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THE NONSQUARE POINT OF GENERALIZED ORLICZ SEQUENCE SPACES WITH LUXEMBURG NORM

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Abstract: In this article, we investigate the (J)nonsquare point of generalized Orlicz sequence spaces. Analysing and combining the structures of the classical Orlicz spaces and the generalized ones, we get sufficient and necessary conditions of this property by the generating function, which not only generalizes the results of the classical Orlicz sequence spaces but also is easy to use.

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1 Introduction

James [1] introduced conception of (J)nonsquareness of linear normed spaces in 1964. Later, Schäffer [2] introduced conception of (S)nonsquareness of linear normed spaces. Then Chen [3] proved that the two conceptions of nonsquareness were equivalent. However, Wang and Shi [4] introduced (J)nonsquare point and (S)nonquare point and proved that they are not same. First, as an important property of linear normed spaces, the nonsquareness has a close relation with flatness, rotundity and basis. Second, it is "uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings" [5] that makes it play an important role in the application of approximation theory. On one hand, the nonsquare point is a meticulous depiction of nonsquareness. On the other hand, the nonsquare point is a powerful tool to study nonsquareness. So the research of nonsquare point of Banach space is very important. Because of the complication of generalized Orlicz sequence spaces, the discussion of nonsquare point has not be seen . In this article, we discuss the properties of generalized Orlicz sequence spaces and get the sufficient and necessary criterion of (J)nonsquare point as well.

In the sequel, let $\mathbb{N}$ be the set of all natural numbers, $\mathrm{Card}I = \mathrm{Cardinal}\ I$ where I is a subset of $\mathbb{N}$.

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Biography: Shi Zhongrui (1955–), male, born in Manzhouli, Inner Mongolia, professor, major in the theory of Orlicz spaces.

2 Some Lemmas

Definition 2.1 Let M be an Orlicz function, that is to say $M:(-\infty,+\infty)\to[0,+\infty]$ (M may reach the value of $+\infty$) is convex, even, left-continued on $[0,+\infty)$ and $M(0)=0$. For $u=\{u(i)\}_{i=1}^{\infty}$ is a sequence of real numbers, denote

$$\rho_M(u)=\sum_{i=1}^{\infty}M(u(i)),\ \ \widetilde{l_M}=\{u:\exists\lambda>0,\ \rho_M(\lambda u)<+\infty\}.$$

Then we can prove that $\widetilde{l_M}$ is a linear set. Define the function

$$\|u\|_{(M)}=\inf\{\lambda>0:\rho_M(\frac{u}{\lambda})\le 1\}$$

on $\widetilde{l_M}$, then $\{\widetilde{l_M},\ \|\cdot\|_{(M)}\}$ is called generalized Orlicz sequence space. Denote

$$\alpha=\sup\{u\ge 0:M(u)=0\},\ \beta=\sup\{u\ge 0:M(u)<+\infty\}.$$

Remark 1 $\forall 0<u_1<u_2$, since $M(u_1)=M(\frac{u_1}{u_2}u_2)\le\frac{u_1}{u_2}M(u_2)\le M(u_2)$, this yields that M is non-decreasing on $[0,+\infty)$. Thus $\lim\limits_{u\to\infty}M(u)$ exists (may be $+\infty$). Denote $M(\infty)=\lim\limits_{u\to\infty}M(u)$.

Remark 2 $\alpha=\beta=0\Longleftrightarrow M(u)=\begin{cases}0, & u=0,\\ +\infty, & u\ne 0.\end{cases}$ In fact, $\forall u>0$, $M(u)=+\infty$, we have $\beta=0$, thus $\alpha=\beta=0$. Conversely, when $\alpha=\beta=0$, $\forall u>0, M(u)=+\infty$. Since M is even, then $\forall u\ne 0, M(u)=+\infty$, combining with $M(0)=0$, we get

$$M(u)=\begin{cases}0, & u=0,\\ \infty, & u\ne 0,\end{cases}$$

meantime, $\widetilde{l_M}=\{0\}$.

Remark 3 $\alpha=\beta=+\infty\Longleftrightarrow M(u)\equiv 0$. In fact, $\forall u>0$, $M(u)=0$, we have $\alpha=+\infty$, thus $\alpha=\beta=+\infty$. Conversely, when $\alpha=\beta=+\infty$, $\forall u\ge 0$, $M(u)=0$. Noticing that M is even, we get $M(u)\equiv 0$. In this case, $\widetilde{l_M}=s$ (the set of all sequence of real number). Moreover $\forall u\in\widetilde{l_M}$, we have $\|u\|_{(M)}=0$, so $\|\cdot\|_{(M)}$ could not be a norm.

Summarily in the following we always assume: $\exists u_1>0, u_2>0$, s.t. $M(u_1)>0, M(u_2)<\infty$. Then we can prove that $\{\widetilde{l_M},\ \|\cdot\|_{(M)}\}$ forms a Banach space, denoted by $l_{(M)}$ and $\|\cdot\|_{(M)}$ is called Luxemburg norm.

Definition 2.2 [1] Let X be a norm linear space, $S(X)$ be the unit sphere. $x\in S(X)$ is called a (J) nonsquare point if $\forall y\in S(X), \min\{\|x+y\|,\|x-y\|\}<2$.

Lemma 2.3 [6] $\forall s\in R$, $M(\frac{s}{2})=\frac{1}{2}M(s)\Leftrightarrow\ \forall\lambda\in(0,1),\ M(\lambda s)=\lambda M(s)$.

Proof See [6].

Denote $s_l=\sup\{s<\beta:M(\frac{s}{2})=\frac{1}{2}M(s)\}$.

Lemma 2.4 If $u\in S(l_{(M)})$ is a (J)nonsquare point, then

(a) $\mathrm{Card}I_\beta(u)\le 1$;

(b) $\exists\lambda_0>1,\ \sum\limits_{i\notin I_\beta(u)} M(\lambda_0 u(i))<\infty$,

where $I_r(u)=\{i\in\mathbb{N}:|u(i)|=r\}$.

Proof First, we will prove (a) is correct. Otherwise, assume $\mathrm{Card}I_\beta(u)>1$, then $\exists i_1,i_2\in\mathbb{N}$ s.t. $|u(i_1)|=|u(i_2)|=\beta$.

Let

$$v(i)=\begin{cases} u(i_1), & i=i_1,\\ -u(i_2), & i=i_2,\\ u(i), & \text{otherwise.}\end{cases}$$

Then $\|v\|_{(M)}=\|u\|_{(M)}=1$. Since $\forall\lambda>1,\rho_M(\lambda\frac{u\pm v}{2})\ge M(\lambda\beta)=\infty(\beta>0)$. Therefore, $\left\|\frac{u\pm v}{2}\right\|_{(M)}\ge\frac{1}{\lambda}$. Hence, $\left\|\frac{u\pm v}{2}\right\|_{(M)}\ge 1$. Considering that $\left\|\frac{u\pm v}{2}\right\|_{(M)}\le 1$, we deduce $\left\|\frac{u\pm v}{2}\right\|_{(M)}=1$, which contradicts the fact that u is a (J)nonsquare point.

Second, we are going to prove (b) is correct.

(b)1 Assume $\mathrm{Card}I_\beta(u)=1$ ($\exists 1\,|u(i)|=\beta$) and $\forall\lambda>1,\sum\limits_{j\ne i}M(\lambda u(j))=\infty$. Set

$$v(j)=\begin{cases} u(i), & j=i,\\ -u(j), & j\ne i.\end{cases}$$

Then $\|v\|_{(M)}=\|u\|_{(M)}=1$. Since $\forall\lambda>1,\rho_M(\lambda\frac{u+v}{2})=M(\lambda u(i))=M(\lambda\beta)=\infty(\beta>0)$. Thus $\left\|\frac{u+v}{2}\right\|_{(M)}\ge 1$. Considering that $\left\|\frac{u+v}{2}\right\|_{(M)}\le 1$, we deduce $\left\|\frac{u+v}{2}\right\|_{(M)}=1$.

Since $\forall\lambda>1,\rho_M(\lambda\frac{u-v}{2})=\sum\limits_{j\ne i}M(\lambda u(j))=\infty$. Then $\left\|\frac{u-v}{2}\right\|_{(M)}\ge 1$. Considering that $\left\|\frac{u-v}{2}\right\|_{(M)}\le 1$, we have $\left\|\frac{u-v}{2}\right\|_{(M)}=1$. So $\left\|\frac{u\pm v}{2}\right\|_{(M)}=1$, which contradicts the fact that u is a (J)nonsquare point.

(b)2 Assume $\mathrm{Card}I_\beta(u)=0$ (i.e., $\forall i\in\mathbb{N}$, $|u(i)|<\beta$) and $\forall\lambda>1,\ \rho_M(\lambda u)=\infty$.

In case of $\beta>\alpha$, we claim that $\exists\lambda_0>1$ s.t. $\forall i,|\lambda_0 u(i)|<\beta$.

In fact, if $\beta=\infty$, since $\forall i\in\mathbb{N}$, $|u(i)|<\infty$, then $|\lambda_0 u(i)|<\infty=\beta$.

If $\beta<\infty$, assume $\forall\lambda>1,\exists i\in\mathbb{N}$ s.t. $|\lambda u(i)|\ge\beta$.

For $\lambda_1=1+1,\exists i_1$ s.t. $|\lambda_1 u(i_1)|\ge\beta$. By $I_\beta(u)=0$, $\exists 1<\lambda_1'<\lambda_1$ s.t. $\forall i\le i_1,|\lambda_1' u(i)|<\beta$.

For $\lambda_2=\min\{1+\frac12,\lambda_1'\},\exists i_2>i_1$ s.t. $|\lambda_2 u(i_2)|\ge\beta$. By $I_\beta(u)=0$, $\exists 1<\lambda_2'<\lambda_2$ s.t. $\forall i\le i_2,|\lambda_2' u(i)|<\beta$.

Continuing this process in such a way, we get a sequence $\lambda_n\searrow 1, i_n\nearrow\infty$ s.t. $|\lambda_n u(i_n)|\ge\beta$. Then $\forall\lambda>1,\exists n_0$ s.t. $\lambda>\lambda_{n_0},|u(i_n)|\ge\frac{\beta}{\lambda_n}>\frac{\beta}{\lambda}$, while $n\ge n_0$.

Set $\varepsilon=\frac{\beta-\alpha}{2},\eta=\frac{\beta}{\beta-\varepsilon}$, then $\eta>1$. So $|u(i_n)|>\frac{\beta}{\eta}=\beta-\varepsilon=\frac{\alpha+\beta}{2}>\alpha$, while $n\ge n_0$.

Therefore, $\rho_M(u)\ge\sum\limits_{n=1}^{\infty}M(u(i_n))\ge\sum\limits_{n=n_0}^{\infty}M(u(i_n))\ge\sum\limits_{n=n_0}^{\infty}M(\beta-\varepsilon)=\infty$, a contradiction to $\rho_M(u)\le 1$.

Take $\lambda_n\searrow 1,\lambda_1<\lambda_0$. Since $\rho_M(\lambda_1 u)=\infty$, then $\exists i_1>0$ s.t. $1\le\sum\limits_{i\le i_1}M(\lambda_1 u(i))<\infty,\sum\limits_{i>i_1}M(\lambda_1 u(i))=\infty$. Analogously, $\exists i_1'>i_1$ s.t. $1\le\sum\limits_{i_1<i\le i_1'}M(\lambda_1 u(i))<\infty$. Since

$\rho_M(\lambda_2 u) = \infty, \sum_{i\le i_1'} M(\lambda_2 u(i)) < \infty$, thus $\sum_{i>i_1'} M(\lambda_2 u(i)) = \infty$. Then $\exists i_2 > i_1'$ s.t. $1 \le \sum_{i_1'<i\le i_2} M(\lambda_2 u(i)) < \infty, \sum_{i>i_2} M(\lambda_2 u(i)) = \infty$. Analogously, $\exists i_2' > i_2$ s.t. $1 \le \sum_{i_2<i\le i_2'} M(\lambda_2 u(i)) < \infty$. Continuing this process in such a way, we get a sequence $i_1 < i_1' < i_2 < i_2' < \cdots$ s.t.

$$1 \le \sum_{i_{n-1}'<i\le i_n} M(\lambda_n u(i)) < \infty,\ \ 1 \le \sum_{i_n<i\le i_n'} M(\lambda_n u(i)) < \infty, n = 1, 2, \cdots, \quad \text{where } i_0' = 0.$$

Set

$$v(i) = \begin{cases} u(i), & i_{n-1}' < i \le i_n, n = 1, 2, \cdots, \\ 0, & \text{otherwise}, \end{cases}$$

$$w(i) = \begin{cases} u(i), & i_n < i \le i_n', n = 1, 2, \cdots, \\ 0, & \text{otherwise}. \end{cases}$$

Then $v + w = u$, $vw = 0$, $\rho_M(v) \le \rho_M(u) \le 1$, $\rho_M(w) \le \rho_M(u) \le 1$. Hence $\|v\|_{(M)} \le 1$, $\|w\|_{(M)} \le 1$. $\forall m \in \mathbb{N}$,

$$\rho_M(\lambda_m v) = \sum_{n=1}^{\infty} \sum_{i_{n-1}'<i\le i_n} M(\lambda_m v(i)) \ge \sum_{n=m}^{\infty} \sum_{i_{n-1}'<i\le i_n} M(\lambda_n v(i)) \ge \sum_{n=m}^{\infty} 1 = \infty.$$

Therefore, $\|v\|_{(M)} \ge 1$, hence $\|v\|_{(M)} = 1$. Similarly, we get $\|w\|_{(M)} = 1$.

Take $u' = v - w$, then $\forall i, |u'(i)| = |u(i)|$, hence $\|u'\|_{(M)} = \|u\|_{(M)} = 1$. Then $\left\|\frac{u+u'}{2}\right\|_{(M)} = \|v\|_{(M)} = 1$, $\left\|\frac{u-u'}{2}\right\|_{(M)} = \|w\|_{(M)} = 1$, which contradicts the fact that u is a (J)nonsquare point.

In case of $0 < \beta = \alpha < +\infty$, firstly, we claim that $\forall \lambda > 1, \exists i \in \mathbb{N}$ s.t. $|\lambda u(i)| > \alpha = \beta$. Otherwise, assume $\exists \lambda > 1$ s.t. $\forall i \in \mathbb{N}, |\lambda u(i)| \le \alpha$. Then

$$\rho_M(\lambda u) = \sum_{i=1}^{\infty} M(\lambda u(i)) \le \sum_{i=1}^{\infty} M(\alpha) = 0,$$

which contradicts $\|u\|_{(M)} = 1$.

Take $\lambda_n \searrow 1$, and $\{u(i_n)\}$, a sequence of$\{u(i)\}$ s.t. $|\lambda_n u(i_n)| \ge \beta = \alpha$. Then $|u(i_n)| \ge \frac{\beta}{\lambda_n}$. Thus $\lim_{n\to\infty} |u(i_n)| = \beta$. Therefore, $\forall \lambda > 1, \exists n_0$ s.t. $|\lambda u(i_n)| > \beta$, for $n \ge n_0$.

Let

$$v(j) = \begin{cases} (-1)^n u(i_n), & j = i_n, n = 1, 2, \cdots, \\ 0, & \text{otherwise}. \end{cases}$$

Hence $\forall \lambda > 1$,

$$\rho_M(\lambda v) = \sum_{n=1}^{\infty} M(\lambda u(i_n)) \ge \sum_{n=n_0}^{\infty} M(\lambda u(i_n)) = \infty,$$

$$\rho_M(\lambda \frac{u-v}{2}) = \sum_{n=1}^{\infty} M(\lambda u(i_{2n-1})) \ge \sum_{n=n_0}^{\infty} M(\lambda u(i_{2n-1})) = \infty,$$

$$\rho_M(\lambda \frac{u+v}{2}) = \sum_{n=1}^{\infty} M(\lambda u(i_{2n})) \ge \sum_{n=n_0}^{\infty} M(\lambda u(i_{2n})) = \infty,$$

then $\|v\|_{(M)} > \frac{1}{\lambda}$, $\left\|\frac{u-v}{2}\right\|_{(M)} > \frac{1}{\lambda}$, $\left\|\frac{u+v}{2}\right\|_{(M)} > \frac{1}{\lambda}$, thus $\|v\|_{(M)} = \left\|\frac{u-v}{2}\right\|_{(M)} = \left\|\frac{u+v}{2}\right\|_{(M)} = 1$, which contradicts the fact that u is a (J)nonsquare point.

3 Main Results

Theorem 3.1 Given $M(s_l) = 0$. $u \in S(l_{(M)})$ is a (J) nonsquare point $\Longleftrightarrow$

(a) $\mathrm{Card} I_\beta(u) \le 1$;

(b) $\exists \lambda_0 > 1$, $\sum\limits_{i \notin I_\beta(u)} M(\lambda_0 u(i)) < \infty$.

Proof Necessity By Lemma 2.4, we can obtain the conclusion immediately.

Sufficiency From $M(s_l) = 0$, we have $s_l \le \sup\{s \ge 0 : M(s) = 0\} = \alpha$. Noticing that $M(\frac{\alpha}{2}) = \frac{1}{2}M(\alpha) = 0$, we have $\alpha \le s_l$, hence $s_l = \alpha$.

(I) If $\mathrm{Card} I_\beta(u) = 1$, i.e., $\exists 1\, |u(i_0)| = \beta$. From (b), we know $\exists \lambda_0 > 1$ s.t. $\sum\limits_{i \ne i_0} M(\lambda_0 u(i)) < \infty$. $\forall \lambda_n \searrow 1$, by dominated convergence theorem, $\lim\limits_{\lambda \to 1^+} \sum\limits_{i \ne i_0} M(\lambda u(i)) = \sum\limits_{i \ne i_0} M(u(i))$.

$\forall v \in S(l_{(M)})$, in case of $u(i_0)v(i_0) \ge 0$, let $\lambda < 2$. Then

$$\left|\frac{\lambda u(i_0) - v(i_0)}{2}\right| \le \max\{\left|\frac{\lambda u(i_0)}{2}\right|, \left|\frac{v(i_0))}{2}\right|\} < \beta.$$

Hence $\lim\limits_{\lambda \to 1^+} M(\frac{\lambda u(i_0) - v(i_0)}{2}) = M(\frac{u(i_0) - v(i_0)}{2})$.

When $\beta > \alpha$, then $|u(i_0)| = \beta > \alpha = s_l$, hence $M(\frac{u(i_0)}{2}) < \frac{1}{2}M(u(i_0))$. Therefore,

$$\begin{aligned}
&\lim_{\lambda \to 1^+} \rho_M(\frac{\lambda u - v}{2}) = \lim_{\lambda \to 1^+} M(\frac{\lambda u(i_0) - v(i_0)}{2}) + \lim_{\lambda \to 1^+} \sum_{i \ne i_0} M(\frac{\lambda u(i) - v(i)}{2}) \\
\le& M(\frac{u(i_0) - v(i_0)}{2}) + \lim_{\lambda \to 1^+} \sum_{i \ne i_0} [\frac{1}{2}M(\lambda u(i)) + \frac{1}{2}M(v(i))] \\
\le& M(\frac{u(i_0)}{2}) + M(\frac{v(i_0)}{2}) + \frac{1}{2}\sum_{i \ne i_0} [M(u(i)) + M(v(i))] \\
<& \frac{1}{2}M(u(i_0)) + \frac{1}{2}M(v(i_0)) + \frac{1}{2}\sum_{i \ne i_0} [M(u(i)) + M(v(i))] = \frac{1}{2}[\rho_M(u) + \rho_M(v)] \le 1.
\end{aligned} \tag{$*$}$$

Hence $\exists \lambda > 1$ s.t. $\rho_M(\frac{\lambda u - v}{2}) < 1$. Then $\left\|\frac{\lambda u - v}{2}\right\|_{(M)} \le 1$, i.e., $\left\|\frac{u - \frac{v}{\lambda}}{2}\right\|_{(M)} \le \frac{1}{\lambda}$. Thus

$$\left\|\frac{u - v}{2}\right\|_{(M)} \le \left\|\frac{u - v}{2} - \frac{u - \frac{v}{\lambda}}{2}\right\|_{(M)} + \left\|\frac{u - \frac{v}{\lambda}}{2}\right\|_{(M)} \le \frac{1}{2}(1 - \frac{1}{\lambda}) + \frac{1}{\lambda} = \frac{1}{2}(1 + \frac{1}{\lambda}) < 1.$$

When $0 < \beta = \alpha < \infty$, then $\rho_M(u) \le 0$.

Therefore,

$$
\begin{aligned}
&\lim_{\lambda\to 1^+}\rho_M(\frac{\lambda u-v}{2})\\
= & \lim_{\lambda\to 1^+}M(\frac{\lambda u(i_0)-v(i_0)}{2})+\lim_{\lambda\to 1^+}\sum_{i\neq i_0}M(\frac{\lambda u(i)-v(i)}{2})\\
\leq & M(\frac{u(i_0)-v(i_0)}{2})+\lim_{\lambda\to 1^+}\sum_{i\neq i_0}[\frac{1}{2}M(\lambda u(i))+\frac{1}{2}M(v(i))]\\
\leq & \frac{1}{2}M(u(i_0))+\frac{1}{2}M(v(i_0))+\frac{1}{2}\sum_{i\neq i_0}[M(u(i))+M(v(i))]=\frac{1}{2}[\rho_M(u)+\rho_M(v)]\leq\frac{1}{2}<1.
\end{aligned}
$$

Hence $\exists\lambda>1$ s.t. $\rho_M(\frac{\lambda u-v}{2})<1$. Thus $\left\|\frac{u-v}{2}\right\|_{(M)}<1$.

In case of $u(i_0)v(i_0)<0$, then $-u(i_0)v(i_0)>0$, by the above arguments,

$$\left\|\frac{u+v}{2}\right\|_{(M)}=\left\|\frac{(-u)-v}{2}\right\|_{(M)}<1.$$

(II) If $\mathrm{Card}I_\beta(u)=0$, from (b), we know $\exists\lambda_0>1$ s.t. $\rho_M(\lambda_0 u)<\infty$. By dominated convergence theorem

$$\lim_{n\to\infty}\rho_M(\lambda_n u)=\sum_{i=1}^{\infty}\lim_{n\to\infty}M(\lambda_n u(i))=\sum_{i=1}^{\infty}M(u(i))=\rho_M(u).$$

From $\|u\|_{(M)}=1$, we have $\rho_M(\lambda_n u)>1$, then $\rho_M(u)\geq 1$, considering that $\rho_M(u)\leq 1$, then $\rho_M(u)=1$. Hence $\exists i_0$ s.t. $|u(i_0)|>\alpha$, then $M(\frac{u(i_0)}{2})<\frac{1}{2}M(u(i_0))$. From $\lim\limits_{n\to\infty}\rho_M(\lambda_n u)=\rho_M(u)$, we have $\lim\limits_{\lambda\to\ 1^+}\rho_M(\lambda u)=\rho_M(u)$.

If $u(i_0)v(i_0)\geq 0$, let $\lambda<2$, then

$$\left|\frac{\lambda u(i_0)-v(i_0)}{2}\right|\leq\max\{\left|\frac{\lambda u(i_0)}{2}\right|,\left|\frac{v(i_0))}{2}\right|\}<\beta.$$

Hence

$$\lim_{\lambda\to 1^+}M(\frac{\lambda u(i_0)-v(i_0)}{2})=M(\frac{u(i_0)-v(i_0)}{2}).$$

Repeating the above argument of (*), we get

$$\lim_{\lambda\to 1^+}\rho_M(\frac{\lambda u-v}{2})\leq 1,$$

then $\left\|\frac{u-v}{2}\right\|_{(M)}<1$.

If $u(i_0)v(i_0)<0$, then $-u(i_0)v(i_0)>0$, by the above arguments,

$$\left\|\frac{u+v}{2}\right\|_{(M)}=\left\|\frac{(-u)-v}{2}\right\|_{(M)}<1.$$

Combining (I) and (II), we get that u is a (J) nonsquare point.

Theorem 3.2 Given $M(s_l) > 0$. $u \in S(l_M)$ is a (J) nonsquare point $\Longleftrightarrow$

(a) $\mathrm{Card} I_\beta(u) \le 1$;

(b) $\exists \lambda_0 > 1$, $\sum\limits_{i \notin I_\beta(u)} M(\lambda_0 u(i)) < \infty$;

(c) (1) $\rho_M(u) < 1$; or (2) $\exists i$ s.t. $|u(i)| > s_l$; or (3) $M(s_l)\mathrm{Card} I_\theta(u) < 1$.

Proof Since $M(s_l) > 0$, we claim $\alpha = 0$. In fact, if $s_l = \infty$, by the hypothesis of M, $\exists u_1 > 0$ s.t. $M(u_1) > 0$. Then $\forall s \in (0, u_1)$, $M(\frac{s}{u_1} u_1) = \frac{s}{u_1} M(u_1) > 0$, hence $\alpha = 0$. If $s_l < \infty$, since $M(s_l) > 0$, then $\forall s \in (0, s_l)$, $M(s) = M(\frac{s}{s_l} s_l) = \frac{s}{s_l} M(s_l) > 0$, thus $\alpha = 0$.

Necessity By Lemma 2.4, (a) and (b) certainly hold.

Assume (c) does not hold, i.e., $\rho_M(u) = 1$, $\forall i, |u(i)| \le s_l$ and $M(s_l)\mathrm{Card} I_\theta(u) \ge 1$. Hence $\mathrm{Card} I_\theta(u) > 0$. Set $I_\theta(u) = \{i_j\}$.

In case of $M(s_l) \ge 1$, we will deduce that $\exists s_1$ s.t. $M(s_1) = 1$.

If $M(s_l) = +\infty$, then $s_l = \infty$. In fact, assume $s_l < \infty$. $\forall \lambda \in (0, 1)$, from Lemma 2.3, we have $M(\lambda s_l) = \lambda M(s_l) = \infty$, hence $\lambda s_l \ge \beta$, then $s_l > \beta$, a contradiction to $s_l \le \beta$. Hence by Remark 1, $\exists s \in (0, \infty)$ s.t. $1 < M(s) < \infty$. Take $\lambda = \frac{1}{M(s)}$, $s_1 = \lambda s$, by Lemma 2.3, we get $M(s_1) = M(\lambda s) = \lambda M(s) = 1$.

If $1 < M(s_l) < +\infty$, then $s_l < \infty$. In fact, assume $s_l = \infty$. From Lemma 2.3, $s_l \le \beta$, then $\beta = \infty$, noticing that $\alpha = 0$, hence $\alpha < \beta$. By Remark 1, $M(s_l) = \lim\limits_{u \to \infty} M(u)$, then $\exists s \in (\alpha, s_l)$ s.t. $1 < M(s) < \infty$. Take $n \in \mathbb{N}$ s.t. $n > \frac{M(s_l)}{M(s)}$, then

$$M(s_l) < nM(s) = nM(\frac{1}{n} ns) = n\frac{1}{n} M(ns) = M(ns).$$

Noticing that $ns < s_l$, M is non-decreased, hence $M(ns) \le M(s_l)$, a contradiction to $M(ns) > M(s_l)$. Take $\lambda = \frac{1}{M(s_l)}$, $s_1 = \lambda s_l$, from Lemma 2.3, we have

$$M(s_1) = M(\lambda s_l) = \lambda M(s_l) = 1.$$

If $M(s_l) = 1$, take $s_1 = s_l$. Set

$$v(i) = \begin{cases} s_1, & i = i_j, j = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\rho_M(v) = M(s_1) = 1$, $\forall \lambda > 1, \rho_M(\lambda v) = M(\lambda s_1) > M(s_1) = 1$, hence $\|v\|_{(M)} = 1$. Since $\forall i, |v(i)| \le s_l$, $\mathrm{supp} u \bigcap \mathrm{supp} v = \emptyset$, thus $\forall \lambda \in (1, 2)$,

$$\rho_M(\lambda \frac{u \pm v}{2}) = \sum_{i \in \mathrm{supp} u} M(\lambda \frac{u(i)}{2}) + \sum_{i \in \mathrm{supp} v} M(\lambda \frac{v(i)}{2}) = \frac{\lambda}{2}[\rho_M(u) + \rho_M(v)] = \lambda > 1,$$

then $\left\|\frac{u \pm v}{2}\right\|_{(M)} \ge \frac{1}{\lambda}$, therefore, $\left\|\frac{u \pm v}{2}\right\|_{(M)} \ge 1$, noticing that $\left\|\frac{u \pm v}{2}\right\|_{(M)} \le 1$, we deduce that $\left\|\frac{u \pm v}{2}\right\|_{(M)} = 1$, which contradicts the fact that u is a (J)nonsquare point.

In case of $M(s_l) < 1$, let $k = [\frac{1}{M(s_l)}] + 1$, then $(k-1)M(s_l) \le 1 < kM(s_l)$.

(I) If $(k-1)M(s_l) = 1$, then $\mathrm{Card} I_\theta(u) \ge \frac{1}{M(s_l)} = k - 1$. Set

$$v(i) = \begin{cases} s_l, & i = i_j, j = 1, 2, \cdots, k-1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\rho_M(v) = (k-1)M(s_l) = 1$,

$$\forall \lambda > 1, \rho_M(\lambda v) = (k-1)M(\lambda s_l) > (k-1)M(s_l) = 1,$$

hence $\|v\|_{(M)} = 1$, since $\forall i, |v(i)| \leq s_l$, supp$v \in I_\theta(u)$. Repeating the above arguments, we get $\left\|\frac{u\pm v}{2}\right\|_{(M)} = 1$, which contradicts the fact that u is a (J)nonsquare point.

(II) If $(k-1)M(s_l) < 1$, by the continuity of M on $[0, s_l]$, $\exists s_1 \in (0, s_l)$ s.t. $(k-1)M(s_l) + M(s_1) = 1$. Then Card$I_\theta(u) \geq \frac{1}{M(s_l)} > k-1$, hence Card$I_\theta(u) \geq k$. Set

$$v(i) = \begin{cases} s_l, & i = i_j, j = 1, 2, \cdots, k-1, \\ s_1, & i = i_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\rho_M(v) = kM(s_l) + M(s_1) = 1$, $\forall \lambda > 1$,

$$\rho_M(\lambda v) = kM(\lambda s_l) + M(\lambda s_1) > kM(s_l) + M(s_1) = 1.$$

Thus $\|v\|_{(M)} = 1$, and supp$v \subset I_\theta(u)$, $\forall i, |v(i)| \leq s_l$. Repeating the above arguments, we get $\left\|\frac{u\pm v}{2}\right\|_{(M)} = 1$, which contradicts the fact that u is a (J)nonsquare point.

Sufficiency $\forall v \in S(l_{(M)})$.

(I) If supp$u \cap$supp$v \neq \emptyset$, take $i_0 \in$supp$u\cap$supp$v$. From $\rho_M(\frac{u\pm v}{2}) \leq 1$, we have

$$\lim_{i\to\infty} M(\frac{u(i) \pm v(i)}{2}) = 0.$$

Noticing that $\alpha = 0$, then $\lim\limits_{i\to\infty} \frac{u(i)\pm v(i)}{2} = 0$. From $M(s_l) > 0$, we get $s_l > 0$, hence $\exists i' > i_0$ s.t. $\left|\frac{u(i)\pm v(i)}{2}\right| < s_l$, $\forall i \geq i'$. Then $\forall 1 < \lambda < 2$, $\left|\lambda\frac{u(i)\pm v(i)}{2}\right| < s_l$, when $i \geq i'$. Therefore,

$$M(\lambda\frac{u(i) \pm v(i)}{2}) = \lambda\frac{1}{\lambda}M(\lambda\frac{u(i) \pm v(i)}{2}) = \lambda M(\frac{1}{\lambda}\lambda\frac{u(i) \pm v(i)}{2}) = \lambda M(\frac{u(i) \pm v(i)}{2}).$$

In case of $u(i_0)v(i_0) > 0$.

(i) If Card$I_\beta(u) = 0$, then $\forall i, |u(i)| < \beta$. Therefore,

$$\forall i, \left|\frac{u(i) - v(i)}{2}\right| \leq \left|\frac{u(i)}{2}\right| + \left|\frac{v(i)}{2}\right| < \frac{1}{2}\beta + \frac{1}{2}\beta = \beta (\beta > 0).$$

Then

$$\begin{aligned} \lim_{\lambda\to 1^+} \rho_M(\lambda\frac{u-v}{2}) &= \lim_{\lambda\to 1^+}\sum_{i<i'} M(\lambda\frac{u(i)-v(i)}{2}) + \lim_{\lambda\to 1^+}\sum_{i\geq i'} M(\lambda\frac{u(i)-v(i)}{2}) \\ &= \sum_{i<i', i\neq i_0} M(\frac{u(i)-v(i)}{2}) + M(\frac{u(i_0)-v(i_0)}{2}) + \lim_{\lambda\to 1^+}\sum_{i\geq i'} \lambda M(\frac{u(i)-v(i)}{2}) \\ &< \sum_{i<i', i\neq i_0} M(\frac{u(i)-v(i)}{2}) + max\{M(\frac{u(i_0)}{2}), M(\frac{v(i_0)}{2})\} + \sum_{i\geq i'} M(\frac{u(i)-v(i)}{2}) \\ &\leq \sum_{i<i', i\neq i_0} \frac{M(u(i)) + M(v(i))}{2} + \frac{M(u(i_0)) + M(v(i_0))}{2} + \sum_{i\geq i'} \frac{M(u(i)) + M(v(i))}{2} \\ &= \frac{1}{2}[\rho_M(u) + \rho_M(u)] \leq 1. \end{aligned}$$

Hence $\exists\lambda > 1$ s.t. $\rho_M(\lambda\frac{u-v}{2}) < 1$, thus $\left\|\frac{u-v}{2}\right\|_{(M)} < 1$.

(ii) If $\mathrm{Card}I_\beta(u) = 1$, i.e., $\exists 1\, |u(i_1)| = \beta$. Then $\forall i \neq i_1, |u(i)| < \beta$.

When $i_1 \notin \mathrm{supp}u \cap \mathrm{supp}v$, then $v(i_1) = 0$, and $\left|\frac{u(i_1)-v(i_1)}{2}\right| = \frac{\beta}{2} < \beta (\beta > 0)$. $\forall i \neq i_1$,

$$\left|\frac{u(i)-v(i)}{2}\right| \leq \left|\frac{u(i)}{2}\right| + \left|\frac{v(i)}{2}\right| < \frac{1}{2}\beta + \frac{1}{2}\beta = \beta\ (\beta > 0).$$

Hence $\forall i \in \mathbb{N}$, $\left|\frac{u(i)-v(i)}{2}\right| < \beta$. By the progress of (i), we get $\left\|\frac{u-v}{2}\right\|_{(M)} < 1$.

When $i_1 \in \mathrm{supp}u \cap$ supp v, without loss of generality, take $i_0 = i_1$, then

$$\left|\frac{u(i_0)-v(i_0)}{2}\right| \leq \max\{\left|\frac{u(i_0)}{2}\right|, \left|\frac{v(i_0)}{2}\right|\} = \frac{1}{2}\beta < \beta.$$

$\forall i \neq i_0, |u(i)| < \beta$. Hence $\forall i, \left|\frac{u(i)-v(i)}{2}\right| < \beta$. By the progress of prove of (i), we get $\left\|\frac{u-v}{2}\right\|_{(M)} < 1$.

In case of $u(i_0)v(i_0) < 0$, then $-u(i_0)v(i_0) > 0$,

$$\left\|\frac{u+v}{2}\right\|_{(M)} = \left\|\frac{(-u)-v}{2}\right\|_{(M)} < 1.$$

(II) If $\mathrm{supp}u \cap \mathrm{supp}v = \emptyset$, i.e., $\mathrm{supp}v \subset \mathbb{N} \backslash$ $\mathrm{supp}u$.

While $\rho_M(u) < 1$, then

$$\begin{aligned}
&\lim_{\lambda\to 1^+}\rho_M(\lambda\frac{u\pm v}{2}) = \lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}u} M(\lambda\frac{u(i)}{2}) + \lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}v} M(\lambda\frac{v(i)}{2})\\
\leq &\lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}u}\frac{\lambda}{2}M(u(i)) + \lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}v}\frac{\lambda}{2}M(v(i)) \qquad (**)\\
= &\lim_{\lambda\to 1^+}\frac{\lambda}{2}[\rho_M(u) + \rho_M(u)] = \frac{1}{2}[\rho_M(u) + \rho_M(u)] < \frac{1}{2}\cdot\ 2\ = 1.
\end{aligned}$$

Hence $\exists\lambda > 1$ s.t. $\rho_M(\lambda\frac{u\pm v}{2}) < 1$, therefore, $\left\|\frac{u\pm v}{2}\right\|_{(M)} < 1$.

While $\exists i_0$, $|u(i_0)| > s_l$. Then

$$\begin{aligned}
&\lim_{i\to 1^+}\rho_M(\lambda\frac{u\pm v}{2})\\
= &\lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}u, i\neq i_0} M(\lambda\frac{u(i)}{2}) + \lim_{\lambda\to 1^+} M(\lambda\frac{u(i_0)}{2}) + \lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}v} M(\lambda\frac{v(i)}{2})\\
\leq &\lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}u, i\neq i_0}\frac{\lambda}{2}M(u(i)) + M(\frac{u(i_0)}{2}) + \lim_{\lambda\to 1^+}\sum_{i\in\mathrm{supp}v}\frac{\lambda}{2}M(v(i)) \qquad (***)\\
< &\sum_{i\in\mathrm{supp}u, i\neq i_0}\frac{1}{2}M(u(i)) + \frac{1}{2}M(u(i_0)) + \sum_{i\in\mathrm{supp}v}\frac{1}{2}M(v(i))\\
= &\frac{1}{2}[\rho_M(u) + \rho_M(v)] \leq 1.
\end{aligned}$$

Hence

$$\exists\lambda > 1 \ \text{s.t.}\rho_M(\lambda\frac{u \pm v}{2}) < 1,$$

therefore, $\left\|\frac{u\pm v}{2}\right\|_{(M)} < 1$.

While $M(s_l)\mathrm{Card}I_\theta(u) < 1$.

From $\mathrm{supp}u \cap \mathrm{supp}v = \emptyset$, we have $\mathrm{Card}I_\theta(u) > 0$, since $M(s_l) > 0$, then

$$\mathrm{Card}I_\theta(u) < \infty.$$

If $\forall i, |v(i)| < \beta$. From $\alpha = 0$, we have $\beta > \alpha = 0$. Combining with $\mathrm{Card}I_\theta(u) < \infty$, we have $\exists\lambda_0 > 1 \ \text{s.t.} |\lambda_0 v(i)| < \beta$,

$$\lim_{\lambda\to 1^+}\rho_M(\lambda v) = \lim_{\lambda\to 1^+}\sum_{\mathrm{supp}v} M(\lambda v(i)) = \sum_{\mathrm{supp}v}\lim_{\lambda\to 1^+} M(\lambda v(i)) = \sum_{\mathrm{supp}v} M(v(i)) = \rho_M(v) \leq 1.$$

Since $\|v\|_{(M)} = 1$, then

$$\forall\lambda > 1, \rho_M(\lambda v) > 1,$$

hence $\lim\limits_{\lambda\to 1^+}\rho_M(\lambda v) \geq 1$, then

$$\rho_M(v) = \lim_{\lambda\to 1^+}\rho_M(\lambda v) = 1.$$

Noticing that $M(s_l)\mathrm{Card}I_\theta(u) < 1$, $\mathrm{supp}v \subset I_\theta(u)$, we get $\exists i_0 \in \mathrm{supp}v$ s.t. $|v(i_0)| > s_l$. In fact, assume $\forall i, |v(i)| < s_l$, then $\rho_M(v) \leq M(s_l)\mathrm{Cardsupp}v \leq M(s_l)\mathrm{Card}I_\theta(u) < 1$, a contradiction to $\rho_M(v) = 1$. Exchange the positions of u and v in $(***)$, we get $\left\|\frac{u\pm v}{2}\right\|_{(M)} < 1$.

If $\exists i_0 \in\mathrm{supp}v$ s.t. $|v(i_0)| = \beta$.

When $s_l < \beta$, exchange the positions of u and v in $(***)$, we get $\left\|\frac{u\pm v}{2}\right\|_{(M)} < 1$.

When $s_l = \beta$, from $M(s_l)\mathrm{Card}I_\theta(u) < 1$, we have

$$\rho_M(v) \leq \sum_{i=1}^{\infty} M(\beta) = M(s_l)\mathrm{Card}I_\theta(u) < 1,$$

exchange the positions of u and v in $(**)$,we get $\left\|\frac{u\pm v}{2}\right\|_{(M)} < 1$.

Summarily, u is a (J)nonsquare point.

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关于赋Luxemburg范数广义Orlicz序列空间的(J)非方点

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摘要: 本文研究了赋Luxemburg范数广义Orlicz序列空间的(J)非方点. 通过分析、综合经典Orlicz空间与广义Orlicz空间的结构, 给出了用生成函数表达该性质的充分必要条件. 该结果不仅推广了经典空间的结果, 也是十分好用的.

关键词: 广义Orlicz序列空间; Luxemburg范数; (J)非方点

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