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# THE CLASSIFICATION OF GRADIENT RICCI ALMOST SOLITONS

ZENG Fan-qi, MA Bing-qing
(*Department of Mathematics, Henan Normal University, Xinxiang 453007, China*)

**Abstract:** We study the classification of a gradient Ricci almost soliton. Using similar methods as in [11] for $n \geq 5$, we obtain that the Weyl curvature tensor is harmonic or Einstein under the assumption that the Bach tensor is flat.

**Keywords:** gradient Ricci almost solitons; Bach tensor; Weyl curvature tensor
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## 1 Introduction

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold. If there exist two smooth functions $f$, $\lambda$ on $(M^n, g)$ such that

$$R_{ij} + f_{ij} = \lambda g_{ij}, \tag{1.1}$$

then $(M^n, g)$ is called a gradient Ricci almost soliton which was introduced by Pigola, Rigoli, Rimoldi and Setti in [1], where $R_{ij}$ denotes the Ricci curvature of $(M^n, g)$. Clearly, the above gradient Ricci almost solitons generalize the concept of gradient Ricci solitons which play a very important role in Hamilton's Ricci flow as it corresponds to the self-similar solutions and often arises as singularity models, for a survey in this subject we refer to the work due to Cao in [2]. When $\lambda = \rho R + \mu$ in (1.1) with $\rho, \mu$ two real constants, $(M^n, g)$ is called the gradient $\rho$-Einstein soliton (see [3]) which is a special case of $(m, \rho)$-quasi-Einstein manifolds defined in [4], where $R$ is the scalar curvature of $(M^n, g)$. For the recent research on this direction, see [5–10] and the references therein.

In this paper, using a similar idea used in [11–13], we derive some formulas, and establish a link between the Cotton tensor $C_{ijk}$ and the 3-tensor $D_{ijk}$, that is, $C_{ijk} = D_{ijk} - W_{ijkl}f^l$, where $W_{ijkl}$ is the Weyl curvature tensor. By virtue of this relationship we give some classifications of gradient Ricci almost solitons.

## 2 Preliminaries

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**Biography:** Zeng Fanqi (1987–), male, born at Xinyang, Henan, master, major in differential geometry.

We use moving frames in all calculations and adopt the following index convention:

$$1 \le i, j, k, \cdots \le n, \qquad 2 \le a, b, c, \cdots \le n$$

throughout this paper.

**Lemma 2.1** Let $(M^n, g)$ be a gradient Ricci almost soliton satisfying (1.1). Then we have

$$\Delta f = n\lambda - R, \tag{2.1}$$
$$(|\nabla f|^2)_i = 2\lambda f_i - 2R_{ij} f^j, \tag{2.2}$$
$$\frac{1}{2} R_{,i} = (n-1)\lambda_i + R_{ij} f^j, \tag{2.3}$$

where $f^j = g^{jk} f_k$.

**Proof** Equations (2.1) and (2.2) are direct consequences of (1.1) and the fact

$$(|\nabla f|^2)_i = 2f_j f_{ij} = 2f_j(\lambda g_{ij} - R_{ij}) = 2\lambda f_i - 2R_{ij} f_j.$$

By the second Bianchi identity, we get

$$\begin{aligned} \frac{1}{2} R_{,i} = R_{ij,j} =& (\lambda g_{ij} - f_{ij})_{,j} \\ =& \lambda_i - f_{ijj} \\ =& \lambda_i - (\Delta f)_i - R_{ij} f_j, \end{aligned}$$

where we used the Ricci identity $f_{ijj} = (\Delta f)_i + R_{ij} f_j$. Insertting (2.1) and (2.2) into the above equation gives (2.3). We complete the proof of Lemma 2.1.

For $n \ge 3$, the Weyl curvature tensor and the Cotton tensor are defined by

$$\begin{aligned} W_{ijkl} =& R_{ijkl} - \frac{1}{n-2}(A_{ik} g_{jl} - A_{il} g_{jk} + A_{jl} g_{ik} - A_{jk} g_{il}) \\ =& R_{ijkl} - \frac{1}{n-2}(R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ik} g_{jl} - g_{il} g_{jk}) \end{aligned} \tag{2.4}$$

and

$$C_{ijk} = A_{kj,i} - A_{ki,j}, \tag{2.5}$$

where $A_{ij}$ is called the Schouten tensor given by

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}.$$

From the definition of the Cotton tensor, we have that $C_{ijk}$ is skew-symmetric in the first two indices and trace-free in any two indices:

$$C_{ijk} = -C_{jik}, \qquad g^{ij} C_{ijk} = g^{ik} C_{ijk} = 0.$$

The divergence of the Weyl curvature tensor is related to the Cotton tensor by

$$-\frac{n-3}{n-2}C_{ijk} = W_{ijkl,}{}^{l}. \tag{2.6}$$

For $n \geq 4$, the Bach tensor is defined by

$$B_{ij} = \frac{1}{n-3}W_{ikjl,}{}^{lk} + \frac{1}{n-2}W_{ikjl}R^{kl}. \tag{2.7}$$

Using (2.6), we may extend the definition of Bach tensor in dimensions including 3 as follows:

$$B_{ij} = \frac{1}{n-2}(C_{kij,}{}^{k} + W_{ikjl}R^{kl}). \tag{2.8}$$

As in [11], see also [8, 12, 13], we define the following 3-tensor $D$ by

$$\begin{aligned} D_{ijk} =& \frac{1}{n-2}(R_{kj}f_i - R_{ki}f_j) + \frac{1}{(n-1)(n-2)}(R_{il}g_{jk}f^l - R_{jl}g_{ik}f^l) \\ & - \frac{R}{(n-1)(n-2)}(g_{kj}f_i - g_{ki}f_j). \end{aligned} \tag{2.9}$$

Then we have that $D_{ijk}$ is skew-symmetric in the first two indices and trace-free in any two indices:

$$D_{ijk} = -D_{jik}, \qquad g^{ij}D_{ijk} = g^{ik}D_{ijk} = 0.$$

**Lemma 2.2** Let $(M^n, g)$ be a gradient Ricci almost soliton satisfying (1.1). Then the Cotton tensor, D-tensor and the Weyl curvature tensor are related by

$$C_{ijk} = D_{ijk} - W_{ijkl}f^l. \tag{2.10}$$

**proof** Using formula (1.1), we have

$$\begin{aligned} R_{kj,i} - R_{ki,j} =& (\lambda g_{kj} - f_{kj})_{,i} - (\lambda g_{ki} - f_{ki})_{,j} \\ =& \lambda_i g_{kj} - \lambda_j g_{ki} + f_{kij} - f_{kji} \\ =& \lambda_i g_{kj} - \lambda_j g_{ki} - R_{ijkl}f^l. \end{aligned}$$

Therefore,

$$\begin{aligned} C_{ijk} =& A_{kj,i} - A_{ki,j} \\ =& \lambda_i g_{kj} - \lambda_j g_{ki} - R_{ijkl}f^l \\ & - \frac{1}{2(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}) \\ =& - \frac{1}{(n-1)}(R_{il}g_{kj}f_l - R_{jl}g_{ki}f_l) - R_{ijkl}f^l \\ =& D_{ijk} - W_{ijkl}f^l, \end{aligned}$$

where the third equality used equation (2.3). It completes the proof of Lemma 2.2.

The next lemma links the norm of $D_{ijk}$ to the geometry of the level surfaces of the function $f$ on $(M^n, g)$. The proof can be found in [15, Proposition 2.3] and [11, Proposition 3.1].

**Lemma 2.3** Let $(M^n, g)$ be a Riemannian manifold and let $\Sigma_c = \{x|f(x) = c\}$ be the level surface with respect to regular value $c$ of $f$. Choose local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ on $(M^n, g)$ such that $e_1 = \nabla f/|\nabla f|$ and $\{e_2, \cdots, e_n\}$ tangent to $\Sigma_c$. Denote by $|D_{ijk}|$ the norm of the 3-tensor $D$, and by $g_{ab}$ the induced metric on $\Sigma_c$. We have

$$|D_{ijk}|^2 = \frac{2|\nabla f|^2}{(n-1)(n-2)^2}\left((n-2)\sum_{a=2}^{n} R_{1a}^2 + (n-1)\Big|R_{ab} - \frac{R-R_{11}}{n-1}g_{ab}\Big|^2\right),$$

where $R_{ij} = \mathrm{Ric}(e_i, e_j)$ are the components of the Ricci curvature on $(M^n, g)$, $R$ is the scalar curvature of $(M^n, g)$. Note that the indices $2 \le a, b, c, \cdots \le n$, then $R_{ab}$ denotes the Ricci tensor of $(M^n, g)$ restricted to the tangent space of $\Sigma_c$ and $g^{ab}R_{ab} = R - R_{11}$.

## 3 Some Results

With the help of Lemma 2.3, we can obtain the following result.

**Proposition 3.1** Let $(M^n, g)$ be a gradient Ricci almost soliton satisfying (1.1) with $D_{ijk} = 0$. Let $\Sigma_c = \{x|f(x) = c\}$ be the level surface with respect to regular value $c$ of $f$. Then for any local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ with $e_1 = \nabla f/|\nabla f|$ and $\{e_2, \cdots, e_n\}$ tangent to $\Sigma_c$, we have

(1) $|\nabla f|$, $\Delta f$, $\lambda$ and the scalar curvature $R$ of $(M^n, g)$ are all constant on $\Sigma_c$;

(2) $R_{1a} = 0$ and $e_1 = \nabla f/|\nabla f|$ is an eigenvector of the Ricci operator;

(3) the second fundamental form $h_{ab}$ of $\Sigma_c$ is of the form $h_{ab} = \frac{H}{n-1}g_{ab}$;

(4) the mean curvature $H = \frac{(n-1)\lambda-(R-R_{11})}{|\nabla f|}$ is constant on $\Sigma_c$;

(5) on $\Sigma_c$ the Ricci tensor of $(M^n, g)$ either has a unique eigenvalue $\nu$, or has two distinct eigenvalues $\nu$ and $\sigma$ of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \nabla f/|\nabla f|$ is an eigenvector of $\nu$. Moreover, both $\nu$ and $\sigma$ are constant on $\Sigma_c$.

**Proof** Under this chosen orthonormal frame, we have $f_1 = |\nabla f|$ and $f_2 = f_3 = \cdots = f_n = 0$. When $D_{ijk} = 0$, we have from Lemma 2.3 that

$$R_{1a} = 0 \tag{3.1}$$

and

$$R_{ab} = \frac{R-R_{11}}{n-1}g_{ab}. \tag{3.2}$$

Therefore, we obtain from (2.2) and (2.3)

$$(|\nabla f|^2)_a = 0, \quad \forall\ a,$$

which show that $|\nabla f|$ is constant on $\Sigma_c$. We derive form (2.2) and (2.3)

$$R_{,i} = 2(n-1)\lambda_i + 2\lambda f_i - (|\nabla f|^2)_i$$

which means that

$$dR = 2(n-1)d\lambda + 2\lambda df - d(|\nabla f|^2). \tag{3.3}$$

Taking exterior differential of the both sides of (3.3), we obtain $d\lambda \wedge df = 0$. Therefore, according to the well-known Cartan's lemma, there exists a smooth function $\varphi$ such that

$$d\lambda = \varphi\, df,$$

which shows that $\lambda$ is also constant on $\Sigma_c$. Hence, (1) is proved.

In particular, (2) can be obtained from (3.1) directly.

By the definition of $h_{ab}$, we have

$$h_{ab} = \langle \nabla_{e_a}\Big(\frac{\nabla f}{|\nabla f|}\Big), e_b\rangle = \frac{1}{|\nabla f|} f_{ab} = \frac{1}{|\nabla f|}\Big(\lambda - \frac{R-R_{11}}{n-1}\Big) g_{ab}, \tag{3.4}$$

where the last equality used (3.2). Hence,

$$H = g^{ab} h_{ab} = \frac{(n-1)\lambda - (R - R_{11})}{|\nabla f|} \tag{3.5}$$

and (3) is proved.

By the Codazzi equation

$$R_{1cab} = \nabla_a^{\Sigma_c} h_{bc} - \nabla_b^{\Sigma_c} h_{ac},$$

we get from tracing over $b$ and $c$

$$R_{1a} = \nabla_a^{\Sigma_c} H - \nabla_b^{\Sigma_c} h_{ab} = \frac{n-2}{n-1} H_{,a}$$

and (4) follows form $R_{1a} = 0$.

Since $H$ is constant on $\Sigma_c$, we have from (3.5)

$$R_{11,a} = 0.$$

Applying

$$R_{11,a} = e_a(R_{11}) - 2R(\nabla_{e_a} e_1, e_1) = e_a(R_{11}) - 2h_{ab}R_{1b} = e_a(R_{11})$$

yields $e_a(R_{11}) = 0$, which shows that $\nu = R_{11}$ is constant on $\Sigma_c$. By (3.2) we know that for distinct $a$, the eigenvalues of $R_{aa}$ are the same. Hence, we have the eigenvalue $\sigma$ is also constant. We obtain (5) and complete the proof of Proposition 3.1.

**Theorem 3.2** Let $(M^n, g)$ be a gradient Ricci almost soliton satisfying (1.1). Then

$$(n-2)B_{ij} = D_{kij,}{}^{k} + \frac{n-3}{n-2} C_{kji} f^k. \tag{3.6}$$

If $(M^n, g)$ is compact, then for $p \geq 0$,

$$\int_{M^n} f^p B_{ij} f^i f^j \, dv_g = -\frac{1}{2} \int_{M^n} f^p |D|^2 \, dv_g. \tag{3.7}$$

In particular, if $B_{ij}=0$, we obtain from (3.7) the 3-tensor $D_{ijk}=0$.

**Proof** By virtue of (2.8) and (2.10), we have

$$\begin{aligned}(n-2)B_{ij}=&C_{kij,k}+W_{ikjl}R_{kl}\\=&(D_{kij}-W_{kijl}f_l)_{,k}+W_{ikjl}R_{kl}\\=&D_{kij,k}-W_{kijl,k}f_l-W_{kijl}f_{kl}+W_{ikjl}R_{kl}\\=&D_{kij,k}-W_{kijl,k}f_l\\=&D_{kij,k}+\frac{n-3}{n-2}C_{lji}f_l.\end{aligned}$$

If $(M^n,g)$ is compact, we obtain using integrating by parts

$$\begin{aligned}&(n-2)\int\limits_{M^n}f^pB_{ij}f_if_j\,dv_g\\=&\int\limits_{M^n}f^p\left(D_{kij,}{}^k+\frac{n-3}{n-2}C_{kji}f^k\right)f_if_j\,dv_g\\=&\int\limits_{M^n}f^pD_{kij,}{}^kf_if_j\,dv_g\\=&-\int\limits_{M^n}f^pD_{kij}f^if^{kj}\,dv_g\\=&-\frac{n-2}{2}\int\limits_{M^n}f^p|D|^2\,dv_g.\end{aligned}$$

Therefore, we obtain (3.7) and complete the proof of Theorem 3.2.

**Proposition 3.3** Let $(M^n,g)$ be a compact gradient Ricci almost soliton satisfying (1.1) with $B_{ij}=0$. If $n\geq 4$, then the Cotton tensor $C_{ijk}=0$ at all points where $\nabla f\neq 0$.

**Proof** From Lemma 2.2 and Theorem 3.2, we conclude that $C_{ijk}=-W_{ijkl}f_l$. Under the orthonormal frame as in Lemma 2.3, we have

$$C_{ijk}=-W_{ijk1}|\nabla f|. \tag{3.8}$$

In particular, we obtain from (3.8)

$$C_{ij1}=0. \tag{3.9}$$

From Theorem 3.2, we get

$$\frac{n-3}{n-2}C_{1ji}|\nabla f|=0.$$

Hence, If $n\geq 4$, then

$$C_{1ji}=C_{j1i}=0. \tag{3.10}$$

Moreover, from (3.8) we also have that $C_{abc}=-W_{abc1}|\nabla f|$. Using (2.4) and Proposition 3.1, we obtain

$$W_{abc1}=R_{abc1}=R_{1cba}=\nabla^{\Sigma}_{e_b}h_{ac}-\nabla^{\Sigma}_{e_a}h_{bc}=0.$$

Therefore, we obtain

$$C_{abc} = 0. \tag{3.11}$$

Combining (3.9) with (3.10) and (3.11), we arrive at the conclusion of Proposition 3.3.

**Proposition 3.4** Let $(M^4, g)$ be a compact gradient Ricci almost soliton satisfying (1.1). If $B_{ij} = 0$, then the Weyl curvature tensor $W_{ijkl} = 0$ at all points where $\nabla f \neq 0$.

**Proof** Since $B_{ij} = 0$, we have $D_{ijk} = C_{ijk} = 0$. Hence, Lemma 2.2 shows that $W_{ijk1} = 0$ for $1 \leq i, j, k \leq 4$. It remains to show that $W_{abcd} = 0$ for $2 \leq a, b, c, d \leq 4$. This essentially reduces to show the Weyl curvature tensor is equal to zero in 3 dimensions (see [14, p.276–277] or [11, p.13]). Therefore, we have $W_{ijkl} = 0$.

**Theorem 3.5** Let $(M^n, g)$ be a compact gradient Ricci almost soliton satisfying (1.1) with $B_{ij} = 0$.

(1) If $n \geq 5$, then the Weyl curvature tensor is harmonic or Einstein.

(2) If $n = 4$ and it has positive sectional curvature, then $(M^4, g)$ is rotational symmetric or Einstein.

**Proof** (1) If $(M^n, g)$ is not Einstein, then from the set $\{p|\nabla f(p) = 0\}$ is of measure zero we have $C_{ijk} = 0$ on $\Omega = \{x|\nabla f \neq 0\}$ everywhere according to Proposition 3.3 and the continuity. Hence, the Weyl curvature tensor is harmonic.

(2) Under the assumption of Theorem 3.1, Proposition 3.4 shows that $(M^4, g)$ has vanishing Weyl curvature tensor at all points where $\nabla f \neq 0$. So if the set $\Omega = \{x|\nabla f \neq 0\}$ is dense, by continuity of the Weyl curvature tensor we have $W_{ijkl} = 0$ everywhere and $(M^4, g)$ is locally conformally flat. Recall that in any neighborhood of the level surface $\Sigma_c$, where $\nabla f \neq 0$, we can express the metric $ds^2$ by

$$ds^2 = \frac{1}{|\nabla f|^2}(f, \theta) df^2 + g_{ab}(f, \theta) d\theta^a \theta^b, \tag{3.12}$$

where $\theta = (\theta^2, \cdots, \theta^n)$ denote the intrinsic coordinates on $\Sigma_c$. Since $(M^4, g)$ has vanishing Weyl curvature tensor and positive sectional curvature, the Gauss equation

$$R^{\Sigma_c}_{abcd} = R_{abcd} + h_{aa} h_{bb} - h^2_{ab}$$

and Proposition 3.1 tells us that $(\Sigma_c, g_{ab})$ is a space form with constant positive sectional curvature and $\frac{1}{|\nabla f|}(f, \theta) = \frac{1}{|\nabla f|}(f)$. Hence on $\Omega$ we have

$$ds^2 = \frac{1}{|\nabla f|^2}(f) df^2 + \varphi^2(f) g_{_{\mathbb{S}^{n-1}}}, \tag{3.13}$$

where $g_{\mathbb{S}^{n-1}}$ denotes the standard metric on unit sphere $\mathbb{S}^{n-1}$. We conclude that $(M^4, g)$ is rotationally symmetric.

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# 近黎奇梯度孤立子的分类


曾凡奇, 马冰清

(河南师范大学数学系, 河南 新乡 453007)



**摘要**: 本文研究黎奇梯度孤立子的分类问题. 利用与文献[11]类似的方法, 在Bach张量等于零的条件下, 对于$n \geq 5$, 证明了流形是Einstein的或者Weyl曲率张量是调和的.



**关键词**: 黎奇梯度孤立子; Bach张量; Weyl曲率张量

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