Vol. 34 (2014)
No. 3

数 学 杂 志
J. of Math. (PRC)

ON DUALLY FLAT AND CONFORMALLY FLAT (α, β)-METRICS

CHENG Xin-yue, ZHANG Ting, YUAN Min-gao
(*School of Math. and Statistics, Chongqing University of Technology, Chongqing 400054, China*)

Abstract: In this paper, from the relation between the sprays of two dually flat and conformally flat (α, β) -metrics, we obtain that locally dually flat and conformally flat Randers metrics are Minkowskian. Further, we extend the result to the non-Randers type and show that the locally dually flat and conformally flat (α, β)-metrics of non-Randers type must be Minkowskian under an extra condition.

Keywords: (α, β)-metric; dually flat Finsler metric; conformally flat Finsler metric, Minkowski metirc

2010 MR Subject Classification: 53B40; 53C60

Document code: A **Article ID:** 0255-7797(2014)03-0417-06

1 Introduction

The notion of dually flat metrics was first introduced by Amari and Nagaoka when they studied the information geometry on Riemannian space [1, 2]. Later on, the notion of locally dually flat Finsler metrics was introduced by Shen [3]. A Finsler metric $F = F(x, y)$ on an n-dimensional manifold M is called the locally dually flat Finsler metric if at every point there is a coordinate system (x^i) in which the geodesic coefficients are in the following form $G^i = -\frac{1}{2} g^{ij} H_{y^j}$, where $H = H(x, y)$ is a local scalar function on the tangent bundle TM of M and satisfies $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system. It is shown that a Finsler metric on an open subset $U \subset \mathbf{R}^n$ is dually flat if and only if it satisfies the following PDE $(F^2)_{x^k y^l} y^k - 2(F^2)_{x^l} = 0$. In this case, $H = -\frac{1}{6}(F^2)_{x^m} y^m$. Recently, Shen, Zhou and the second author studied locally dually flat Randers metrics $F = \alpha + \beta$ and classified locally dually flat Randers metrics $F = \alpha + \beta$ with isotropic S -curvature [4]. Later, Xia characterized locally dually flat (α, β)-metrics on an n-dimensional manifold $M (n \geq 3)$ [5].

The study on conformal properties has a long history. Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be conformally related if there is a scalar function $\sigma(x)$ on M such that $F = e^{\sigma(x)} \bar{F}$. A Finsler metric which is conformally related to a Minkowski metric is called conformally flat Finsler metric. In 1989, Ichijyō and Hashiguchi defined a

* **Received date:** 2012-04-13 **Accepted date:** 2013-01-04
Foundation item: Supported by National Natural Science Foundation of China (10971239).
Biography: Cheng Xinyue (1958–), male, born at Chongqing, professor, major in differential geometry and its applications.

conformally invariant Finsler connection in a Finsler space with (α, β)-metric and gave the condition for a Randers space to be conformally flat based on their connection (see [6]). Later, S. Kikuchi found a conformally invariant Finsler connection and gave a necessary and sufficient condition for a Finsler metric to be conformally flat by a system of partial differential equations under an extra condition (see [7]). By using Kikuchi's conformally invariant Finsler connection, Hojo, Matsumoto and Okubo studied conformally Berwald Finsler spaces and its applications to (α, β)-metrics (see [8]). Recently, Kang proved that any conformally flat Randers metric of scalar flag curvature is projectively flat and classified completely conformally flat Randers metrics of scalar flag curvature (see [9]). On the other hand, Bacso and the second author studied the global conformal transformations on a Finsler space (M, F). They obtain the relations between some important geometric quantities of F and their correspondences respectively, including Riemann curvatures, Ricci curvatures and S-curvatures (see [10, 11]). The Weyl theorem states that the projective and conformal properties of a Finsler metric determine the metric properties uniquely. Thus the conformal properties of a Finsler metric deserve extra attention.

In this paper, we study and classify locally dually flat and conformally flat (α, β)-metrics. Firstly, we can prove the following theorem.

Theorem 1.1 Let $F = \alpha + \beta$ be a locally dually flat Randers metric on an n-dimensional manifold M $(n \geq 3)$. Assume that F is conformally flat. Then it must be Minkowskian.

Further, following Xia's main result on locally dually flat (α, β)-metrics in [5], we study and characterize locally dually flat and conformally flat (α, β)-metrics of non-Randers type. We get the following theorem.

Theorem 1.2 Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β)-metric on an n-dimensional manifold M $(n \geq 3)$. Suppose that ϕ satisfies one of the following conditions:

(i) $\phi(s)$ is a polynomial of s with $\phi'(0) = 0$;

(ii) $\phi(s)$ is an analytic function with $\phi'(0) = \phi''(0) = 0$;

(iii) $\phi'(0) \neq 0,\ s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0$,

where $k_1,\ k_2$ and k_3 are constants. Then, if F is locally dually flat with α conformally flat, F must be Minkowskian.

2 Preliminary

Let M be an n-dimensional C^∞ mainfold and TM denotes the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(a) F is C^∞ on $TM\backslash\{0\}$;

(b) At any point $x \in M, F_x(y) := F(x, y)$ is a Minkowski norm on T_xM,

we call the pair (M, F) an n-dimensional Finsler manifold.

Let (M, F) be a Finsler manifold and $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^i y^j}$. For any non-zero vector $y = y^i \frac{\partial}{\partial x^i} \in T_xM$, F induces an inner product $\mathbf{g}_y$ on T_xM as $\mathbf{g}_y(u, v) := g_{ij}(x, y)u^i v^j$, where $u = u^i \frac{\partial}{\partial x^i} \in T_xM$, $v = v^i \frac{\partial}{\partial x^i} \in T_xM$.

The geodesic $\sigma=\sigma(t)$ of a Finsler metric F is characterized by the following system of 2nd order ordinary differential equations

$$\frac{d^2\sigma^i(t)}{dt^2}+2G^i(\sigma(t),\frac{d}{dt}\sigma(t))=0,$$

where $G^i:=\frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\}$, where $(g^{ij})=(g_{ij})^{-1}$. G^i are called the geodesic coefficients of F.

By the definition, an (α,β)-metric is a Finsler metric expressed in the following form

$$F=\alpha\phi(s),\ s=\frac{\beta}{\alpha},$$

where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form with $\|\beta_x\|_\alpha<b_0,\ x\in M$. It is proved (see [12]) that $F=\alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_0,b_0)$ satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\quad |s|\le b<b_0.$$

In particular, when $\phi=1+s$, the metirc $F=\alpha\phi(\beta/\alpha)$ is just the Randers metric $F=\alpha+\beta$. Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively. Denote

$$\begin{aligned}&r_{ij}:=(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),\\&s^i{}_j:=a^{il}s_{lj},\quad s_i:=b^js_{ji},\quad s_0:=s_iy^i,\quad r_{00}:=r_{ij}y^iy^j,\end{aligned}$$

where $(a^{ij}):=(a_{ij})^{-1}$ and $b_{i|j}$ denote the covariant derivative of β with respect to α. Then we have

Lemma 2.1 (see [12]) The geodesic coefficients of G^i are related to G^i_α by

$$G^i=G^i_\alpha+\alpha Qs^i{}_0+\{-2Q\alpha s_0+r_{00}\}\{\Psi b^i+\Theta\alpha^{-1}y^i\},\tag{2.1}$$

where $s^i{}_0:=s^i{}_jy^j$ and

$$Q:=\frac{\phi'}{\phi-s\phi'},\Theta:=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi\big[(\phi-s\phi')+(b^2-s^2)\phi''\big]},\Psi:=\frac{\phi''}{2\big[(\phi-s\phi')+(b^2-s^2)\phi'')\big]}.$$

In order to prove our theorems, we need some lemmas about locally dually flat (α,β)-metrics. Shen, Zhou and the second author first characterized locally dually flat Randers metrics and obtained the following lemma.

Lemma 2.2 (see [4]) Let $F=\alpha+\beta$ be a Randers metric on an n-dimensional manifold M. Then F is locally dually flat if and only if in an adapted coordinate system, β and α satisfy

$$r_{00}=\frac{2}{3}\theta\beta-\frac{5}{3}\tau\beta^2+[\tau+\frac{2}{3}(\tau b^2-b_m\theta^m)]\alpha^2,\tag{2.2}$$

$$s_{k0}=-\frac{1}{3}(\theta b_k-\beta\theta_k),\tag{2.3}$$

$$G^m_\alpha=\frac{1}{3}(2\theta+\tau\beta)y^m-\frac{1}{3}(\tau b^m-\theta^m)\alpha^2,\tag{2.4}$$

where $\tau=\tau(x)$ is a scalar function and $\theta=\theta_k y^k$ is a 1-form on M and $\theta^m := a^{mk}\theta_k$.

Later, Xia characterized locally dually flat (α,β) -metrics.

Lemma 2.3 (see [5]) Let $F=\alpha\phi(\beta/\alpha)$ be an (α,β) -metric on an n -dimensional manifold M $(n\geq 3)$. Suppose F is not Riemannian and ϕ satisfies one of the following:

(i) $\phi(s)$ is a polynomial of s with $\phi'(0)=0$;

(ii) $\phi(s)$ is an analytic function with $\phi'(0)=\phi''(0)=0$;

(iii) $\phi'(0)\neq 0,\ s(k_2-k_3s^2)(\phi\phi'-s\phi'^2-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0$,

where $k_1,\ k_2$ and k_3 are constants. Then F is locally dually flat on M if and only if α and β satisfy

$$s_{l0} = \frac{1}{3}(\beta\theta_l-\theta b_l), \tag{2.5}$$

$$r_{00} = \frac{2}{3}[\theta\beta-(\theta_l b^l)\alpha^2], \tag{2.6}$$

$$G^l_\alpha = \frac{1}{3}(2\theta y^l+\theta^l\alpha^2), \tag{2.7}$$

where $\theta:=\theta_i(x)y^i$ is a 1-form on M and $\theta^l:=a^{lk}\theta_k$.

3 Proof of Theorems

Now we are in the position to prove the theorems. First, we prove Theorem 1.1.

Proof of Theorem 1.1 Let $F=\alpha\phi(\beta/\alpha)$ and $\bar F=\bar\alpha\phi(\bar\beta/\bar\alpha)$ be two (α,β) -metrics. If F and $\bar F$ are conformally related, that is $F=e^{\sigma(x)}\bar F$, then we have the following relations:

$$\begin{aligned}
\bar\alpha &= e^{-\sigma(x)}\alpha, \qquad \bar\beta=e^{-\sigma(x)}\beta, \bar a_{ij}=e^{-2\sigma(x)}a_{ij}, \qquad \bar b_i=e^{-\sigma(x)}b_i,\\
\bar b_{i\|j} &= e^{-\sigma}(b_{i|j}+b_j\sigma_i-b_r\sigma^r a_{ij}), && (3.1)\\
\bar r_{ij} &= e^{-\sigma(x)}(r_{ij}+\frac{1}{2}\sigma_i b_j+\frac{1}{2}\sigma_j b_i-b_r\sigma^r a_{ij}), && (3.2)\\
\bar s_{ij} &= e^{-\sigma(x)}(s_{ij}+\frac{1}{2}\sigma_i b_j-\frac{1}{2}\sigma_j b_i), && (3.3)
\end{aligned}$$

where $\sigma_i:=\frac{\partial\sigma}{\partial x^i}$, $\sigma^i:=a^{ij}\sigma_j$, and "$\|$" denotes the covariant derivative with respect to $\bar\alpha$.

Let $F=\alpha+\beta$ and $\bar F=\bar\alpha+\bar\beta$ be two Randers metrics and $F=e^{\sigma(x)}\bar F$. Then the above relations still hold. Assume F is conformally flat, then $\bar F$ is Minkowskian. In this case, $\bar b_{i\|j}=0$ and (3.1), (3.2), (3.3) are reduced to:

$$b_{i|j} = b_r\sigma^r a_{ij}-b_j\sigma_i, \tag{3.4}$$

$$r_{ij} = b_r\sigma^r a_{ij}-\frac{1}{2}\sigma_i b_j-\frac{1}{2}\sigma_j b_i, \tag{3.5}$$

$$s_{ij} = \frac{1}{2}\sigma_j b_i-\frac{1}{2}\sigma_i b_j. \tag{3.6}$$

For any Finsler metric F, the geodesic coefficients G^i can be expressed as:

$$G^i=\frac{1}{4}g^{il}\{(F^2)_{x^k y^l}y^k-(F^2)_{x^l}\}. \tag{3.7}$$

In particular, for $\bar{\alpha}$ and α, by (3.7), their geodesic coefficients $G^i_{\bar{\alpha}}$ and G^i_{α} have the relation

$$G^i_{\bar{\alpha}} = G^i_{\alpha} - \sigma_0 y^i + \frac{1}{2}\alpha^2\sigma^i, \tag{3.8}$$

where $\sigma_0 := \sigma_k y^k$ and $\sigma^i := a^{il}\sigma_l$.

If F is locally dually flat, then Lemma 2.2 holds for F. Note that α is also conformally flat since F is conformally flat, then $\bar{\alpha}$ is Euclidean and $G^i_{\bar{\alpha}} = 0$. Combining (2.4) and (3.8) yields

$$\{\frac{1}{3}(2\theta + \tau\beta) - \sigma_0\}y^i = \{\frac{1}{3}(\tau b^i - \theta^i) - \frac{1}{2}\sigma^i\}\alpha^2.$$

For the dimension of manifold M satisfies $n \geq 3$ and α^2 is not divisible in this circumstances, we immediately have $\sigma^i = \frac{2}{3}(\tau b^i - \theta^i)$, $\sigma_0 = \frac{1}{3}(2\theta + \tau\beta)$. Comparing the above two equations, one easily has

$$\theta_i = \frac{1}{4}\tau b_i. \tag{3.9}$$

Combining (2.2), (3.5) and (3.9) we get

$$(\frac{3}{2}\tau\beta - \sigma_0)\beta = (t + \tau + \frac{1}{2}\tau b^2)\alpha^2, \tag{3.10}$$

where $t := b_i\sigma^i$.

When $n \geq 3$, α^2 is indivisible, then from (3.10) we have

$$\sigma_i = \frac{3}{2}\tau b_i, \tag{3.11}$$

$$t + \tau + \frac{1}{2}\tau b^2 = 0. \tag{3.12}$$

Plugging (3.11) into (3.12) yields $\tau(1 + 2b^2) = 0$. Considering that $1 + 2b^2 \neq 0$, one has $\tau = 0$. Then $\sigma_i = 0$, i.e., σ is a constant. In this case, F is Minkowskian.

In the end, we are going to prove Theorem 1.2.

Proof of Theorem 1.2 Assume that $F = \alpha\phi(\beta/\alpha)$ is an (α, β) -metric satisfying the conditions in Theorem 1.2, $\alpha = e^{\sigma(x)}\bar{\alpha}$ and α is conformally flat. Then $\bar{\alpha}$ is Euclidean and (2.5), (2.6), (2.7) in Lemma 2.3 hold. By (2.7) and (3.8) we have

$$(\frac{2}{3}\theta - \sigma_0)y^i = (-\frac{1}{2}\sigma^i - \frac{1}{3}\theta^i)\alpha^2. \tag{3.13}$$

Then by (3.13) and the fact that α^2 is indivisible when $n \geq 3$ again, naturally we get

$$\theta_i = \frac{3}{2}\sigma_i, \tag{3.14}$$

$$\theta^i = -\frac{3}{2}\sigma^i. \tag{3.15}$$

We use a_{ij} to lower the index of (3.15) and obtain

$$\theta_i = -\frac{3}{2}\sigma_i. \tag{3.16}$$

Comparing (3.14) with (3.16), instantly we conclude $\sigma_i = 0$ and $\theta_i = 0$. Then σ is a constant and obviously α is Euclidean. According to (2.5) and (2.6), we get $s_{ij} = 0$ and $r_{ij} = 0$, which implies that β is parallel with respect to α. Therefore, F is Minkowskian.

References

[1] Amari S-I. Differential-geometrical methods in statistics[M]. Springer Lecture Notes in Statistics, 28, Berlin: Springer-Verlag, 1985.

[2] Amari S-I, Nagaoka H. Methods of information geometry[M]. AMS Translation of Math. Monographs, 191, Oxford: Oxford University Press, 2000.

[3] Shen Z. Riemann-Finsler geometry with application to information geometry[M]. Chin. Ann. Math., 2006, 27B(1): 73–94.

[4] Cheng X, Shen Z, Zhou Y. On locally dually flat Finsler metrics[J]. International Journal of Mathematics, 2010, 21(11): 1531–1543.

[5] Xia Q. On locally dually flat (α, β)-metrics[J]. Differential Geometry and its Applications, 2011, 29: 233–243.

[6] Ichijyō Y, Hashuiguchi M. On the condition that a Randers space be conformally flat[J]. Rep. Fac. Sci. Kagoshima Univ., no. 1989, 22: 7–14.

[7] Kikuchi S. On the condition that a Finsler space be conformally flat[J]. Tensor, N. S., 1994, 55: 97–100.

[8] Hojo S-I, Matsumoto M, Okubo K. Theory of conformally Berwald Finsler spaces and its applications to (α, β)- metrics[J]. Balkan J. of Geometry and Its Applications, 2000, 5(1): 107–118.

[9] Kang L. On conformally flat Randers metrics[J]. Sci. Sin. Math., 2011, 41(5): 439-446.

[10] Bacso S, Cheng X. Finsler conformal transformalations and the curvature invariances[J]. Publ. Math. Debrecen, 2007, 70(1–2): 221–231.

[11] Cheng X, Shen Z. Finsler geometry-an approach via Randers spaces[M]. Beijing: Science Press, 2012.

[12] Chern S S, Shen Z. Rieman-Finsler geometry[M]. Nankai Tracts in Mathematics, Vol. 6, Beijing: World Scientific, 2005.

对偶平坦和共形平坦的(α, β)-度量

程新跃, 张 婷, 袁敏高

(重庆理工大学数学与统计学院, 重庆 400054)

摘要: 本文主要研究了对偶平坦和共形平坦的(α, β) -度量. 利用对偶平坦和共形平坦与其测地线的关系, 得到了局部对偶平坦和共形平坦的Randers度量是Minkowskian度量的结论. 进一步, 推广到非Randers型的情形, 我们证明了局部对偶平坦和共形平坦的非Randers型的(α, β) -度量在附加的条件下一定是Minkowskian度量.

关键词: (α, β)度量; 对偶平坦的Finsler度量; 共形平坦的Finsler度量; Minkowskian度量

MR(2010)**主题分类号**: 53B40; 53C60 **中图分类号**: O186.1