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常利率下复合泊松 - 更新风险模型的破产问题

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摘 要: 本文研究了常数利率下, 保费收入为复合 Poisson 过程, 理赔到达过程为一般更新过程的风险模型. 利用离散化的方法, 获得了该风险模型的破产概率、破产时余额分布及破产前瞬间余额分布的级数展开式, 推广了文 [1] 和文 [2] 中的相关结果.

关键词: 更新过程; 破产概率; 破产时余额分布; 破产前瞬间余额分布
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1 引言

经典的复合 Poisson 风险模型 $R(t) = u + ct + \sum\limits_{i=1}^{N(t)} X_i$ 假定按照单位时间常速率取得保单, 且每张保单的保险费也相同, 但在实际应用中收到的保单数和每张保单的保费都是随机的, 所以近年来许多学者对其进行了推广, 一方面用复合 Poisson 过程来描述保费收入, 且用更一般的点过程（更新过程、Cox 过程）来描述理赔次数；另一方面也考虑在经济环境中利率、通货膨胀率、随机干扰等因素的影响, 其中带利率的风险模型是当前风险理论研究的热点之一 [1]−[7].

吴荣 [1] 讨论了保费在单位时间内以常数速率连续收取、理赔到达过程为一般更新过程、带常数利率的风险模型 $R(t) = ue^{\delta t} + \frac{c(e^{\delta t}-1)}{\delta} - \sum\limits_{i=1}^{N(t)} X_i e^{\delta(t-T_i)}$, 给出了几个重要的精算诊断量如破产概率、破产时余额的分布、破产前瞬间余额分布等的级数展式；朱柘琍 [2] 对保费收取次数为 Poisson 过程、每次收取保费为常数 C、理赔到达过程为一般更新过程、带常数利率的风险模型 $U_\delta(t) = ue^{\delta t} + C\sum\limits_{i=1}^{M(t)} e^{\delta(t-K_i)} - \sum\limits_{i=1}^{N(t)} X_i e^{\delta(t-T_i)}$ 讨论了类似的问题. 本文将研究带常数利率的保费收入为复合 Poisson 过程, 理赔到达过程为一般更新过程的风险模型 $U_\delta(t) = ue^{\delta t} + \sum\limits_{i=1}^{M(t)} Y_i e^{\delta(t-K_i)} - \sum\limits_{i=1}^{N(t)} X_i e^{\delta(t-T_i)}$, 推广吴荣 [1] 和朱柘琍 [2] 的相关结论.

2 模型的建立和几个主要引理

设 $(\Omega, \mathcal{F}, P)$ 为一个完备的概率空间, 以下所涉及到的随机变量均为该空间上的随机变量. $\{N(t), t \geqslant 0\}$ 为一普通的更新过程, 它表示 $(0, t]$ 时间间隔内的理赔次数, 相应的更新流为 $0 = T_0 < T_1 < T_2 < \cdots < T_n < \cdots$, 它表示各次理赔时刻, 更新间隔

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$W_n = T_n - T_{n-1}, n = 1, 2, \cdots$ 独立同分布, 其分布函数为 $\gamma(x)$. $\{M(t), t \geq 0\}$ 为一参数为 λ 的 Poisson 过程, 它表示 $(0, t]$ 时间间隔内收取保费的次数, 相应的 Poisson 流为 $0 = K_0 < K_1 < K_2 < \cdots < K_n < \cdots$, 它表示各次保费收取时刻. $\{X_n, n = 1, 2, \cdots\}$, $\{Y_n, n = 1, 2, \cdots\}$ 均为独立同分布的非负随机变量序列, 分别表示各次理赔额和各次投保额, 其分布函数分别为 $F(x)$ 和 $G(x)$. 于是 $X(t) = \sum\limits_{i=1}^{N(t)} X_i$ 表示 $(0, t]$ 时间间隔内的理赔总额, $Y(t) = \sum\limits_{i=1}^{M(t)} X_i$ 表示 $(0, t]$ 时间间隔内收取的保费总额. 假定 $\{X_n, n = 1, 2, \cdots\}$, $\{Y_n, n = 1, 2, \cdots\}$, $\{N_t, t \geq 0\}$, $\{M_t, t \geq 0\}$ 相互独立.

若保险公司除保费收入之外, 还以常数利率 $\delta(\delta > 0)$ 获取资产余额的利息, 于是考虑如下带常数利率的一般更新风险模型 $dU_\delta(t) = \delta U_\delta(t)dt + dY(t) - dX(t), U_\delta(0) = u \geq 0$. 即

$$\begin{aligned} U_\delta(t) &= ue^{\delta t} + \int_0^t e^{\delta(t-v)} dY(v) - \int_0^t e^{\delta(t-v)} dX(v) \\ &= ue^{\delta t} + \sum_{i=1}^{M(t)} Y_i e^{\delta(t-K_i)} - \sum_{i=1}^{N(t)} X_i e^{\delta(t-T_i)}. \end{aligned}$$

记 $\psi_\delta(u) = P\{\bigcup\limits_{t\geq 0}(U_\delta(t) < 0) \mid U_\delta(0) = u\}$, $\psi_\delta(u)$ 表示保险公司初始准备金为 u 时风险过程 $\{U_\delta(t), t \geq 0\}$ 的破产概率. 用 T 表示相应的破产时, 即 $T = \inf\{t > 0 : U_\delta(t) < 0\}$, 则 T 是一个停时, 也有 $\psi_\delta(u) = P\{T < \infty \mid U_\delta(0) = u\}$. 注意到破产只有可能在索赔时刻发生, 所以

$$\psi_\delta(u) = P\{\bigcup_{n=1}^{\infty}(T = T_n) \mid U_\delta(0) = u\} = P\{\bigcup_{n=1}^{\infty}(U_\delta(T_n) < 0) \mid U_\delta(0) = u\}.$$

初始准备金为 u 时风险过程 $\{U_\delta(t), t \geq 0\}$ 的生存概率记为 $\Phi_\delta(u)$, 即 $\Phi_\delta(u) = 1 - \psi_\delta(u)$.

首先我们准备几个引理.

引理 1 [7] 对所有的 $0 \leq s < t$ 及整数 $m, n(m < n)$, 在 $M(s) = m, M(t) = n$ 的条件下 $(K_{m+1} - s, K_{m+2} - s, \cdots, K_n - s)$ 的条件分布密度为

$$f(s_1, s_2, \cdots, s_{n-m}) = \frac{(n-m)!}{(t-s)^{n-m}} I_{0<s_1<s_2<\cdots<s_{n-m}\leq t-s}\ (s_1, s_2, \cdots, s_{n-m}).$$

也就是说, 若 $U_1, U_2, \cdots, U_{n-m}$ 为 $[0, t-s]$ 上 $n-m$ 个均匀分布的独立的随机变量, $U_{(i)}$, $i = 1, 2, \cdots, n-m$ 为相应的顺序统计量, 则在 $M(s) = m, M(t) = n$ 的条件下 $(K_{m+1} - s, K_{m+2} - s, \cdots, K_n - s)$ 的条件分布密度与 $(U_{(1)}, U_{(2)}, \cdots, U_{(n-m)})$ 的分布密度相同. 特别地, 若令 $s = 0, m = 0$, 则有在 $M(t) = n$ 的条件下 $(K_1, K_2, \cdots, K_n)$ 的条件分布密度为

$$f(s_1, s_2, \cdots, s_n) = \frac{n!}{t^n} I_{0<s_1<s_2<\cdots<s_n\leq t}(s_1, s_2, \cdots, s_n).$$

即在 $M(t) = n$ 的条件下 $(K_1, K_2, \cdots, K_n)$ 的条件分布密度与 $(U_{(1)}, U_{(2)}, \cdots, U_{(n)})$ 的分布密度相同, 其中 $U_{(i)}, i = 1, 2, \cdots, n$ 表示 $[0, t]$ 上 n 个均匀分布的独立的随机变量 $U_1, U_2, \cdots, U_n$ 的第 i 个次序统计量.

引理 2 $P(\sum\limits_{i=m+1}^{n} Y_i e^{-\delta(K_i-t)} \le z \mid M(t)=m, M(t+s)=n) = P(\sum\limits_{i=1}^{n-m} Y_i e^{-\delta U_{(i)}} \le z)$, 其中 $U_{(i)}, i=1,2,\cdots,n-m$ 表示 $[0,s]$ 上 $n-m$ 个均匀分布的独立的随机变量 $U_1, U_2, \cdots, U_{n-m}$ 的第 i 个次序统计量. 特别地, 取 $t=0, m=0$, 则有

$$P(\sum_{i=1}^{n} Y_i e^{-\delta K_i} \le z \mid M(s)=n) = P(\sum_{i=1}^{n} Y_i e^{-\delta U_{(i)}} \le z),$$

其中 $U_{(i)}, i=1,2,\cdots,n$ 表示 $[0,s]$ 上 n 个均匀分布的独立的随机变量 $U_1, U_2, \cdots, U_n$ 的第 i 个次序统计量.

证

$$\begin{aligned}
& P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A, (K_{m+1}-t, K_{m+2}-t, K_n-t) \in B \mid M(t)=m, M(t+s)=n) \\
= \ & P((K_{m+1}-t, K_{m+2}-t, K_n-t) \in B \mid M(t)=m, M(t+s)=n) \\
& \times P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A \mid (K_{m+1}-t, K_{m+2}-t, K_n-t) \in B, M(t)=m, M(t+s)=n).
\end{aligned}$$

注意到 $\{Y_n, n=1,2,\cdots\}$ 与 $\{M_t, t\ge 0\}, \{K_n, n=1,2,\cdots\}$ 均相互独立, 所以

$$\begin{aligned}
& P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A \mid (K_{m+1}-t, K_{m+2}-t, K_n-t) \in B, M(t)=m, M(t+s)=n) \\
= \ & P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A).
\end{aligned}$$

从而由引理 1 知

$$\begin{aligned}
& P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A, (K_{m+1}-t, K_{m+2}-t, K_n-t) \in B \mid M(t)=m, M(t+s)=n) \\
= \ & P((U_{(1)}, U_{(2)}, \cdots, U_{(n-m)}) \in B) P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A),
\end{aligned}$$

其中 $U_{(i)}, i=1,2,\cdots,n-m$ 表示 $[0,s]$ 上 $n-m$ 个均匀分布的独立的随机变量 $U_1, U_2, \cdots, U_{n-m}$ 的第 i 个次序统计量. 注意到 $\{Y_n, n=1,2,\cdots\}$ 独立同分布, 则有

$$\begin{aligned}
& P((Y_{m+1}, Y_{m+2}, \cdots, Y_n) \in A, \\
& (K_{m+1}-t, K_{m+2}-t, K_n-t) \in B \mid M(t)=m, M(t+s)=n) \\
= \ & P((U_{(1)}, U_{(2)}, \cdots, U_{(n-m)}) \in B) P((Y_1, Y_2, \cdots, Y_{n-m}) \in A),
\end{aligned}$$

于是

$$P(\sum_{i=m+1}^{n} Y_i e^{-\delta(K_i-t)} \le z \mid M(t)=m, M(t+s)=n) = P(\sum_{i=1}^{n-m} Y_i e^{-\delta U_{(i)}} \le z).$$

引理 3 保费总额折现过程 $\{L_\delta(t) = \sum\limits_{i=1}^{M(t)} Y_i e^{-\delta K_i}, t \ge 0\}$ 满足

$$L_\delta(T_n) - L_\delta(T_{n-1}) \stackrel{d}{=} e^{-\delta T_{n-1}} L_\delta(W_n),$$

其中 $X \stackrel{d}{=} Y$ 表示随机变量 X、Y 有相同的分布函数.

证 首先证明下面 (2.1) 式

$$
\begin{aligned}
& P(L_\delta(t+s)-L_\delta(t)\le z\mid M(t)=m,M(t+s)=n)\\
= \quad & P(e^{-\delta t}\sum_{i=1}^{n-m}Y_ie^{-\delta K_i}\le z\mid M(s)=n-m), \qquad (2.1)\\
& P(L_\delta(t+s)-L_\delta(t)\le z\mid M(t)=m,M(t+s)=n)\\
= \quad & P(\sum_{i=M(t)+1}^{M(t+s)}Y_ie^{-\delta K_i}\le z\mid M(t)=m,M(t+s)=n)\\
= \quad & P(\sum_{i=m+1}^{n}Y_ie^{-\delta(K_i-t)}\le ze^{\delta t}\mid M(t)=m,M(t+s)=n)=P(\sum_{i=1}^{n-m}Y_ie^{-\delta U_{(i)}}\le ze^{\delta t})\\
= \quad & P(\sum_{i=1}^{n-m}Y_ie^{-\delta K_i}\le ze^{\delta t}\mid M(s)=n-m) \qquad (2.2)\\
= \quad & P(e^{-\delta t}\sum_{i=1}^{n-m}Y_ie^{-\delta K_i}\le z\mid M(s)=n-m), \qquad (2.3)
\end{aligned}
$$

其中式 (2.2), (2.3) 由引理 2 得到, 于是 (2.1) 式得证.

记 $F(s,t)=P(W_n\le s,T_{n-1}\le t)$, 则有

$$
\begin{aligned}
& P(L_\delta(T_n)-L_\delta(T_{n-1})\le z)=P(L_\delta(T_{n-1}+W_n)-L_\delta(T_{n-1})\le z)\\
= \quad & \int P(L_\delta(T_{n-1}+W_n)-L_\delta(T_{n-1})\le z\mid W_n=s,T_{n-1}=t)F(ds,dt)\\
= \quad & \int P(L_\delta(t+s)-L_\delta(t)\le z\mid W_n=s,T_{n-1}=t)F(ds,dt)\\
= \quad & \int P(\sum_{i=M(t)+1}^{M(t+s)}Y_ie^{-\delta K_i}\le z\mid W_n=s,T_{n-1}=t)F(ds,dt)\\
= \quad & \int P(\sum_{i=M(t)+1}^{M(t+s)}Y_ie^{-\delta K_i}\le z)F(ds,dt)=\int P(L_\delta(t+s)-L_\delta(t)\le z)F(ds,dt)\\
= \quad & \int\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}P(L_\delta(t+s)-L_\delta(t)\le z\mid M(t)=m,M(t+s)=n)\\
& P(M(t)=m,M(t+s)=n)F(ds,dt)\\
= \quad & \int\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}P(e^{-\delta t}\sum_{i=1}^{n-m}Y_ie^{-\delta K_i}\le z\mid M(s)=n-m)\\
& P(M(t)=m,M(t+s)=n)F(ds,dt)(\text{由 (2.1) 式知})\\
= \quad & \int\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}P(e^{-\delta t}\sum_{i=1}^{M(s)}Y_ie^{-\delta K_i}\le z\mid M(s)=n-m)\\
& P(M(t)=m)P(M(s)=n-m)F(ds,dt)
\end{aligned}
$$

$$\begin{aligned}
&= \int P(e^{-\delta t}\sum_{i=1}^{M(s)} Y_i e^{-\delta K_i} \le z) F(ds, dt) \\
&= \int P(e^{-\delta t}\sum_{i=1}^{M(s)} Y_i e^{-\delta K_i} \le z \mid W_n = s, T_{n-1} = t) F(ds, dt) \\
&= \int P(e^{-\delta T_{n-1}}\sum_{i=1}^{M(W_n)} Y_i e^{-\delta K_i} \le z \mid W_n = s, T_{n-1} = t) F(ds, dt) \\
&= P(e^{-\delta T_{n-1}}\sum_{i=1}^{M(W_n)} Y_i e^{-\delta K_i} \le z) = P(e^{-\delta T_{n-1}} L_\delta(W_n) \le z),
\end{aligned}$$

从而引理 3 得证.

注意到 $N(T_i) = i, i = 1, 2, \cdots$, 则有

$$\begin{aligned}
U_\delta(T_n) &= ue^{\delta T_n} + \sum_{i=1}^{M(T_n)} Y_i e^{\delta(T_n - K_i)} - \sum_{i=1}^{n} X_i e^{\delta(T_n - T_i)} \\
&= e^{\delta W_n} U_\delta(T_{n-1}) + e^{\delta W_n} e^{\delta T_{n-1}} \sum_{i=M(T_{n-1})+1}^{M(T_n)} Y_i e^{-\delta K_i} - X_n.
\end{aligned}$$

用与引理 3 类似的方法可以证明

$$U_\delta(T_n) \stackrel{d}{=} e^{\delta W_n} U_\delta(T_{n-1}) + e^{\delta W_n} L_\delta(W_n) - X_n. \tag{2.4}$$

引理 4 记 $\{L_\delta(t) = \sum\limits_{i=1}^{M(t)} Y_i e^{-\delta K_i}, t \ge 0\}$ 的分布函数为 $F_t(x)$, 则

$$\begin{aligned}
F_t(x) &= e^{-\lambda t}\sum_{n=1}^{\infty}\{\lambda^n \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_0^\infty \int_0^\infty \\
&\quad \cdots \int_0^\infty I_{(0,x]}(\sum_{i=1}^n y_i e^{-\delta s_i}) dG(y_1) dG(y_2) \cdots dG(y_n)\}.
\end{aligned}$$

证 由引理 1 及 $\{Y_n, n = 1, 2, \cdots\}$ 与 $\{K_n, n = 1, 2, \cdots\}$ 和 $\{M_t, t \ge 0\}$ 互相独立知

$$P(L_\delta(t) \le x \mid M(t) = n) = P(\sum_{i=1}^n Y_i e^{-\delta K_i} \le x \mid M(t) = n)$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 \cdots \int_{-\infty}^{\infty} ds_n P(\sum_{i=1}^n Y_i e^{-\delta s_i} \le x \mid M(t) = n, K_i = s_i, i = 1, 2, \cdots, n) \frac{n!}{t^n}, \\
&\quad I_{0<s_1<s_2<\cdots<s_n\le t}(s_1, s_2, \cdots, s_n) \\
&= \frac{n!}{t^n} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n P(\sum_{i=1}^n Y_i e^{-\delta s_i} \le x) \\
&= \frac{n!}{t^n} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n E I_{(0,x]}(\sum_{i=1}^n Y_i e^{-\delta s_i}),
\end{aligned}$$

于是

$$\begin{aligned} F_t(x) &= P(L_\delta(t)\le x)=\sum_{n=1}^{\infty}P(L_\delta(t)\le x\mid M(t)=n)P(M(t)=n)\\ &= \sum_{n=1}^{\infty}\frac{n!}{t^n}\int_0^t ds_1\int_{s_1}^t ds_2\cdots\int_{s_{n-1}}^t ds_n\int_0^\infty\int_0^\infty\\ &\quad \cdots\int_0^\infty I_{(0,x]}(\sum_{i=1}^n y_i e^{-\delta s_i})dG(y_1)dG(y_2)\cdots dG(y_n)\frac{(\lambda t)^n e^{-\lambda t}}{n!}\\ &= e^{-\lambda t}\sum_{n=1}^{\infty}\{\lambda^n\int_0^t ds_1\int_{s_1}^t ds_2\cdots\int_{s_{n-1}}^t ds_n\int_0^\infty\int_0^\infty\\ &\quad \cdots\int_0^\infty I_{(0,x]}(\sum_{i=1}^n y_i e^{-\delta s_i})dG(y_1)dG(y_2)\cdots dG(y_n)\}. \end{aligned}$$

3 主要结果

记 $\Im_n=\sigma\{U_\delta(T_i),i\le n\},n=1,2,\cdots$, 则对任意 Borel 可测集 $B\subset R$, 有

$$\begin{aligned} &P(U_\delta(T_n)\epsilon B\mid \Im_{n-1})\\ =\ &P(e^{\delta W_n}U_\delta(T_{n-1})+e^{\delta W_n}e^{\delta T_{n-1}}\sum_{i=M(T_{n-1})+1}^{M(T_n)}Y_i e^{-\delta K_i}-X_n\epsilon B\mid \Im_{n-1})\\ =\ &P(e^{\delta W_n}U_\delta(T_{n-1})+e^{\delta W_n}e^{\delta T_{n-1}}\sum_{i=M(T_{n-1})+1}^{M(T_n)}Y_i e^{-\delta K_i}-X_n\epsilon B\mid U_\delta(T_{n-1}))\\ =\ &P(U_\delta(T_n)\epsilon B\mid U_\delta(T_{n-1})), \end{aligned}$$

上述第二个等式成立是因为 $e^{\delta W_n}$, $\sum\limits_{i=M(T_{n-1})}^{M(T_n)}Y_i e^{-\delta K_i}$ 和 X_n 关于 $\Im_{n-1}$ 独立, 而 $U_\delta(T_{n-1})$ 和 $e^{\delta T_{n-1}}$ 关于 $\sigma(U_\delta(T_{n-1}))$ 可测, 且 $\sigma(U_\delta(T_{n-1}))\subset\Im_{n-1}$.

于是 $\{U_\delta(T_n),n=1,2,\cdots\}$ 为一马氏过程, 下面考虑其转移概率. 由 (2.4) 式, 有

$$\begin{aligned} &Q(n-1,x,n,A)=P(U_\delta(T_n)\epsilon A\mid U_\delta(T_{n-1})=x)\\ =\ &P(e^{\delta W_n}U_\delta(T_{n-1})+e^{\delta W_n}e^{\delta T_{n-1}}\sum_{i=M(T_{n-1})+1}^{M(T_n)}Y_i e^{-\delta K_i}-X_n\epsilon A\mid U_\delta(T_{n-1})=x)\\ =\ &P(e^{\delta W_n}U_\delta(T_{n-1})+e^{\delta W_n}L_\delta(W_n)-X_n\epsilon A\mid U_\delta(T_{n-1})=x)\\ =\ &P(xe^{\delta W_n}+e^{\delta W_n}L_\delta(W_n)-X_n\epsilon A\mid U_\delta(T_{n-1})=x)\\ =\ &P(xe^{\delta W_n}+e^{\delta W_n}L_\delta(W_n)-X_n\epsilon A)\\ =\ &P(xe^{\delta W_1}+e^{\delta W_1}L_\delta(W_1)-X_1\epsilon A). \end{aligned}$$

由于转移概率与时间无关, 于是可记 $Q(n-1,x,n,A)=Q(x,A)$. 综上, 有

定理 1 $\{U_\delta(T_n), n=0,1,2,\cdots\}$ 为一时齐马氏链, 其转移概率为

$$Q(x,A)=P(xe^{\delta W_1}+e^{\delta W_1}L_\delta(W_1)-X_1\in A).$$

下面利用转移概率 $Q(x,A)$ 刻画破产概率、破产时余额分布及破产前瞬间余额的分布, 给出其级数展开式.

定理 2 (1) 破产概率 $\psi_\delta(u)$ 有如下展式:

$$\begin{aligned}\psi_\delta(u) &= \int_{-\infty}^{0} Q(u,dx)+\sum_{n=2}^{\infty}\int_0^\infty Q(u,dx_1)\int_0^\infty Q(x_1,dx_2)\\ &\quad \cdots\int_0^\infty Q(x_{n-2},dx_{n-1})\int_{-\infty}^{0} Q(x_{n-1},dx_n);\end{aligned}$$

(2) 生存概率 $\Phi_\delta(u)$ 满足 $\Phi_\delta(u)=\int_0^\infty \Phi_\delta(x)Q(u,dx)$;

(3) 生存概率 $\Phi_\delta(u)$ 满足

$$\Phi_\delta(u)=\int_0^\infty d\gamma(t)\int_0^\infty dF_t(y)\int_0^{(u+y)e^{\delta t}}\Phi_\delta((u+y)e^{\delta t}-x)dF(x).$$

证 (1)

$$\begin{aligned}&\psi_\delta(u)\\ =\ & P\{\bigcup_{n=1}^{\infty}(T=T_n)|U_\delta(0)=u\}=\sum_{n=1}^{\infty}P\{T=T_n|U_\delta(0)=u\}=P\{U_\delta(T_1)<0|U_\delta(0)=u\}\\ &+\sum_{n=2}^{\infty}P\{U_\delta(T_1)>0,U_\delta(T_2)>0,\cdots,U_\delta(T_{n-1})>0,U_\delta(T_n)<0|U_\delta(0)=u\}\\ =\ & \int_{-\infty}^{0}Q(u,dx)+\sum_{n=2}^{\infty}\int_0^\infty Q(u,dx_1)\int_0^\infty Q(x_1,dx_2)\\ &\cdots\int_0^\infty Q(x_{n-2},dx_{n-1})\int_{-\infty}^{0}Q(x_{n-1},dx_n).\end{aligned}$$

(2) 由定理 1 知

$$P(T=T_n|U_\delta(0)=u)=\int_0^\infty P(T=T_{n-1}|U_\delta(0)=x)Q(u,dx), n\geq 2,$$

两边对 $n\geq 2$ 求和并注意到 $\psi_\delta(u)=1-\Phi_\delta(u)$ 可得 $\Phi_\delta(u)=\int_0^\infty \Phi_\delta(x)Q(u,dx)$.

(3) 注意到 $\Phi_\delta(y)=0, y<0$, 由定理 1 知

$$\begin{aligned}\Phi_\delta(u)=&E\Phi_\delta(U_\delta(T_1))\\ =&E\Phi_\delta(ue^{\delta T_1}+e^{\delta T_1}L_\delta(T_1)-X_1)\\ =&\int_0^\infty d\gamma(t)\int_0^\infty dF_t(y)\int_0^{(u+y)e^{\delta t}}\Phi_\delta((u+y)e^{\delta t}-x)dF(x)\end{aligned}$$

Gerber, Goovaerts 和 Kass[8] 引入破产时余额的分布函数亦即破产时赤字的分布函数 $G_\delta(u,y)$ 来描述"破产"的严重程度, 即

$$G_\delta(u,y)=P(T<\infty,-y\le U_\delta(T)<0|U_\delta(0)=u),y>0.$$

定理 3 破产时余额的分布 $G_\delta(u,y)$ 有如下展式

$$\begin{aligned}G_\delta(u,y) &= \int_{-y}^{0}Q(u,dx)+\sum_{n=2}^{\infty}\int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\\ &\quad\cdots\int_0^{\infty}Q(x_{n-2},dx_{n-1})\int_{-y}^{0}Q(x_{n-1},dx_n).\end{aligned}$$

证

$$\begin{aligned}&G_\delta(u,y)=P(T<\infty,-y\le U_\delta(T)<0|U_\delta(0)=u)\\ =\ &P(\bigcup_{n=1}^{\infty}(T=T_n),-y\le U_\delta(T)<0|U_\delta(0)=u)\\ =\ &\sum_{n=1}^{\infty}P(T=T_n,-y\le U_\delta(T)<0|U_\delta(0)=u)\\ =\ &P\{-y\le U_\delta(T_1)<0|U_\delta(0)=u\}\\ &+\sum_{n=2}^{\infty}P(U_\delta(T_1)>0,U_\delta(T_2)>0,\cdots,U_\delta(T_{n-1})>0,-y\le U_\delta(T_n)<0|U_\delta(0)=u)\\ =\ &\int_{-y}^{0}Q(u,dx)+\sum_{n=2}^{\infty}\int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\cdots\int_0^{\infty}Q(x_{n-2},dx_{n-1})\int_{-y}^{0}Q(x_{n-1},dx_n).\end{aligned}$$

在 Gerber, Goovaerts 和 Kass[8] 工作的基础上, Dufresne 和 Gerber[9] 引入了如下破产前瞬间余额分布函数

$$F_\delta(u,y)=P(T<\infty,0<U_\delta(T-)<y|U_\delta(0)=u),y>0.$$

定理 4 破产前瞬间余额的分布 $F_\delta(u,y)$ 有如下展式

$$\begin{aligned}&F_\delta(u,y)\\ =\ &I_{(u<y)}[\int_0^{\frac{\ln y-\ln u}{\delta}}d\gamma(t)\int_{ue^{\delta t}}^{y}F_t(xe^{-\delta t}-u)dF(x)+\int_0^{\frac{\ln y-\ln u}{\delta}}(1-F(y))F_t(ye^{-\delta t}-u)d\gamma(t)]\\ &+\sum_{n=2}^{\infty}\int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\\ &\cdots\int_0^{y}Q(x_{n-2},dx_{n-1})\int_0^{\frac{\ln y-\ln x_{n-1}}{\delta}}d\gamma(t)\int_{x_{n-1}e^{\delta t}}^{y}F_t(xe^{-\delta t}-x_{n-1})dF(x)\\ &+\sum_{n=2}^{\infty}\int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\\ &\cdots\int_0^{y}Q(x_{n-2},dx_{n-1})\int_0^{\frac{\ln y-\ln x_{n-1}}{\delta}}(1-F(y))F_t(ye^{-\delta t}-x_{n-1})d\gamma(t).\end{aligned}$$

证

$$
\begin{aligned}
& F_\delta(u,y)=P(T<\infty,0<U_\delta(T-)\le y|U_\delta(0)=u)\\
= & I_{(u<y)}P(T=T_1,0<U_\delta(T_1-)\le y|U_\delta(0)=u)\\
& +\sum_{n=2}^{\infty}P(T=T_n,0<U_\delta(T_n-)\le y|U_\delta(0)=u)\\
= & I_{(u<y)}P(U_\delta(T_1)<0,0<U_\delta(T_1-)\le y|U_\delta(0)=u)\\
& +\sum_{n=2}^{\infty}P(U_\delta(T_1)>0,U_\delta(T_2)>0,\cdots,U_\delta(T_{n-1})>0,U_\delta(T_n)<0,0<U_\delta(T_n-)\le y|U_\delta(0)=u),
\end{aligned}
$$

而

$$
\begin{aligned}
& I_{(u<y)}P(T=T_1,0<U_\delta(T_1-)\le y|U_\delta(0)=u)\\
= & I_{(u<y)}P(ue^{\delta T_1}+e^{\delta T_1}L_\delta(T_1)-X_1<0,0<ue^{\delta T_1}+e^{\delta T_1}L_\delta(T_1)\le y)\\
= & I_{(u<y)}\int_0^{\frac{\ln y-\ln u}{\delta}}P(ue^{\delta t}+e^{\delta t}L_\delta(t)-X_1<0,0<ue^{\delta t}+e^{\delta t}L_\delta(t)\le y)d\gamma(t)\\
= & I_{(u<y)}\int_0^{\frac{\ln y-\ln u}{\delta}}P(L_\delta(t)<X_1e^{-\delta t}-u,L_\delta(t)\le ye^{-\delta t}-u)d\gamma(t)\\
= & I_{(u<y)}\int_0^{\frac{\ln y-\ln u}{\delta}}P(ue^{\delta t}<X_1\le y,L_\delta(t)<X_1e^{-\delta t}-u)d\gamma(t)\\
& +I_{(u<y)}\int_0^{\frac{\ln y-\ln u}{\delta}}P(X_1>y,L_\delta(t)<ye^{-\delta t}-u)d\gamma(t)\\
= & I_{(u<y)}\int_0^{\frac{\ln y-\ln u}{\delta}}d\gamma(t)\int_{ue^{\delta t}}^{y}F_t(xe^{-\delta t}-u)dF(x)\\
& +I_{(u<y)}\int_0^{\frac{\ln y-\ln u}{\delta}}(1-F(y))F_t(ye^{-\delta t}-u)d\gamma(t),
\end{aligned}
$$

又由定理 1 知

$$
\begin{aligned}
& P(U_\delta(T_1)>0,U_\delta(T_2)>0,\cdots,U_\delta(T_{n-1})>0,U_\delta(T_n)<0,0<U_\delta(T_n-)\le y|U_\delta(0)=u)\\
= & \int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\cdots\int_0^{y}Q(x_{n-2},dx_{n-1})\\
& P(U_\delta(T_n)<0,0<U_\delta(T_n-)\le y|U_\delta(T_{n-1})=x_{n-1})\\
= & \int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\\
& \cdots\int_0^{y}Q(x_{n-2},dx_{n-1})\int_0^{\frac{\ln y-\ln x_{n-1}}{\delta}}d\gamma(t)\int_{x_{n-1}e^{\delta t}}^{y}F_t(xe^{-\delta t}-x_{n-1})dF(x)\\
& +\int_0^{\infty}Q(u,dx_1)\int_0^{\infty}Q(x_1,dx_2)\\
& \cdots\int_0^{y}Q(x_{n-2},dx_{n-1})\int_0^{\frac{\ln y-\ln x_{n-1}}{\delta}}(1-F(y))F_t(ye^{-\delta t}-x_{n-1})d\gamma(t).
\end{aligned}
$$

于是定理得证.

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RUIN PROBLEMS IN A COMPOUND POISSON-RENEWAL RISK MODEL WITH A CONSTANT INTEREST RATE

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Abstract: The compound Poisson-renewal risk model is such that the aggregate premium is a compound Poisson process and the aggregate claim is a compound renewal process. This paper considers the compound Poisson-renewal risk model with a constant interest rate. Using discretization method, we give the series expansions of ruin probability and the distributions of the surplus at ruin and immediately before ruin. So the recent relevant results of [1] and [2] are extended.

Keywords: renewal process; ruin probability; the deficit at ruin; the surplus immediately before ruin

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