THE PERFECT INTEGER $k$-MATCHINGS AND $k$-FACTOR-CRITICAL GRAPHS

ZHANG Yan-hong$^1$, ZHANG Lei$^{1,2,3}$, REN Hai-zhen$^{1,2,3}$

(1. School of Mathematics and Statistics, Qinghai Normal University, Xining 810008, China)
(2. The State Key Laboratory of Tibetan Information Processing and Application, Xining 810008, China)
(3. Academy of Plateau, Science and Sustainability, Xining 810008, China)

Abstract: This article investigates the existence of perfect integer $k$-matchings and $k$-factor critical graphs. The extension constant represents the connectivity strength of a graph. For regular graphs, a sufficient condition for the existence of perfect integer $k$-matching is given using the extension constant, which extends the results of Hamers et al. and Cioab˘a et al. In addition, for regular graphs, a sufficient condition for the existence of $k$-factor-critical graphs based on extension constant is also given.

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1 Introduction

All graphs considered here are undirected, connected and simple. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The order of $G$ is $|V(G)|$. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $\Gamma(v) = \{e \in E(G) : e$ is incident with $v$ in $G\}$. An integer $k$-matching of a graph $G = (V, E)$ is a function $f$ that assigns to each edge an integer in $\{0, \cdots, k\}$ such that $\sum_{e \in \Gamma(v)} f(e) \leq k$ for each $v \in V$. A vertex $v$ of $G$ is saturated by an integer $k$-matching $f$ or $v$ is $f$-saturated if $\sum_{e \in \Gamma(v)} f(e) = k$, otherwise, $v$ is $f$-unsaturated. An integer $k$-matching is perfect if $\sum_{e \in \Gamma(v)} f(e) = k$ for every vertex $v \in V(G)$.

Clearly, an integer $k$-matching is perfect if and only if its size is $\frac{k|V(G)|}{2}$. Note that when $k$ is odd, a graph with a perfect integer $k$-matching has an even number of vertices. For $k = 1$, the integer (or perfect) 1-matching is just the matching (or perfect matching) in usual sense. If $k = 2$, then the perfect 2-matching coincides with the one defined by Tutte ([1]).

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Biography: Zhang Yanhong(1998-), female, born at Shandong, postgraduate, major in graph theory with applications. E-mail: yhzhang1693@163.com.
Corresponding author: Ren Haizhen
Integer $k$-matchings have attracted many researchers’ attention, such as Lu et al. ([2]) studied the perfect $k$-matchings of general graph and gave a sufficient and necessary condition for its existence, which is the Tutte’s Theorem for perfect integer $k$-matching of a graph. Liu et al. ([3]) proved that when $k$ is even, the integer $k$-matching number of $G$ equals $k$ times its fractional matching number. Hence, by the Berge-Tutte formula of fractional matching in ([4]), they can easily obtained the integer $k$-matching analogue of the Berge-Tutte Formula when $k$ is even.

In addition, Gallai (1963, [5]) introduced the concepts of factor-critical graphs, i.e. a graph $G$ is factor-critical if $G - v$ has a perfect matching for every vertex $v$ of $G$. Very recently, Liu et al. ([6]) extended this definition and defined $k$-factor-critical graph. A connected graph $G$ with at least three vertices is said to be $k$-factor-critical if for any $v \in V(G)$, there exists an integer $k$-matching $h$ such that $\sum_{e \in \Gamma(v)} f(e) = k - 1$ and other vertices are $h$-saturated. We note that this definition is different from the one defined by Favaron and Odile ([7]). Moreover, Liu et al. ([6]) proved the integer $k$-matching analogue of the Berge-Tutte Formula when $k$ is odd, they also gave sufficient and necessary conditions for the existence of $k$-factor-critical.

For two subsets $S, T \subset V(G)$, let $e(S, T)$ denotes the number of edges of $G$ joining $S$ to $T$. For a set $X$, we denote the cardinality of $X$ by $|X|$. Given a subset $S \subset V(G)$, we let $e(S, S^c)$ denote the number of edges with exactly one endpoint in $S$, where $S^c$ denotes the complement of $S$ in $K_{|V(G)|}$. The expansion constant of a graph $G$ is $h(G) = \min \frac{e(S, S^c)}{|S|}$, where the minimum is taken over all $S \subset V(G)$ with $|S| \leq \frac{|V(G)|}{2}$, the expansion constant of a graph represents a strength of connection of the graph ([8]).

In [9], Cioabă proved a sufficient condition for the existence of a perfect matching in a regular graph in terms of expansion constant and $\lambda_3$, which improved the result of Brouwer and Haemers ([10]). For regular graphs, Lu et al. ([2]) provided a sufficient condition for the existence of perfect $k$-matching in terms of the edge connectivity. Based on this, we will consider the existence of perfect integer $k$-matching and $k$-factor-critical graph in terms of the expansion constant. We present a sufficient condition for the existence of a perfect integer $k$-matching in a regular graph in terms of its expansion constant, which generalizes the results of Haemers et al. ([10]) and Cioabă et al. ([9]). Furthermore, a sufficient condition for the existence of $k$-factor-critical graphs for regular graphs in terms of its expansion constant is obtained.

2 Expansion and Perfect Integer $k$-Matching

In this section, we determine a lower bound on the expansion constant of a regular graph, which implies the existence of a perfect integer $k$-matching.

Given a subset $S \subseteq V(G)$, we say $G - S$ is the subgraph obtained from $G$ by deleting all vertices of $S$ and their incident edges. In 2014, Lu et al. ([2]) obtained the Tutte’s Theorem for perfect integer $k$-matching of a graph.

Lemma 2.1 (see [2]) Let $k \geq 2$ be even. A graph $G$ has a perfect integer $k$-matching
if and only if
\[ i(G - S) \leq |S| \text{ for all } S \subseteq V(G), \]
where \( i(G - S) \) is the number of isolated vertices of \( G - S \).

Let \( \text{odd}(G) \) denote the number of odd components with order at least three in \( G \).

**Lemma 2.2** (see [2]) Let \( k \geq 1 \) be odd. A graph \( G \) has a perfect integer \( k \)-matching if and only if
\[ \text{odd}(G - S) + k \cdot i(G - S) \leq k|S| \text{ for all } S \subseteq V(G), \]
where \( \text{odd}(G - S) \) is the number of odd components with order at least three of \( G - S \) and \( i(G - S) \) is the number of isolated vertices of \( G - S \).

**Theorem 2.1** Let \( G \) be a \( d \)-regular graph with \( n \) vertices.

1. If \( k \) is even, and
   \[ h(G) \geq \begin{cases} \frac{d-2}{d+1}, & \text{if } d \text{ is even}, \\ \frac{d-1}{d+1}, & \text{otherwise}, \end{cases} \]
then \( G \) has a perfect integer \( k \)-matching;

2. For even \( n \). If \( k \) is odd, and
   \[ h(G) \geq \begin{cases} \frac{d-2}{d+2}, & \text{if } d \text{ is even}, \\ \frac{d-1}{d+2}, & \text{otherwise}, \end{cases} \]
then \( G \) has a perfect integer \( k \)-matching.

**Proof** (1) Assume \( G \) has no perfect integer \( k \)-matching. By Lemma 2.1, there is a vertex set \( S \) such that \( i(G - S) > |S| \). Let \( |S| = s \) and \( i(G - S) = q \). We easily observe \( q \geq s + 1 \). Let \( V' = V(G) \setminus S \). Thus \( |V'| = n - s \). We denote the \( q \) isolated vertices of \( G - S \) by \( v_1, v_2, \ldots, v_q \). Let \( G_1, G_2, \ldots, G_q \) be the subgraphs induced by \( v_i \) and \( N_G(v_i) \), where \( N_G(v_i) \) is the set of neighbours of \( v_i \) and \( 1 \leq i \leq q \). Denote by \( n_i \) and \( e_i \) the order and the size of \( G_i \), respectively.

For \( i \in [q] \), let \( t_i \) denote the number of edges in \( G \) between \( G_i \) and \( S - N_G(v_i) \). Since \( G \) is connected, it follows that \( t_i \geq 1 \) for each \( i \in [q] \). Because \( v_i \) is adjacent only to vertices in \( S \), we deduce that \( 2e_i = d(d + 1) - t_i \). Clearly \( t_i \) is even.

The sum of the degrees of the vertices in \( S \) is at least the number of edges between \( S \) and \( \bigcup_{i=1}^{q} v_i \). Thus, \( ds \geq \sum_{i=1}^{q} t_i \). Since \( q \geq s + 1 \), it follows that there are at least two \( t_i \)'s such that \( t_i < d \). This implies there are at least two \( t_i \)'s satisfying \( t_i \leq d - 2 \) if \( d \) is even and \( t_i \leq d - 1 \) if \( d \) is odd.

The fact \( n_1 + n_2 < n \) implies that there is at least one \( i \) such that \( n_i < \frac{q}{2} \). Without losing generality, assume \( n_1 < \frac{q}{2} \). Then we obtain \( \frac{d}{n_1} < \frac{d-2}{d+1} \) if \( d \) is even and \( \frac{d}{n_1} \leq \frac{d-1}{d+1} \) if \( d \) is odd. This contradicts the assumption made on the expansion constant, which finishes the proof.

(2) By the definition of perfect integer \( k \)-matching, we know that a graph with a perfect integer \( k \)-matching has an even number of vertices when \( k \) is odd. So we only discuss the case \( n \) is even.
Assume $G$ has no perfect integer $k$-matching. By Lemma 2.2, there is a vertex set $S$ such that \( \text{odd}(G - S) + k \cdot i(G - S) > k|S| \). Let \( |S| = s \) and \( \text{odd}(G - S) + i(G - S) = q \). By easily process, we can obtain \( q > s \). But since $n$ is even, \( s + q \) is even, hence \( q \geq s + 2 \). Let $V' = V(G) \setminus S$. Thus \( |V'| = n - s \). We denote the $q$ components by $G_1, G_2, \ldots, G_q$. Denote by $n_i$ and $e_i$ the order and the size of $G_i$ respectively.

For $i \in [q]$, let $t_i$ denotes the number of edges with one endpoint in $G_i$ and another in $S$. Since $G$ is connected, it follows that $t_i \geq 1$ for each $i \in [q]$. Because vertices in $G_i$ are adjacent only to vertices in $G_i$ or $S$, we deduce that $2e_i = dn_i - t_i = dn_i - t_i - 1 + d - t_i$. Because $n_i$ is odd, it follows that $d - t_i$ is even. Hence, $t_i$ and $d$ have the same parity for each $i$.

The sum of the degrees of the vertices in $S$ is at least the number of edges between $S$ and $\bigcup_{i=1}^{q} G_i$. Thus, \( ds \geq \sum_{i=1}^{q} t_i \). Since $q \geq s + 2$, it follows that there are at least three $t_i$’s such that $t_i < d$. This implies there are at least three $t_i$’s satisfying $t_i \leq d - 2$. If $t_i \leq d - 2$, then $n_i > 1$. Assume $t_i \leq d - 2$ for $i \in [3]$. If $t_i \leq d - 2$, then $n_i(n_i - 1) \geq 2e_i = dn_i - t_i \geq d n_i - d + 2$. Thus, $n_i \geq d \cdot \frac{2}{n_i - 1}$. Hence, $n_i \geq d + 1$. Now, since $d$ and $n_i$ are both odd, we obtain $n_i \geq d + 2$.

The fact $n_1 + n_2 + n_3 < n$ implies that there is at least one $i$ such that $n_i < \frac{n}{2}$. Without losing generality, assume $n_1 < \frac{n}{2}$. Then we obtain $\frac{n_1}{n_1} \leq \frac{d + 2}{d + 1}$ if $d$ is even and $\frac{n_1}{n_1} \leq \frac{d + 2}{d + 2}$ if $d$ is odd. This contradicts the assumption made on the expansion constant, which finishes the proof.

From the above Theorem 2.1, if $n$ is even and let $k = 1$, then we may deduce the following results obtained by Cioabă [9], which provides an expansion constant condition to guarantee that there exists a perfect matching in a regular graph $G$.

**Corollary 2.1** (see [9, 10]) Let $G$ be a $d$-regular graph. If $n$ is even and

$$h(G) \geq \begin{cases} \frac{d - 2}{d + 1}, & \text{if } d \text{ is even}, \\ \frac{d - 2}{d + 2}, & \text{if } d \text{ is odd}, \end{cases}$$

then $G$ has a perfect matching.

### 3 Expansion and $k$-Factor-Critical Graph

In this section, for a positive integer $k$ and a regular graph $G$, we obtain a condition of $k$-factor-critical according to a lower bound of the expansion constant $h(G)$.

In 2021, Liu et al. ([6]) gave a sufficient and necessary condition for the existence of $k$-factor-critical graph.

**Lemma 2.3** (see [6]) A connected graph $G$ with at least three vertices is $k$-factor-critical if and only if $G$ has odd number of vertices and $\text{odd}(G - S) + k \cdot i(G - S) \leq k|S| - 1$ for any $\emptyset \neq S \subseteq V(G)$. 
Theorem 2.2  Let $G$ be a $d$-regular graph with $n$ vertices. If $n$ is odd and

$$h(G) \geq \begin{cases} \frac{d-2}{d+1}, & \text{if } d \text{ is even}, \\ \frac{d-2}{d+2}, & \text{if } d \text{ is odd}, \end{cases}$$

then $G$ is $k$-factor-critical.

Proof Suppose $G$ satisfies the conditions of this theorem, but it is not $k$-factor-critical. By Lemma 2.3, there is a vertex set $S$ such that $\text{odd}(G-S) + k \cdot i(G-S) \geq k|S|$. Let $|S| = s$ and $\text{odd}(G-S) + i(G-S) = q$, we can get $q \geq s$. And since $n$ is odd, $s + q$ is odd, hence $q \geq s + 1$.

Let $G_1, G_2, \ldots, G_q$ be the $q$ odd components with odd vertices of $G - S$. We denote the order and size of $G_i$ by $n_i$ and $e_i$, respectively. For $i \in [q]$, let $t_i$ denote the number of edges with one endpoint in $G_i$ and another in $S$. We know $t_i \geq 1$ for $i \in [q]$ because $G$ is connected. Since vertices in $G_i$ are adjacent only to vertices in $G_i$ or $S$, we can get that $2e_i = dn_i - t_i = d(n_i - 1) + d - t_i$. This implies that $d - t_i$ is even since $n_i$ is odd. Therefore, $t_i$ and $d$ have the same parity for each $i$.

The sum of degrees of the vertices of $S$ is at least the number of edges between $S$ and the odd component $G_i$ for $i \in [q]$. Hence, $ds \geq \sum_{i=1}^{q} t_i$. Because $q \geq s + 1$, it follows that there are at least two $t_i$'s such that $t_i < d$. Hence, there are at least two $t_i$'s such that $t_i < d - 2$. If $t_i \leq d - 2$, then $n_i > 1$. Without loss of generality, we assume $t_i \leq d - 2$ for $i \in [2]$, then $n_i(n_i - 1) \geq 2e_i = dn_i - t_i$. Hence, $n_i \geq d + \frac{2}{n_i - 1}$. Then $n_i \geq d + 1$. If $d$ is odd, we get $n_i \geq d + 2$ since $n_i$ is odd.

There is at least one $i$ such that $n_i < \frac{d}{2}$ because of $n_1 + n_2 < n$. We assume $n_1 < \frac{n}{2}$. Therefore, we get $\frac{n_1}{n_1} \leq \frac{d-2}{d+1}$ if $d$ is even and $\frac{n_1}{n_1} \leq \frac{d-2}{d+2}$ if $d$ is odd. This contradicts the condition made on the expansion constant, which completes the proof.

References


完美整数 $k$-匹配和 $k$-因子临界图

张燕红¹, 张 磊¹²³, 任海珍¹²³
(1. 青海师范大学数学与统计学院, 西宁 810008)
(2. 藏文信息处理与应用国家重点实验室, 西宁 810008)
(3. 高原科学与可持续发展研究院, 西宁 810008)

摘 要: 本文研究完美整数 $k$-匹配和 $k$-因子临界图的存在性. 扩张常数表示图的连通强度, 对于正则图, 利用扩张常数给出了完美整数 $k$-匹配存在的一个充分条件, 这推广了 Hamers 等人和 Cioabă 等人的结果. 此外, 对于正则图, 基于扩张常数还给出了 $k$-因子临界图存在的一个充分条件.

关键词: 完美整数 $k$-匹配; $k$-因子临界图; 连通性; 扩张常数

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