Vol. 34 (2014)
No. 2

数 学 杂 志
J. of Math. (PRC)

一类三阶时滞微分方程解的渐进稳定性

赵 静, 邓艳平

(广西民族大学理学院, 广西 南宁 530006)

摘 要: 本文研究了三阶非线性时滞微分方程解的渐近稳定性. 利用 Lyapunov 泛函, 得到了微分方程的零解是渐进稳定的, 这一结果推广了文献 [2] 的结果.

关键词: 时滞微分方程; 渐近稳定; Lyapunov 泛函

MR(2010) **主题分类号**: 34D20 **中图分类号**: O175.13

文献标识码: A **文章编号**: 0255-7797(2014)02-0319-05

1 引言

在时滞微分方程理论和应用领域中, 稳定性是非常重要的问题. 至今研究时滞微分方程有效的方法还是 Lyapunov 直接法. 对于时滞微分方程解的稳定性研究也得到了很多很好的结果 [1−6].

2007 年, Zhang 和 Si[3] 研究了形如

$$x'''(t)+g(x'(t))x''(t)+f(x(t),x'(t))+h(x(t))=0$$

的三阶非线性时滞微分方程解的渐进稳定性.

2010 年, Cemil Tunc [4] 研究了形如

$$\begin{aligned} & x'''(t)+g(x(t),x'(t))x''(t)+f(x(t-r(t)),x'(t-r(t)))+h(x(t-r(t))) \\ = \;& p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t)) \end{aligned}$$

的三阶非线性时滞微分方程，得到了当 $p=0$ 时此方程零解渐进稳定性的充分性判别准则.

本文, 我们将研究如下三阶非线性时滞微分方程零解的渐进稳定性

$$\begin{aligned} & x'''(t)+g_1(x(t),x'(t))x''(t)+g_2(x(t),x'(t))x'(t) \\ & +f(x(t-r(t)),x'(t-r(t)))+h(x(t-r(t))) \\ = \;& p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t)). \end{aligned} \tag{1.1}$$

*收稿日期: 2012-06-29 接收日期: 2013-01-14

基金项目: 广西民族大学校级科研项目 (2011MDYB034); 广西教育厅一般项目 (2013YB074); 国家民委项目 (12GXZ001).

作者简介: 赵静 (1979–), 女, 河北安国, 博士, 副教授, 主要研究方向: 微分方程及其应用.

显然, (1.1) 式等价于系统

$$\begin{aligned}
&x'(t)=y(t),y'(t)=z(t),\\
&z'(t)=-g_1(x(t),y(t))z(t)-g_2(x(t),y(t))y(t)-f(x(t),y(t))+h(x(t))\\
&+\int_{t-r(t)}^{t}f_x(x(s),y(s))y(s)ds+\int_{t-r(t)}^{t}f_y(x(s),y(s))z(s)ds+\int_{t-r(t)}^{t}h'(x(s))y(s)ds\\
&+p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t)), \qquad (1.2)
\end{aligned}$$

其中 $0\le r(t)\le\alpha$, α 为正常数, r,g_1,g_2,f,h,p 均为连续函数, $f(x,0)=h(0)=0$. 导函数 $r'(t)=\frac{dr}{dt}$, $g_{1x}(x,y)=\frac{\partial g_1(x,y)}{\partial x}$, $g_{2x}(x,y)=\frac{\partial g_2(x,y)}{\partial x}$, $f_x(x,y)=\frac{\partial f(x,y)}{\partial x}$, $f_y(x,y)=\frac{\partial f(x,y)}{\partial y},h'(t)=\frac{dh}{dt}$ 存在且连续. 假设 $r'(t)\le\beta,0<\beta<1$, $f(x(t-r(t)),y(t-r(t)))$, $h(x(t-r(t))$, $g_1(x(t),y(t))$, $g_2(x(t),y(t))$, $p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t))$ 关于 $x(t),y(t),x(t-r(t),y(t-r(t))$ 和 $z(t)$ 满足 Lipschitz 条件, 则方程有唯一解. 最后, 假设所有的解都是实值的.

2 主要结论

定理 设存在正常数 $a,b,c,\mu,\delta,\lambda_1,\lambda_2,K$ 和 L, 使得下列条件成立:

(1) $g_1(x,y)\ge\mu+a$, $g_2(x,y)\ge c$, $yg_{1x}(x,y)\le 0$, $yg_{2x}(x,y)\le 0$;

(2) $f(x,y)\mathrm{sgn}y\ge(b+\delta)|y|$, $-K\le f_x(x,y)\le 0$, $|f_y(x,y)|\le L$;

(3) $0<h'(x)<ab,\mathrm{sgn}h(x)=\mathrm{sgn}x$.

当 $\alpha<\min\left\{\frac{2a(\delta+c)}{a^2b+aK+aL+2\lambda_1},\frac{\mu}{ab+K+L+2\lambda_2}\right\}$ 时, 其中 $\lambda_1=\frac{a^2b+ab+aK+K}{2(1-\beta)},\lambda_2=\frac{aL+L}{2(1-\beta)}$, 则方程 (1.1) 的零解是渐进稳定的.

证 先定义一个 Lyapunov 函数

$$\begin{aligned}
V\ =\ & V(x_t,y_t,z_t)=a\int_0^x h(\xi)d\xi+h(x)y+\frac{1}{2}(ay+z)^2+a\int_0^y[g_1(x,\eta)-a]\eta d\eta\\
&+\int_0^y g_2(x,\eta)\eta d\eta+\int_0^y f(x,\eta)d\eta+\lambda_1\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds\\
&+\lambda_2\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds. \qquad (2.1)
\end{aligned}$$

显然, $V(0,0,0)=0$, 并且 (2.1) 式可以改写为

$$\begin{aligned}
&V=a\int_0^x h(\xi)d\xi-\frac{1}{2b}h(x)^2+\frac{b}{2}\left[y+\frac{h(x)}{b}\right]^2+\frac{1}{2}(ay+z)^2\\
&+a\int_0^y\left[g_1(x,\eta)+\frac{1}{a}g_2(x,\eta)-a\right]\eta d\eta\\
&+\int_0^y f(x,\eta)d\eta-\frac{b}{2}y^2+\lambda_1\int_{-r(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds+\lambda_2\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds. \qquad (2.2)
\end{aligned}$$

由条件 $0 < h'(x) < ab$, $\mathrm{sgn} h(x) = \mathrm{sgn} x$ 和 $g_1(x,y) \geq \mu + a$ 可得

$$\begin{aligned} a\int_0^x h(\xi)d\xi - \frac{1}{2b}h(x)^2 &= a\int_0^x h(\xi)d\xi - \frac{1}{b}\int_0^x h(\xi)h'(\xi)d\xi \\ &= \int_0^x [a-b^{-1}h'(\xi)]h(\xi)d\xi = b^{-1}\int_0^x [ab-h'(\xi)]h(\xi)d\xi > 0 \end{aligned}$$

和

$$a\int_0^y \left[g_1(x,\eta) + \frac{1}{a}g_2(x,\eta) - a\right]\eta d\eta \geq \frac{a\mu + c}{2}y^2.$$

另一方面, 由条件 $f(x,y)\mathrm{sgn} y \geq (b+\delta)|y|$, 可得

当 $y > 0$ 时, $f(x,y) \geq (b+\delta)y$, $f(x,y)y \geq (b+\delta)y^2$.

当 $y < 0$ 时, $f(x,y) \leq (b+\delta)y$, $f(x,y)y \geq (b+\delta)y^2$.

从而必有

$$\int_0^y f(x,\eta)d\eta - \frac{b}{2}y^2 = \int_0^y [f(x,\eta) - b\eta]d\eta \geq 0.$$

综上述

$$\begin{aligned} V &\geq b^{-1}\int_0^x [ab-h'(\xi)]h(\xi)d\xi + \frac{a\mu+c}{2}y^2 + \frac{1}{2}(ay+z)^2 \\ &\quad +\lambda_1\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \lambda_2\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds. \end{aligned}$$

显然对于这个不等式, 存在足够小的正常数 D_i $(i = 1,2,3)$, 使得

$$\begin{aligned} V &= D_1x^2 + D_2y^2 + D_3z^2 + \lambda_1\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \lambda_2\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds \\ &\geq D_4(x^2+y^2+z^2) + \lambda_1\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \lambda_2\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds \\ &\geq D_4(x^2+y^2+z^2), \end{aligned} \tag{2.3}$$

其中 $D_4 = \min\{D_1, D_2, D_3\}$, 从而 $x^2+y^2+z^2 \leq D_4^{-1}V(x_t,y_t,z_t)$, $y^2+z^2 \leq D_4^{-1}V(x_t,y_t,z_t)$.

接下来, 证明存在一个连续函数 $u(x) \geq 0$ 且 $u(|\varphi(0)|) \geq 0$, 使得 $u(|\varphi(0)|) \leq V(\varphi)$.

事实上, V 沿系统 (1.2) 的导数为

$$\begin{aligned} \frac{dV}{dt} &= -af(x,y)y + h'(x)y^2 - [g_{1x}(x,y) - a]z^2 + ay\int_0^y g_{1x}(x,\eta)\eta d\eta + y\int_0^y g_{2x}(x,\eta)\eta d\eta \\ &\quad +y\int_0^y f_x(x,\eta)d\eta - ag_2(x,y)y^2 + (ay+z)\int_{t-r(t)}^t h'(x(s))y(s)ds \\ &\quad +(ay+z)\int_{t-r(t)}^t f_x(x(s),y(s))y(s)ds + (ay+z)\int_{t-r(t)}^t f_y(x(s),y(s))z(s)ds + \lambda_1 r(t)y^2 \\ &\quad -\lambda_1(1-r'(t))\int_{t-r(t)}^t y^2(s)ds + \lambda_2 r(t)z^2 - \lambda_2(1-r'(t))\int_{t-r(t)}^t z^2(s)ds. \end{aligned} \tag{2.4}$$

根据定理假设和等式 $2|st|=s^2+t^2$, 可得

$$\begin{aligned}\frac{dV}{dt} \leq\ & [a(\delta+c)-(\frac{a^2b}{2}+\frac{aK}{2}+\frac{aL}{2}+\lambda_1)]y^2-[\mu-(\frac{ab}{2}+\frac{K}{2}+\frac{L}{2}+\lambda_2)]z^2 \\ & +[\frac{a^2b}{2}+\frac{ab}{2}+\frac{aK}{2}+\frac{K}{2}-\lambda_1(1-\beta)]\int_{t-r(t)}^{t} y^2(s)ds \\ & +[\frac{aL}{2}+\frac{L}{2}-\lambda_2(1-\beta)]\int_{t-r(t)}^{t} z^2(s)ds, \end{aligned} \tag{2.5}$$

从而 $\frac{dV}{dt}\leq 0$. 易证集合 $Z=\{\varphi\in C:\dot{V}(\varphi)=0\}$ 的最大不变集 $Q=\{0\}$. 于是方程 (1.1) 满足 $\frac{dV(x_t,y_t,z_t)}{dt}=0$ 的唯一解是 $x=y=z=0$. 因此, 方程 (1.1) 的零解是渐进稳定的 (根据见参考文献 [5]). 证毕.

下面给出以下的一个三阶非线性时滞微分方程来阐述主要结论的具体应用. 例如三阶非线性时滞微分方程零解的渐进稳定性

$$\begin{aligned} & x'''(t)+[1+e^{-x(t)x'(t)}+\frac{1}{3+\cos(x'(t))}]x''(t)+[3+e^{-x(t)x'(t)}]x'(t) \\ & +x'(t-r)+\frac{1}{10}x(t-r(t))=0. \end{aligned} \tag{2.6}$$

显然, (1.1) 式等价于系统

$$\begin{aligned} & x'(t)=y(t), y'(t)=z(t), \\ & z'(t)=-[1+e^{-x(t)x'(t)}+\frac{1}{3+\cos(x'(t))}]z(t)-[3+e^{-x(t)x'(t)}]y(t)-x'(t-r) \\ & +\frac{1}{10}x(t-r(t))+\int_{t-r(t)}^{t}\frac{4-y^2(s)-y(s)}{(4+y^2(s))^2}z(s)ds+\frac{1}{10}\int_{t-r(t)}^{t}y(s)ds, \end{aligned}$$

其中

$$g_1(x,y)=2+\sin(x)+\frac{1}{3+\cos(y)}, g_2(x,y)=3+\frac{1}{1+x^2+y^2}, f(x,y)=y, h(x)=\frac{1}{10}x.$$

显然

(1) $g_1(x,y)=1+e^{-xy}+\frac{1}{3+\cos(y)}\geq 1+\frac{1}{4}:=\mu+a,\ \ yg_{1x}(x,y)=-y^2e^{-xy}\leq 0,$

$$g_2(x,y)=3+e^{-xy}\geq 3:=c, yg_{2x}(x,y)=-y^2e^{-xy}\leq 0;$$

(2) $f(x,y)\mathrm{sgn}y=y\ \mathrm{sgn}y=|y|=(\frac{1}{2}+\frac{1}{2})|y|:=(b+\delta)|y|,$

$$-1:=-K<f_x(x,y)=0, |f_y(x,y)|=1:=L;$$

(3) $h(x)=\frac{1}{10}x,\ \ h(0)=0,\ \ 0<h'(x)=\frac{1}{10}<ab=\frac{1}{8}$, $\mathrm{sgn}h(x)=\mathrm{sgn}x$.
即定理的所有条件被满足, 因此方程 (2.6) 的零解是渐近稳定的.

参 考 文 献

[1] Cemil T. Global stability of solutions of certain third-order nonlinear differential equations[J]. Pan-American Mathematical Journal, 2004, 14(4): 31–37.

[2] Cemil T. Some new stability and boundedness results of solutions of Liénard type equations with a deviating argument[J]. Nonlinear Analysis: Hybrid Systems, 2010, 4(1): 85–91.

[3] Zhang Lijuan, Si Ligeng. Globally asymptotic stability of a class of third order nonlinear system[J]. Acta Math. Appl. Sin., 2007, 30(1): 99–103.

[4] Cemil T. On the stability and boundedness of nonlinear third order delay differential equations with delay[J]. Filomat, 2010, 24(3): 1–10.

[5] Sinha A S C. On stability of solutions of some third and fourth order delay-differential equations [J]. Infor. Control, 1973, 23: 165–172.

[6] Zhang Zaiyun, Liu Zhenhai. Stability analysis of heat flow with boundary time-varying delay effect[J]. Nonlinear Analysis, 2010, 73: 1878–1889.

[7] 郑祖庥. 泛函微分方程理论 [M]. 安徽: 安徽教育出版社, 1994.

[8] Liu Zhenhai, Han Jiangfeng. Boundary value problems for second order impulsive functional differential equations[J]. Dynamic Systems and Applications, 2011, 20: 369–382.

[9] Hale J. Theory of functional differential equations[M]. New York, Heidelberg: Springer-Verlag, 1977.

[10] Burton T A. Stability and periodic solutions of ordinary and functional differential equations[M]. Orlando: Academic Press, 1985.

ASYMPTOTIC STABILITY OF A CERTAIN THIRD-ORDER DELAY DIFFERENTIAL EQUATION

ZHAO Jing, DENG Yan-ping

(*School of Science, Guangxi University for Natinonalities, Nanning 530006, China*)

Abstract: In this paper, we study the asymptotic stability of zero solution of the third-order nonlinear delay differential equation. By defining a Lyapunov function, we obtain that the zero solution of the differential equation is asymptotic stable, which extends the results in [2].

Keywords: delay differential; asymptotical stability; Lyapunov function

2010 MR Subject Classification: 34D20