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BANACH ALGEBRA STRUCTURE IN DERIVATIVE HARDY SPACES

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Abstract: In this paper, we consider the algebraic structure of derivative Hardy Spaces. By using the method of [6, 12, 15], we get the Duhamel product forming Banach algebra in derivative Hardy Spaces, and invertibility criterion, and describe the extended eigenvalue of the integral operator V. We generalize the results in [1, 2, 6, 11, 16].

Keywords: Duhamel product; banach algebra; extended eigenvalue

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1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of analytic functions on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}} < \infty.$$

In this paper, we are more interested in Derivative Hardy spaces $S^p(\mathbb{D})$ which are defined by

$$S^p(\mathbb{D}) = \{f \in H^p(\mathbb{D}) : \, f'(z) \in H^p(\mathbb{D})\},$$

with norm

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

The main objective of this paper is to study the Duhamel product $\circledast$ on the $S^p(\mathbb{D})$ space. The Duhamel product is defined as follows (see [1]):

$$(f \circledast g)(z) := \frac{d}{dz} \int_0^z f(z-t)g(t) \, dt = \int_0^z f'(z-t)g(t) \, dt + f(0)g(z).$$

The problem about the Duhamel product is not new. Wigley[1] introduced this product and showed that the complex-valued function space is a Banach algebra with respect to the Duhamel product $\circledast$. In 1975, Wigley[2] showed that the Hardy space $H^p(\mathbb{D})$ is a Banach

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algebra with respect to the Duhamel product $\circledast$. Afterward, Merryfield and Watson extended this result to H^p of the polydisc[3]. Then, the study of Duhamel products $\circledast$ attracted many scholars. Karaev and Tuna[4], showed that $C^{(n)}(\overline{\mathbb{D}})$ is a commutative unital Banach algebra with respect to the Duhamel product. Gürdal[5] extended some of Duhamel product properties to the analytic sequence space $l^p_A(\mathbb{D})$. Recently, Guediri and his collaborators[6] showed that the Bergman space is a commutative unital Banach algebra with respect to the Duhamel product. For more information about the Duhamel product, see [4, 6–10].

This paper gives a Banach algebra structure with respect to the Duhamel product $\circledast$ in $S^p(\mathbb{D})$ spaces. Then, we consider the invertibility criterion of the Duhamel product $\circledast$. For the invertibility criterion, Volterra integration operator V is essential in the proof process. Moreover, the invertibility criterion has two interesting results, the maximal ideal space consists of one homomorphism and the spectrum of f consists of one point $f(0)$. In section 3, we will consider the extended eigenvalue of Volterra integration operator V. In fact, the study of the extended eigenvalue of Volterra integration operator V has a lot of rich results. Biswas and his collaborators[11] considered the case of $L^2(0,1)$. Karaev[7] described the extended eigenvalues of V in n-th differentiable functions space. For more information, we recommend [4, 6, 8–10]. Then, we also give some characterizations of the bounded linear operator on $S^p(\mathbb{D})$.

For convenience, we use the notation $X \lesssim Y$ or $Y \lesssim X$ to mean $X < CY$ or $Y < CX$, where C is positive constant. Similarly, we use the notation $X \approx Y$ if both $X \lesssim$ and $Y \lesssim$ hold.

2 The Duhamel Product

Elementary calculation shows that the Duhamel product-$\circledast$ can also be given by

$$\begin{aligned}(f \circledast g)(z) :&= \frac{d}{dz}\int_0^z f(z-t)g(t)\,dt\\ &= \int_0^z f'(z-t)g(t)\,dt + f(0)g(z)\\ &= \int_0^z f(z-t)g'(t)\,dt + f(z)g(0).\end{aligned}$$

Apparently, it is a commutative product. Then, take the derivative of the above equation (see [1])

$$\begin{aligned}(\mathcal{D}_f g)'(z) = (f \circledast g)'(z) &= \int_0^z f'(z-t)g'(t)\,dt + f(0)g'(z) + f'(z)g(0)\\ &= \int_0^{\frac{z}{2}} f'(z-t)g'(t)\,dt + \int_0^{\frac{z}{2}} f'(t)g'(z-t)\,dt + f(0)g'(z) + f'(z)g(0)\\ &= \int_0^{\frac{z}{2}} \big(f'(z-t)g'(t) + f'(t)g'(z-t)\big)\,dt + f(0)g'(z) + f'(z)g(0).\end{aligned}$$

Lemma 2.1 Let $1 \le p < \infty$, for any $f \in S^p(\mathbb{D})$, then (see [12])

$$|f'(z)| \lesssim \frac{\|f\|_{S^p}}{(1-|z|^2)^{\frac{1}{p}}}.$$

Since $t \in [0, \frac{z}{2}]$, by the above Lemma, we have $|f'(t)| \lesssim \|f\|_{S^p}$.

$$\begin{aligned}\big|(\mathcal{D}_f g)'(z)\big| \lesssim & \|g\|_{S^p} \int_0^{\frac{z}{2}} |f'(z-t)|\,|dt| + |f'(z)||g(0)| \\ & + \|f\|_{S^p} \int_0^{\frac{z}{2}} |g'(z-t)|\,|dt| + |f(0)||g'(z)|.\end{aligned} \tag{2.1}$$

To simplify the notation, we denote $\int_0^{\frac{z}{2}} |f'(z-t)|\,|dt|$ and $\int_0^{\frac{z}{2}} |g'(z-t)|\,|dt|$ by $F(z)$ and $G(z)$. Next we consider

$$M_p^p(F,r) = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^p\,d\theta.$$

Lemma 2.2 Let $p > 1$, then $M_p(F,r) \lesssim (1-|z|)^{1-\frac{1}{p}}\|f\|_{S^p}$.

Proof Let $z = re^{i\theta}$, $0 \le r < 1$, and $t = \rho e^{i\theta}$, then we have

$$\begin{aligned}F(z) = & \int_0^{\frac{z}{2}} |f'(z-t)|\,|dt| = \int_{\frac{r}{2}}^{r} |f'(\rho e^{i\theta})|\,d\rho \\ \le & \|f\|_{S^p}\int_{\frac{r}{2}}^{r} \frac{d\rho}{(1-\rho^2)^{\frac{1}{p}}} \le \|f\|_{S^p}\int_{\frac{r}{2}}^{r} \frac{d\rho}{(1-\rho)^{\frac{1}{p}}} \\ \lesssim & (1-|z|)^{1-\frac{1}{p}}\|f\|_{S^p}.\end{aligned}$$

Thus, we get

$$\begin{aligned}M_p^p(F,r) = & \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\frac{r}{2}e^{i\theta}} |f'(re^{i\theta}-t)|^p|dt|\right)^p d\theta \\ \lesssim & (1-|z|)^{p-1}\|f\|_{S^p}^p.\end{aligned}$$

The proof of Lemma is complete.

Theorem 2.3 Suppose that $1 < p < \infty$, the $S^p(\mathbb{D})$ space is a unital and commutative Banach algebra with respect to the Duhamel product $\circledast$, which we will denote $(S^p(\mathbb{D}), \circledast)$.

Proof Firstly, $S^p(\mathbb{D})$ is a Banach space. By (2.1), Lemma 2.1 and Minkowski inequality, we have

$$\begin{aligned}\|\mathcal{D}_f g\|_{S^p} = & |f(0)||g(0)| + \|(f \circledast g)'\|_{H^p} \\ = & |f(0)||g(0)| + \sup_{0<r<1} M_p((f\circledast g)', r) \\ \le & 3\|f\|_{S^p}\|g\|_{S^p} + \|f\|_{S^p}\sup_{0<r<1} M_p(F,r) + \|g\|_{S^p}\sup_{0<r<1} M_p(G,r) \\ \lesssim & \|f\|_{S^p}\|g\|_{S^p}.\end{aligned}$$

The proof is complete.

Corollary 2.4 $(S^p(\mathbb{D}), \circledast)$ is an involutive algebra (cf. [13]), the involution is a map

$$\begin{aligned} * :\quad & \big(S^p(\mathbb{D})\big), \circledast) \to \big(S^p(\mathbb{D}), \circledast\big) \\ & f \mapsto f^* \end{aligned}$$

where $f^* = \overline{f(\overline{z})}$.

Proof Let $f \in S^p(\mathbb{D})$, then we have

$$\begin{aligned} \|f\|_{S^p} &= |f(0)| + \|f'(z)\|_{H^p} \\ &= |f(0)| + \bigg(\sup_{0<r<1} \int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta\bigg)^{\frac{1}{p}} \\ &= |f(0)| + \bigg(\sup_{0<r<1} \int_0^{2\pi} |f'(re^{-i\theta})|^p \, d\theta\bigg)^{\frac{1}{p}} \\ &= |f(0)| + \bigg(\sup_{0<r<1} \int_0^{2\pi} |\overline{f'(re^{-i\theta})}|^p \, d\theta\bigg)^{\frac{1}{p}} = \|f^*\|_{S^p}. \end{aligned}$$

We can get $f^* \in S^p(\mathbb{D})$. Clearly, the involution on $(S^p(\mathbb{D}), \circledast)$ is satisfying $(f^*)^* = f$, and $(\alpha \circledast f + \beta \circledast g)^* = \overline{\alpha} \circledast f^* + \overline{\beta} \circledast g^*$ for all $f, g \in S^p(\mathbb{D})$, $\alpha, \beta \in \mathbb{C}$. Next we prove that $(f \circledast g)^* = f^* \circledast g^*$,

$$\begin{aligned} (f^* \circledast g^*)(z) &= \int_0^z \overline{f'(\overline{z-t})}\,\overline{g(\overline{t})}\, dt + \overline{f(0)}\,\overline{g(\overline{z})} \\ &= \overline{\int_0^z f'(\overline{z-t}) g(\overline{t})\, d\overline{t}} + \overline{f(0)}\,\overline{g(\overline{z})} \\ &= \overline{\int_0^{\overline{z}} f'(\overline{z}-t) g(t)\, dt} + \overline{f(0)}\,\overline{g(\overline{z})} = (f \circledast g)^*(z), \end{aligned}$$

which implies that $(S^p(\mathbb{D}), \circledast)$ is an involutive algebra, or a $*$-algebra.

In [6], Guediri and his collaborators, see the Duhamel product as a special operator, which is called the Duhamel convolution operator with analytic symbol f. For convenience, we call this operators as Duhamel convolution operator and denote it by

$$\mathcal{D}_f g := (f \circledast g)(z) = \int_0^z f'(z-t) g(t)\, dt + f(0) g(z).$$

Clearly, if $f(z) = z$ then $\mathcal{D}_f g = Vg = \int_0^z g(t)dt$ is the Volterra integration operator. Moreover, $\mathcal{D}_f V = V \mathcal{D}_f$. Next we consider the operator norm of $\mathcal{D}_f$.

Theorem 2.5 Suppose that $1 < p < \infty$, then $\mathcal{D}_f$ from $S^p(\mathbb{D})$ to $S^p(\mathbb{D})$ is bounded if and only if $f \in S^p(\mathbb{D})$. Moreover, the norm of operator satisfies $\|\mathcal{D}_f\| \approx \|f\|_{S^p}$.

Proof Firstly, by Theorem 2.3 and Minkowski inequality we have

$$\|\mathcal{D}_f g\|_{S^p} \lesssim \|g\|_{S^p}.$$

Conversely, if $\mathcal{D}_f$ is bounded. Take test function $g = 1$, then

$$\|\mathcal{D}_f\| \geq \|\mathcal{D}_f 1\|_{S^p} = \|f\|_{S^p}.$$

So we get $\|\mathcal{D}_f\| \approx \|f\|_{S^p}$. The proof of Theorem 2.5 is complete.

For the invertibility criterion of the Duhamel product, the Volterra integration operator acts as a bond. In the next Theorem we need the following convolutions(see [14])

$$(V^n g)(z) = \int_0^z \frac{(z-t)^{n-1}}{(n-1)!} g(t)\,dt = \left(\frac{\omega^{n-1}}{(n-1)!} * g\right)(z), \tag{2.2}$$

where the notation $*$ means

$$(f * g)(z) := \int_0^z f(z-t)g(t)\,dt.$$

Before we start proving the theorem, we need the following lemma.

Lemma 2.6 (see [15]) The Taylor series of every function $f \in S^p(\mathbb{D})$ with $f(0) = 0$ converges in norm if and only if $1 < p < \infty$.

Proof Let $1 < p < \infty$, if $f \in S^p(\mathbb{D})$, then $f'(z) \in H^p(\mathbb{D})$. By Corollary 3 of [15], the Taylor series of every function in $H^p(\mathbb{D})$ converges in norm $\|\cdot\|_{H^p}$. We get the Taylor series of $f'(z)$ converges in norm $\|\cdot\|_{H^p}$, i.e.

$$\lim_{n\to\infty} \|f(z) - f_n(z)\|_{S^p} = \lim_{n\to\infty} \|f'(z) - f_n'(z)\|_{H^p} = 0.$$

Theorem 2.7 Let $1 < p < \infty$. Then $f \in S_p(\mathbb{D})$ is $\circledast$-invertible if and only if $f(0) \neq 0$.

Proof If $f(z) \in S^p(\mathbb{D})$ is $\circledast$-invertible, then there exists $g \in S^p(\mathbb{D})$ such that

$$(f \circledast g)(z) = 1, \quad \forall z \in \mathbb{D}.$$

Clearly, if $z = 0$ we have $f(0)g(0) = 1$, i.e. $f(0) \neq 0$.

On the other hand, if $f(0) \neq 0$, let $F(z) = f(z) - f(0)$. Then, we consider the operator $\mathcal{D}_F : S^p(\mathbb{D}) \to S^p(\mathbb{D})$, defined by as follow

$$\mathcal{D}_F g = (F \circledast g)(z) = \int_0^z F'(z-t)g(t)\,dt = \int_0^z f'(z-t)g(t)\,dt.$$

Next, we show that $\mathcal{D}_F$ is compact. For the partial Taylor series

$$F_n = \sum_{n=0}^{N} \hat{F}(n) z^n = \sum_{n=1}^{N} \hat{f}(n) z^n.$$

Then,

$$\begin{aligned} \mathcal{D}_{F_N} g(z) &= \int_0^z F_N'(z-t)g(t)\,dt = \int_0^z F_N'(t)g(z-t)\,dt \\ &= \sum_{n=1}^{N} (n!)\hat{f}(n) \int_0^z \frac{(z-t)^{n-1}}{(n-1)!} g(z-t)\,dt. \end{aligned}$$

So, we deduced that

$$\mathcal{D}_{F_N}g(z)=\sum_{n=1}^{N}(n!)\hat{f}(n)V^n g(z).$$

Thus, $\mathcal{D}_{F_N}$ is compact on $S^p(\mathbb{D})$. Since

$$\|(\mathcal{D}_F-\mathcal{D}_{F_N})g\|_{S^p}=\|\mathcal{D}_{(F-F_N)}g(z)\|_{S^p}\lesssim\|F-F_N\|_{S^p}\|g\|_{S^p},$$

we have

$$\|\mathcal{D}_F-\mathcal{D}_{F_N}\|\lesssim\|F-F_N\|_{S^p}.$$

Let $N\to\infty$, then by Lemma 2.6 and Theorem 1.18 of [18], we deduced that $\mathcal{D}_F$ is compact. Now, suppose that $g\in\ker\mathcal{D}_f$, i.e.

$$\mathcal{D}_f g(z)=\int_0^z f'(z-t)g(t)\,dt+f(0)g(z)=0.$$

We get $g(0)=0$. Then derivative of the above equation, we have

$$\frac{d}{dz}\mathcal{D}_f g(z)=\int_0^z f''(z-t)g(t)dt+f(0)g'(z)+f'(0)g(z)=0.$$

We get $g'(0)=0$. Similarly, by repeating the derivative process, we deduced that $g^{(n)}(0)=0$. So we get $g=0$, i.e. $\ker\mathcal{D}_f=\{0\}$. Notice that

$$\mathcal{D}_f=f(0)I+\mathfrak{D}_F.$$

Since $\mathcal{D}_F$ is compact, $\mathcal{D}_f$ is injective and $f(0)\neq0$. Then, according to Fredholm alternative theorem (see [16, Theorem 3.22]). We infer that $\mathcal{D}_f$ is invertible in $S^p(\mathbb{D})$. The proof is complete.

Corollary 2.8 The maximal ideal space $\mathcal{M}(S^p,\circledast)$ consists of only one homomorphism.

Proof Apparently, the spectrum of any $f\in S^p(\mathbb{D})$ consists of only one point $f(0)$. The functions which vanish at the origin form a maximal ideal, since no proper ideal of $(S^p(\mathbb{D}),\circledast)$ contains any invertible f, and hence $\mathcal{M}$ is only one homomorphism, and Gelfand transform is trivial.

3 The Duhamel Convolution Operator

In this section, we consider the extended eigenvalue of Volterra integration operator V. Let λ be a complex number, if there exists a nonzero operator X satisfying the equation $XV=\lambda VX$, then we say λ is an extended eigenvalue of V. In the next assertion, we consider the extended eigenvector of the Volterra integration operator V. For all bounded linear operators in the $S^p(\mathbb{D})$ space we denote by $\mathcal{B}(S^p(\mathbb{D}))$.

Theorem 3.1 Let $\lambda\neq0\in\mathbb{C}$, $1\leq p<\infty$, and $X\in\mathcal{B}(S^p(\mathbb{D}))$, then we have

(1) Suppose that $|\lambda| \leq 1$, then, $XV = \lambda VX$ if and only if $X = \mathcal{D}_{X1}\mathcal{C}_\lambda$.

(2) Suppose that $|\lambda| > 1$, then, $XV = \lambda VX$ if and only if $X\mathcal{C}_{\frac{1}{\lambda}} = \mathcal{D}_{X1}$.

Proof Firstly we consider (1), if $XV = \lambda VX$, then $XV^n = \lambda^n V^n X$. In particular, take the function 1, we have $XV^n 1 = \lambda^n V^n X1$. By (2.2), we get

$$X\frac{z^n}{n!} = \lambda^n \frac{z^{n-1}}{(n-1)!} * X1,$$

then

$$1 * X\frac{z^n}{n!} = 1 * \lambda^n \frac{z^{n-1}}{(n-1)!} * X1 = \frac{(\lambda n)^n}{n!} * X1.$$

Similarly, for any polynomial $P(z)$

$$1 * XP(z) = P(\lambda z) * X1.$$

Since the polynomial is dense, we get

$$1 * Xf(z) = f(\lambda z) * X1.$$

We deduce that

$$VXf = \int_0^z X1(z-t)C_\lambda f(t)\,dt.$$

Then take derivative of the above equation,

$$Xf(z) = \frac{d}{dz}\int_0^z X1(z-t)C_\lambda f(t)\,dt = \mathcal{D}_{X1}C_\lambda f(z).$$

Conversely, suppose that $X = \mathcal{D}_{X1}C_\lambda$, then

$$\begin{aligned} XVf(z) &= \mathcal{D}_{X1}C_\lambda Vf(z) = X1 \circledast (Vf)(\lambda z) \\ &= X1 \circledast \lambda z \circledast f(\lambda z) = \lambda z \circledast X1 \circledast C_\lambda f(z) \\ &= \lambda\big(z \circledast \mathcal{D}_{X1}C_\lambda f(z)\big) = \lambda VXf(z). \end{aligned}$$

Thus we have $XV = \lambda VX$.

The proof of (2) is similar to that of (1), and a brief proof is given for completeness. Suppose that $|\lambda| > 1$, then $V^n X = \frac{1}{\lambda^n}XV^n$. Choose the function 1, we get $V^n X1 = \frac{1}{\lambda^n}XV^n 1$. By (2.2) we have

$$1 * V^n X1 = 1 * \frac{1}{\lambda^n}XV^n 1,$$

i.e.

$$\frac{z^n}{n!} * X1 = 1 * X\frac{\frac{z^n}{\lambda^n}}{n!}.$$

Since ploynoimal is dense, we deduce that

$$1 * Xf(\frac{z}{\lambda}) = f(z) * X1.$$

Taking the derivative of the above equation, we get $XC_{\frac{1}{\lambda}} = \mathcal{D}_{X1}$.

Conversely, suppose that $XC_{\frac{1}{\lambda}} = \mathcal{D}_{X1}$, then

$$\lambda VXz = \lambda VXC_{\frac{1}{\lambda}}C_{\lambda}z = \lambda V\mathcal{D}_{X1}C_{\lambda}z.$$

Since

$$V\mathcal{D}_{X1}f = z \circledast X1 \circledast f = X1 \circledast z \circledast f = \mathcal{D}_{X1}Vf,$$

so, for any polynomials $P(z)$ we get

$$\begin{aligned}\lambda VXP(z) &= \lambda V\mathcal{D}_{X1}C_{\lambda}P(z) = \lambda\mathcal{D}_{X1}VP(\lambda z)\\ &= \lambda XC_{\frac{1}{\lambda}}VP(\lambda z) = XC_{\frac{1}{\lambda}}(\lambda z \circledast P(\lambda z))\\ &= X(z \circledast P(z)) = XVP(z).\end{aligned}$$

By density, we get $\lambda VXf = XVf$. The proof of the Theorem 2.5 is complete.

If $\lambda = 1$, we have $X = \mathcal{D}_{X1}$, i.e the commutant of the Volterra operator on the $S^p(\mathbb{D})$ is the Duhamel product.

Remark 1 The commutants of the Volterra integration operator on the $S^p(\mathbb{D})$ is characterized by $\{V\}' = \{\mathcal{D}_f : f \in S^p(\mathbb{D})\}$.

Corollary 3.2 Let $f \in S^2(\mathbb{D})$, the Duhamel convolution operator $\mathfrak{D}_f = \overline{\mathrm{alg}\{V\}}$, where

$$\mathrm{alg}\{V\} := \{p(V) : p \text{ is a poloynomial}\}.$$

Clearly, $V\mathfrak{D}_f = \mathfrak{D}_f V$ and $S^2(\mathbb{D})$ is a Hilbert space. By Theorem 2.5 we have $\mathfrak{D}_f \in \mathcal{B}(S^2(\mathbb{D}))$. Every bounded linear operator commuting with V is in the strongly closed algebra generated by V (see [14, 17]). So, $\mathfrak{D}_f = \overline{\mathrm{alg}\{V\}}$.

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导数哈代空间中的巴拿赫代数结构

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摘要: 本文研究了导数哈代空间中的代数结构问题, 利用[6, 12, 15]中方法, 得到了Duhamel乘积在导数哈代空间中构成巴拿赫代数以及可逆的充要条件, 并且刻画了积分算子V的拓展特征值. 推广了[1, 2, 6, 11, 16]中的结果.

关键词: Duhamel乘积; 巴拿赫代数; 拓展特征值

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