gamma(t)}(X_m) < 0$. It follows that $I_{\gamma(t)}(X_m) < 0$ for some m, so the N-variation $\Gamma(s, t)$ is length-decreasing. Therefore, γ could not be minimal.

Next, suppose $\alpha \in \pi_1(M, \bar{\mathbb{U}})$ and let $\gamma \in \alpha$. It is easily seen that γ is $\bar{\mathbb{U}}$ -homotopic to a path-product $\gamma_1\theta_1\gamma_2\theta_2\cdots\theta_{k-1}\gamma_k$, where each $\gamma_i : [a_i, b_i] \to M$ is a minimal N -geodesic intersecting N only at its endpoints, while $\theta_i : [b_i, a_{i+1}] \to M$ lies in N. By the argument given above, it is impossible that $\dot{\gamma}_i(a_i)$ points out of $\mathbb{U}$ while $\dot{\gamma}_i(b_i)$ points into $\mathbb{U}$, since γ_i is to be minimal. Therefore, γ lies in $\mathbb{U}$, α is trivial and so in $\pi_1(\mathbb{U}) \to \pi_1(M, \bar{\mathbb{U}})$ is trivial, which completes the proof.

Now let M be a 3 -dimensional compact Finsler manifold with positive Ricci curvature and suppose $\pi_1(M)$ cannot be generated by less than r elements. Suppose $M = \mathrm{B}_1 \cup \mathrm{B}_2, N = \mathrm{B}_1 \cap \mathrm{B}_2$, B_1 and B_2 are bounded by the smooth surface N of genus f. If B_1 is a homeomorph of a standard solid torus, then $\pi_1(\mathrm{B}_1)$ is a free group on f generators. Applying Theorem 4.1, we obtain

Corollary 4.2 Let M be a 3 -dimensional compact Finsler manifold with positive Ricci curvature, whose fundamental group cannot be generated by less than r elements, and let B be a standard solid torus of genus f. If $r > f$, there is no smooth embedding of B into M such that the boundary of B becomes a surface with semidefinite mean normal and sur-tangent curvatures and the interior of B goes onto the exterior of that surface.

5 The Intersection of Submanifolds in Finsler Manifold

Theorem 5.1 Let M^n be a n -dimensional complete connected Finsler manifold with positive flag curvature and let $\mathbb{V}^r$ and $\mathbb{W}^s$ be compact submanifolds with vanished normal and sur-tangent curvatures. If $r + s \geq n$, then $\mathbb{V}^r \cap \mathbb{W}^s \neq \emptyset$.

Proof Let $\mathbb{V}^r$ and $\mathbb{W}^s$ be any compact submanifolds with $r + s \geq n$. Suppose that $\mathbb{V}^r \cap \mathbb{W}^s = \emptyset$. Let $\gamma : [a, b] \to M$ be a minimal geodesic parametrized by arc-length, $\gamma(a) = p \in \mathbb{V}^r$, $\gamma(b) = q \in \mathbb{W}^s$, and it strikes $\mathbb{V}^r$ and $\mathbb{W}^s$ orthogonally. Let $\mathfrak{V}_a$ be the tangent space of $\mathbb{V}^r$ at p. By parallel translation along $\gamma(t)$, we get a submanifold $\mathfrak{V}_b$ of $\mathfrak{U}$, that is the tangent space to M^n at q. Since $\mathfrak{V}_a$ is orthogonal to $\gamma(t)$ at p, $\mathfrak{V}_b$ is also orthogonal to $\gamma(t)$ at q. Let $\mathfrak{W}$ be the tangent space to $\mathbb{W}^s$ at q. Then $\mathfrak{V}_b$ and $\mathfrak{W}$ are two submanifolds of the linear space $\mathfrak{U}$. Moreover, both $\mathfrak{V}_b$ and $\mathfrak{W}$ are orthogonal to γ at q. Thus the dimension of their intersection is

$$\dim(\mathfrak{V}_b \cap \mathfrak{W}) \geq r + s - (n - 1) \geq 1, \tag{5.1}$$

then there is at least a 1-dimensional submanifold in common between $\mathfrak{V}_b$ and $\mathfrak{W}$. But this simply means that there is a unit vector X_a tangent to $\mathbb{V}^r$ at p whose parallel translate is tangent to $\mathbb{W}^s$ at q. Let X be the parallel translate of X_a along γ. The term $-\int_a^b K(P(t), \dot{\gamma}(t))dt$ of the second variation formula is strictly negative by the flag curvature assumption, where $P(t) = \mathrm{span}\{\dot{\gamma}(t), X(t)\}$ is a family of tangent planes along $\gamma(t)$.

Since $\mathbb{V}^r$ and $\mathbb{W}^s$ are compact submanifolds with vanished normal and sur-tangent curvatures, we have $I_{\gamma(t)}(X) < 0$, thus $\gamma(t)$ cannot be minimizing, which leads to a contradiction.

6 k-th Ricci Curvature and Submanifolds in Finsler Manifold

Let M be an n-dimensional Finsler manifold. For any point $p \in M$, let $\mathrm{H}^m \subset T_pM$ be a m-plane spanned by m -mutually orthogonal unit tangent vectors $X_1, \cdots, X_m \in T_pM$, and $X \in T_pM$ be a tangent vector orthogonal to H^m. Then the H^m-X flag curvature of M is defined by

$$K(\mathrm{H}^m, X) := \sum_{i=1}^{m} K(P_i, X), \tag{6.1}$$

where $P_i = \mathrm{span}(X_i, X)$. Obviously, $K(\mathrm{H}^m, X)$ is independent of the choice of $X_1, \cdots, X_m$.

If $\gamma : [0, d] \to M$ is a geodesic and $\mathrm{H}_0^m \subset T_{\gamma(0)}M$ is a plane through $\gamma(0)$, we denote by H_t^m the parallel translate of H_0^m to $\gamma(d)$ along $\gamma(t)$. Now, we can state our theorem as follows:

Theorem 6.1 Let M be an n -dimensional complete connected Finsler manifold and N be a r -dimensional submanifold with nonpositive mean normal and sur-tangent curvatures. Let $p \in M$, assume that along each minimizing geodesic $\gamma : [0, d] \to M$ starting from p, we have for all m-plane $(m \le r)$ $\mathrm{H}_0^m \subset \dot\gamma(0)^\perp$, $K(\mathrm{H}_t^m, \dot\gamma(t)) \ge mc > 0$, where $\dot\gamma(0)^\perp$ is the orthogonal complement of $\dot\gamma(0)$ in $T_{\gamma(0)}M$. Then the distance $d(p, N) \le \frac{\pi}{2\sqrt{c}}$.

Proof Let the distance $d(p, N) := d$. Then let $\gamma(s), s \in [0, d]$ be a minimal geodesic in M parametrized by arc-length from p to $p_0 \in N$ which realizes the minimum distance from p to N. Since $\gamma(t)$ is the minimal geodesic, it strikes N orthogonally. Take a unit orthogonal basis $X_1, \cdots, X_r$ of $T_{p_0}N$ and let $E_i(t)$ be the parallel translate of X_i along $\gamma(t)$ and define $W_i(t) = \sin \frac{\pi t}{2d} E_i(t), i = 1, \cdots, r$. Each vector field $W_i(t)$ gives rise to a variation of the variational curves of the geodesic $\gamma(t)$ by keeping one end point p fixed and other end points on N. By the second variation formula of arc-length, for $i = 1, \cdots, r$, we have

$$\begin{aligned} I_{\gamma(t)}(W_i) &= \mathrm{g}_{\dot\gamma(t)}(D_{W_i}W_i, \dot\gamma(t))(d) + \mathbf{T}_{\dot\gamma(t)}(W_i)(d) \\ &\quad + \int_0^d \{\mathrm{g}_{\dot\gamma(t)}(D_{\dot\gamma(t)}W_i, D_{\dot\gamma(t)}W_i) - \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(W_i), W_i)\}dt. \end{aligned} \tag{6.2}$$

Since N is a r -dimensional compact submanifold with nonpositive mean normal and sur-tangent curvatures, we have

$$\sum_{i=1}^{r} I_{\gamma(t)}(W_i) \le \int_0^d \{r(\frac{\pi}{2d}\cos\frac{\pi t}{2d})^2 - (\sin\frac{\pi t}{2d})^2 \sum_{i=1}^{r} \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(E_i), E_i)\}dt. \tag{6.3}$$

By the assumption on the curvature of M, we know that

$$\sum_{j=1}^{m} \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(E_{i_j}), E_{i_j}) \ge mc \tag{6.4}$$

for indices $i_1, i_2, \cdots, i_m \in \{1, \cdots, r\}, i_j \ne i_l \ \ (j \ne l), m \le r$, we have

$$\begin{aligned} &\sum_{i=1}^{r} \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(E_i), E_i) = \frac{r}{mC_r^m} \sum_{1 \le i_1 < i_2 < \cdots < i_m \le r} \sum_{j=1}^{m} \mathrm{g}_{\dot\gamma(t)}(R_{\dot\gamma(t)}(E_{i_j}), E_{i_j}) \\ \ge\ & \frac{r}{mC_r^m} \sum_{1 \le i_1 < i_2 < \cdots < i_m \le r} mc \\ =\ & rc. \end{aligned} \tag{6.5}$$

Substituting (6.5) into (6.3), we find

$$\sum_{i=1}^{r} I_{\gamma(t)}(W_i) \leq \int_0^d \{r(\frac{\pi}{2d}\cos\frac{\pi t}{2d})^2 - rc(\sin\frac{\pi t}{2d})^2\}dt. \tag{6.6}$$

Suppose that $d(p,N) > \frac{\pi}{2\sqrt{c}}$, then

$$\sum_{i=1}^{r} I_{\gamma(t)}(W_i) \leq \int_0^d \{rc(\cos\frac{\pi t}{2d})^2 - rc(\sin\frac{\pi t}{2d})^2\}dt = 0. \tag{6.7}$$

Hence $I_{\gamma(t)}(W_i) < 0$, for some i, this leads to a contradiction that $\gamma(t)$ is the minimal length from p to N. Thus, we must have $d(p,N) \leq \frac{\pi}{2\sqrt{c}}$.

The following Corollary 6.2 is an immediate consequence of Theorem 6.1.

Corollary 6.2 Let M be an n -dimensional complete connected Finsler manifold and N be a r -dimensional submanifold with nonpositive mean normal and sur-tangent curvatures. If the k-th Ricci curvature $\mathrm{Ric}_{(k)}(M) \geq kc(k \leq r)$, then $d(p,N) \leq \frac{\pi}{2\sqrt{c}}$, for any $p \in M$.

Let N be a compact submanifold and d_0 be its diameter. For any point $p, q \in M$, there are $p_0, q_0 \in N$, such that $d(p,p_0) = d(p,N)$ and $d(q,q_0) = d(q,N)$, then

$$d(p,q) \leq d(p,p_0) + d(p_0,q_0) + d(q_0,q) \leq \frac{\pi}{2\sqrt{c}} + d_0 + \frac{\pi}{2\sqrt{c}} = \frac{\pi}{\sqrt{c}} + d_0.$$

Note that M is complete, then M is compact. Hence we obtain

Corollary 6.3 Let M be an n-dimensional complete connected Finsler manifold and N be a r-dimensional compact submanifold with nonpositive mean normal and sur-tangent curvatures. If the k-th Ricci curvature $\mathrm{Ric}_{(k)}(M) \geq kc(k \leq r)$, then M is compact.

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Finsler 流形中弧长第二变分与子流形

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摘 要: 本文研究了Finsler 流形中的子流形的相关问题. 利用文[23, 24]中引入的Finsler 流形中的切曲率和法曲率的概念, 计算出Finsler 流形中测地线的一个新的第二变分公式, 获得了关于Finsler子流形中几何不变量和拓扑不变量的一些新的关系, 推广了文[4]的许多结果.

关键词: 超切曲率; 法曲率; 第二变分公式; 紧致性

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