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SOLVABILITY OF COUPLED SYSTEMS OF FRACTIONAL p-LAPLACIAN EQUATION WITH IMPULSIVE EFFECTS

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Abstract: This article investigates the problem of a coupled system of fractional order p-Laplacian equations with impulsive effects. Some new results for the existence of solutions of this system are obtained by using variational method. In the process of proof, the conditions of variable coefficient and nonlinear term in the system are weakened, and the existing results are extended.

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1 Introduction and main results

Fractional differential equations have been extensively applied in mathematical modeling. The theory of fractional differential equations is a hot topic in recent decades. Many scholars have developed a strong interest in this kind of problem and achieved some excellent results [1-8]. It is well known that left and right fractional differential operators are widely used in physical phenomena of anomalous diffusion, such as fractional convection diffusion equation [9-10]. In recent years, the equations containing left and right fractional differential operators have become a new research field in the theory of fractional differential equations. For example, Ervin and Roop [11] first proposed a class of steady-state fractional convection-diffusion equations with variational structure

$$\begin{cases} -aD\left(p_0 D_t^{-\beta}+q_t D_T^{-\beta}\right)Du+b\left(t\right)Du+c\left(t\right)u=f,\ 0\le\beta<1, \\ u\left(0\right)=u\left(T\right)=0, \end{cases}$$

where D is the classical first derivative, ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional derivatives. The authors constructed a suitable fractional derivative space. By

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using Lax-Milgram theorem, the solution of problem was studied. The following Dirichlet problems were discussed in [12]

$$\begin{cases} \frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}\left(u'\left(t\right)\right)+\frac{1}{2}{}_tD_T^{-\beta}\left(u'\left(t\right)\right)\right)+\nabla F(t,u\left(t\right))=0,\ 0\leq\beta<1,\\ u(0)=u(T)=0.\end{cases}$$

The existence result of the solution was obtained by Mountain pass theorem and the minimization principle under the Ambrosetti-Rabinowtiz condition. The following year, the authors [13] used the critical point theory to further discuss the following problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t))=\nabla F(t,u(t)),\quad \text{a.e. } t\in[0,T],\ \frac{1}{2}<\alpha\leq 1,\\ u(0)=u(T)=0.\end{cases}$$

Under the Ambrosetti-Rabinowtiz condition, the existence of the weak solution was obtained by using Mountain pass theorem. In addition, the authors also discussed the regularity of the weak solution.

In recent decades, impulsive differential equations have been the focus of mathematicians' research. Impulsive differential equation is an effective method to describe the instantaneous change of the state of things, and it can reflect the changing law of things more deeply and accurately. It has practical significance and application value in many fields of science and technology, such as signal communication, economic regulation, aerospace technology, management science, engineering science, chaos theory, information science, life science and so on. Many scholars at home and abroad have studied this kind of problem. For example, in [14-15], the authors considered the following fractional impulsive problems

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}u(t))+a(t)u(t)=\lambda f(t,u(t)),\ t\neq t_j,\ \text{a.e.}\ t\in[0,T],\\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u))(t_j)=\mu I_j(u(t_j)),\ j=1,2,\cdots,n,\\ u(0)=u(T)=0,\end{cases}$$

where $\alpha\in(\frac{1}{2},1]$, $\lambda,\ \mu\in(0,+\infty)$, $I_j\in C(\mathbb{R},\mathbb{R})$, $j=1,2,\cdots,n$. $a\in C([0,T])$ and there exist two positive constants a_1, a_2 such that $0<a_1\leq a(t)\leq a_2$. In addition,

$$\Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u))(t_j)={}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^+)-{}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^-),$$

$${}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^+)=\lim_{t\to t_j^+}({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t)),$$

$${}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t_j^-)=\lim_{t\to t_j^-}({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}u)(t)).$$

The main tools used in this paper are variational method and three critical points theorem. Torres and Nyamoradi [16] explored fractional p-Laplacian problems with impulsive effects

$$\begin{cases} {}_tD_T^{\alpha}\left(|{}_0D_t^{\alpha}u(t)|^{p-2}{}_0D_t^{\alpha}u(t)\right)+a(t)|u(t)|^{p-2}u(t)=f(t,u(t)),\ t\neq t_j,\ \text{a.e. } t\in[0,T],\\ \Delta\left({}_tI_T^{1-\alpha}\left(|{}_0D_t^{\alpha}u(t_j)|^{p-2}{}_0D_t^{\alpha}u(t_j)\right)\right)=I_j(u(t_j)),\ j=1,2,\cdots,n,\ n\in\mathbb{N},\\ u(0)=u(T)=0,\end{cases}$$

where $\alpha \in (\frac{1}{p}, 1]$, $p \in (1, \infty)$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $I_j \in C(\mathbb{R}, \mathbb{R})$. The solution of the problem was discussed under the condition of Ambrosetti-Rabinowtiz by using Mountain pass theorem. On the other hand, the coupled systems of fractional differential equations have gained importance due to their applications in many fields of science and engineering. For example, Zhao et al. [17] investigated the following coupled system of fractional differential equations

$$\begin{cases} {}_tD_T^{\alpha}(a(t){}_0D_t^{\alpha}u(t)) = \lambda f_u(t, u(t), v(t)), \ 0 < t < T, \\ {}_tD_T^{\beta}(b(t){}_0D_t^{\beta}v(t)) = \lambda f_v(t, u(t), v(t)), \ 0 < t < T, \\ u(0) = u(T) = 0, v(0) = v(T) = 0, \end{cases}$$

where $\lambda > 0$, $0 < \alpha$, $\beta \leq 1$, a, $b \in L^{\infty}[0, T]$ with $a_0 := \operatorname{es\,sin\,f}_{[0,T]} a(t) > 0$ and $b_0 := \operatorname{es\,sin\,f}_{[0,T]} b(t) > 0$. By the variational methods, the existence results were obtained.

Inspired by the above literature, we study the following fractional impulsive coupled systems

$$\begin{cases} {}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) + a(t)\phi_p(u(t)) = \chi f_u(t, u(t), v\,(t)), \ t \neq t_j, \ \text{a.e. } t \in [0, T], \\ {}_tD_T^{\beta}\phi_p({}_0D_t^{\beta}v(t)) + b(t)\phi_p(v(t)) = \chi f_v(t, u(t), v\,(t)), \ t \neq t'_i, \ \text{a.e. } t \in [0, T], \\ \Delta({}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), \Delta({}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v))(t'_i) = \mu S_i(v(t'_i)), \\ u(0) = u(T) = v(0) = v(T) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\alpha, \beta \in (1/p, 1]$, $\chi > 0$, $\mu \in \mathbb{R}$, $\phi_p(x) = |x|^{p-2}x \ (x \neq 0)$, $\phi_p(0) = 0$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, u, v)$ is continuous in $[0, T]$ for every $(u, v) \in \mathbb{R}^2$ and $f(t, \cdot, \cdot)$ is a C^1 function in $\mathbb{R}^2$ for any $t \in [0, T]$, and f_s denotes the partial derivative of f with respect to s. $I_j, S_i \in C(\mathbb{R}, \mathbb{R})$, $j = 1, 2, \cdots, m$, $m, \in \mathbb{N}$, $i = 1, 2, \cdots, n$, $n \in \mathbb{N}$, $a(t), b(t) \in C([0, T], \mathbb{R})$, $T > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $0 = t'_0 < t'_1 < \cdots < t'_n < t'_{n+1} = T$, and

$$\Delta({}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u))(t_j) = {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^-),$$
$$\Delta({}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v))(t'_i) = {}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v)(t'_i{}^+) - {}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v)(t'_i{}^-),$$

$${}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^+) = \lim_{t \to t_j^+} {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t),$$

$${}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^-) = \lim_{t \to t_j^-} {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t),$$

$${}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v)(t'_i{}^+) = \lim_{t \to t'_i{}^+} {}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v)(t),$$

$${}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v)(t'_i{}^-) = \lim_{t \to t'_i{}^-} {}_tD_T^{\beta-1}\phi_p({}_0^CD_t^{\beta}v)(t).$$

For ease of reading, here are some additional definitions of fractional order derivatives. Let $n - 1 \leq \gamma < n$, $n \in \mathbb{N}$, then ${}_0D_t^{\gamma}u(t)$ and ${}_tD_T^{\gamma}u(t)$ represent the left and right Riemann-Liouville fractional order derivatives, respectively, in the following form:

$${}_0D_t^{\gamma}u(t) = \frac{d^n}{dt^n}{}_0I_t^{n-\gamma}u = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\gamma-1}uds,$$

$$ {}_tD_T^{\gamma}u(t) = (-1)^n \frac{d^n}{dt^n} {}_tI_T^{n-\gamma}u = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_t^T (s-t)^{n-\gamma-1} u ds. $$

${}_0^CD_t^{\gamma}u(t)$ represents the left Caputo fractional order derivative, in the following form:

$$ {}_0^CD_t^{\gamma}u(t) = {}_0I_t^{n-\gamma} \frac{d^n u(t)}{dt^n} = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1} u^{(n)}(s) ds. $$

If $\alpha = \beta = 1$, $p = 2$, $a(t) = b(t) = 1$, $\chi = \mu = 1$, then the above fractional coupled systems with impulsive effects are reduced to a famous second order impulsive coupled systems

$$ \begin{cases} \ddot{u}(t) + u(t) = f_u(t, u(t), v(t)), \ t \neq t_j, \ \text{a.e. } t \in [0, T], \\ \ddot{v}(t) + v(t) = f_v(t, u(t), v(t)), \ t \neq t'_i, \ \text{a.e. } t \in [0, T], \\ \Delta(\dot{u}(t_j)) = I_j(u(t_j)), \ j = 1, 2, \cdots, m, \\ \Delta(\dot{v}(t'_i)) = S_i(v(t'_i)), \ i = 1, 2, \cdots, n, \\ u(0) = u(T) = v(0) = v(T) = 0, \ \text{a.e. } t \in [0, T]. \end{cases} $$

This paper studies a class of fractional impulsive coupled systems with p-Laplacian operator. Under the condition that the nonlinear term satisfies a new class of conditions and the impulse function satisfies a sub-linear condition, the existence of at least three weak solutions for the coupled system is obtained by using the three critical points theorem. In literatures [14-16], the authors only study the existence of solutions for boundary value problems of fractional differential equations with impulsive effects by using the critical point theory, while this paper studies the coupled systems of fractional differential equations with impulsive effects. To some extent, it generalizes the existing results of [14-16]. At the same time, this paper requires $\text{essinf}_{t\in[0,T]} a(t) > -\lambda_1$, $\lambda_1 > 0$, which weakens the relevant condition $0 < a_1 \leq a(t) \leq a_2$ in [14-15], thereby improving the existing results in [14-15].

2 Preliminaries

For basic concepts and lemmas of fractional derivatives and integrals, please see [18-19]. Here, we give some important lemmas and definitions.

Proposition 2.1 ([18]) Let u be a function defined on $[a, b]$, $0 < a < b$. If ${}_a^cD_t^{\gamma}u(t)$, ${}_t^cD_b^{\gamma}u(t)$, ${}_aD_t^{\gamma}u(t)$ and ${}_tD_b^{\gamma}u(t)$ all exist, then

$$ {}_a^cD_t^{\gamma}u(t) = {}_aD_t^{\gamma}u(t) - \sum_{j=0}^{n-1} \frac{u^j(a)}{\Gamma(j-\gamma+1)} (t-a)^{j-\gamma}, t \in [a, b], $$

$$ {}_t^cD_b^{\gamma}u(t) = {}_tD_b^{\gamma}u(t) - \sum_{j=0}^{n-1} \frac{u^j(b)}{\Gamma(j-\gamma+1)} (b-t)^{j-\gamma}, t \in [a, b], $$

where $n \in \mathbb{N}$, $n - 1 < \gamma < n$, $\Gamma(j - \gamma + 1)$ is the Euler gamma function, in the following form:

$$ \Gamma(j - \gamma + 1) = \int_0^{\infty} t^{j-\gamma} e^{-t} dt, \ t^{j-\gamma} = e^{(j-\gamma)\log(t)}. $$

Definition 2.1 ([19]) Let $0<\alpha\leq 1$, $1<p<\infty$. Define the fractional derivative space $E^{\alpha,p}$ as follows

$$E^{\alpha,p}=\left\{u\in L^p\left([0,T],\mathbb{R}\right)|_0D_t^\alpha u\in L^p\left([0,T],\mathbb{R}\right)\right\},$$

with the norm

$$\|u\|_{E^{\alpha,p}}=(\|u\|_{L^p}^p+\|_0D_t^\alpha u\|_{L^p}^p)^{\frac{1}{p}}, \tag{2.1}$$

where $\|u\|_{L^p}=(\int_0^T|u\left(t\right)|^pdt)^{1/p}$ is the norm of $L^p\left([0,T],\mathbb{R}\right)$. $E_0^{\alpha,p}$ is defined by closure of $C_0^\infty\left([0,T],\mathbb{R}\right)$ with respect to the norm $\|u\|_{E^{\alpha,p}}$.

Remark 2.1 For any $u\in E_0^{\alpha,p}$, according to Proposition 2.1, when $0<\alpha<1$ and the boundary conditions $u(0)=u(T)=0$ are satisfied, we can get ${}_0^cD_t^\alpha u(t)={}_0D_t^\alpha u(t)$, ${}_t^cD_T^\alpha u(t)={}_tD_T^\alpha u(t)$, $t\in[0,\ T]$.

Lemma 2.1 ([19]) Let $0<\alpha\leq 1$, $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ with respect to the norm $\|u\|_{E^{\alpha,p}}$ is a reflexive and separable Banach space.

Lemma 2.2 ([13]) Let $0<\alpha\leq 1$, $1<p<\infty$. If $u\in E_0^{\alpha,p}$, then

$$\|u\|_{L^p}\leq\frac{T^\alpha}{\Gamma\left(\alpha+1\right)}\|_0D_t^\alpha u\|_{L^p}. \tag{2.2}$$

If $\alpha>1/p$, then

$$\|u\|_\infty\leq C_\infty\|_0D_t^\alpha u\|_{L^p}, \tag{2.3}$$

where $\|u\|_\infty=\max_{t\in[0,T]}|u\left(t\right)|$ is the norm of $C\left([0,T],\mathbb{R}\right)$, and

$$C_\infty=\frac{T^{\alpha-\frac{1}{p}}}{\Gamma\left(\alpha\right)\left(\alpha\mathrm{p}^*-\mathrm{p}^*+1\right)^{\frac{1}{\mathrm{p}^*}}}>0,\quad \mathrm{p}^*=\frac{p}{p-1}>1.$$

According to (2.2), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{E^{\alpha,p}}=(\int_0^T|_0D_t^\alpha u\left(t\right)|^pdt)^{\frac{1}{p}}=\|_0D_t^\alpha u\|_{L^p},\ \forall u\in E_0^{\alpha,p}. \tag{2.4}$$

Lemma 2.3 ([13]) Assume that $1/p<\alpha\leq 1,\ 1<p<\infty$, then $E_0^{\alpha,p}$ is compactly embedded in $C\left([0,T],\mathbb{R}\right)$.

Lemma 2.4 ([13]) Let $1/p<\alpha\leq 1,\ 1<p<\infty$. Assume that the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_k\rightharpoonup u$, then $u_k\to u$ in $C\left([0,T],\mathbb{R}\right)$, i.e., $\|u_k-u\|_\infty\to 0,\ k\to\infty$.

To investigate problem (1.1), this article defines a new norm on the space $E_0^{\alpha,p}$, as follows

$$\|u\|_\alpha=(\int_0^T|_0D_t^\alpha u\left(t\right)|^pdt+\int_0^T a(t)|u\left(t\right)|^pdt)^{\frac{1}{p}}. \tag{2.5}$$

Lemma 2.5 ([16]) If $\mathrm{essinf}_{t\in[0,T]}a(t)>-\lambda_1$, where $\lambda_1=\inf\limits_{u\in E_0^{\alpha,p}\setminus\{0\}}\frac{\int_0^T|_0D_t^\alpha u(t)|^pdt}{\int_0^T|u(t)|^pdt}>0$. Then the norm $\|u\|_\alpha$ is equivalent to $\|u\|_{E^{\alpha,p}}$, that is, there exist two positive constants Λ_1, Λ_2, such that $\Lambda_1\|u\|_{E^{\alpha,p}}\leq\|u\|_\alpha\leq\Lambda_2\|u\|_{E^{\alpha,p}}$, $\forall u\in E_0^{\alpha,p}$, where $\|u\|_{E^{\alpha,p}}$ is defined in (2.4).

Lemma 2.6 Let $0 < \alpha \le 1$, $1 < p < \infty$. By Lemmas 2.2, 2.5 and (2.4), for $u \in E_0^{\alpha,p}$, one has

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_{E^{\alpha,p}} \le \Lambda_p \|u\|_\alpha, \tag{2.6}$$

where $\Lambda_p = \frac{T^\alpha}{\Lambda_1 \Gamma(\alpha+1)}$. If $\alpha > 1/p$, then

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha \mathrm{p}^* - \mathrm{p}^* + 1)^{\frac{1}{\mathrm{p}^*}}} \|u\|_{E^{\alpha,p}} \le \Lambda_\infty \|u\|_\alpha, \tag{2.7}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ is the norm of $C([0,T], \mathbb{R})$, and

$$\Lambda_\infty = \frac{T^{\alpha-\frac{1}{p}}}{\Lambda_1 \Gamma(\alpha)(\alpha \mathrm{p}^* - \mathrm{p}^* + 1)^{\frac{1}{\mathrm{p}^*}}}, \quad \mathrm{p}^* = \frac{p}{p-1} > 1.$$

Define a new norm on the space $E_0^{\beta,p}$, as follows

$$\|v\|_\beta = (\int_0^T \left|{}_0D_t^\beta v(t)\right|^p dt + \int_0^T b(t)|v(t)|^p dt)^{\frac{1}{p}}, \tag{2.8}$$

where the definition of $E_0^{\beta,p}$ is similar to that of $E_0^{\alpha,p}$, see Definition 2.1. Similar to Lemma 2.5, the relationship between $\|v\|_\beta$ and $\|v\|_{E^{\beta,p}}$ is given below, where the definition of $\|v\|_{E^{\beta,p}}$ is similar to the definition of $\|u\|_{E^{\alpha,p}}$, as shown in (2.4).

Lemma 2.7 If $\operatorname{essinf}_{t\in[0,T]} b(t) > -\lambda_1{}'$, where $\lambda_1{}' = \inf\limits_{v \in E_0^{\beta,p}\setminus\{0\}} \frac{\int_0^T \left|{}_0D_t^\beta v(t)\right|^p dt}{\int_0^T |v(t)|^p dt} > 0$, then the norm $\|v\|_\beta$ is equivalent to $\|v\|_{E^{\beta,p}}$, in other words, there exist $\Lambda_1{}', \Lambda_2{}' > 0$, such that $\Lambda_1{}' \|v\|_{E^{\beta,p}} \le \|v\|_\beta \le \Lambda_2{}' \|v\|_{E^{\beta,p}}$, $\forall v \in E_0^{\beta,p}$. So

$$\|v\|_{L^p} \le \frac{T^\beta}{\Gamma(\beta+1)} \|v\|_{E^{\beta,p}} \le \Lambda_p{}' \|v\|_\beta, \tag{2.9}$$

$$\|v\|_\infty \le \frac{T^{\beta-\frac{1}{p}}}{\Gamma(\beta)(\beta p^* - p^* + 1)^{\frac{1}{p^*}}} \|v\|_{E^{\beta,p}} \le \Lambda_\infty{}' \|v\|_\beta, \tag{2.10}$$

where $\Lambda_p{}' = \frac{T^\beta}{\Lambda_1{}' \Gamma(\beta+1)}$, $\Lambda_\infty{}' = \frac{T^{\beta-\frac{1}{p}}}{\Lambda_1{}' \Gamma(\beta)(\beta p^* - p^* + 1)^{\frac{1}{p^*}}}$, $p^* = \frac{p}{p-1} > 1$.

Define the fractional derivative space

$$X = E_0^{\alpha,p} \times E_0^{\beta,p}, \tag{2.11}$$

whose norm is as follows

$$\|(u,v)\|_X = \|u\|_\alpha + \|v\|_\beta, \ \forall (u,v) \in X. \tag{2.12}$$

From Lemma 2.1, we can see that X is a separable reflexive Banach space. According to Lemma 2.3, X compactly embedded in $C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R})$. By (2.7), (2.10), we have

$$\|(u,v)\|_\infty = \max_{t\in[0,T]} |u(t)| + \max_{t\in[0,T]} |v(t)| \le \Lambda_\infty \|u\|_\alpha + \Lambda_\infty{}' \|v\|_\beta \le M \|(u,v)\|_X, \tag{2.13}$$

where $M = \max\left\{\Lambda_\infty, {\Lambda_\infty}'\right\}$.

Lemma 2.8 ([18]) (Integration by parts) Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, $1/p + 1/q < 1 + \alpha$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \alpha$. If the function $u \in L^p([a,b], \mathbb{R})$, $v \in L^q([a,b], \mathbb{R})$, then

$$\int_a^b \left[{}_aD_t^{-\alpha} u(t)\right] v(t)\, dt = \int_a^b u(t) \left[{}_tD_b^{-\alpha} v(t)\right] dt. \tag{2.14}$$

By multiplying the first equation in problem (1.1) by any $x \in E_0^{\alpha,p}$ and integrating on $[0,T]$, we can obtain

$$\int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) x(t) dt + \int_0^T a(t)\phi_p(u(t))x(t)dt - \chi \int_0^T f_u(t, u(t), v(t)) x(t) dt = 0.$$

By Lemma 2.8, one has

$$\begin{aligned}
&\int_0^T {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) x(t) dt = -\sum_{j=0}^n \int_{t_j}^{t_{j+1}} x(t) d[{}_tD_T^{\alpha-1} \phi_p({}_0D_t^\alpha u(t))] \\
=& -\sum_{j=0}^n {}_tD_T^{\alpha-1} \phi_p({}_0D_t^\alpha u(t)) x(t)|_{t_j}^{t_{j+1}} + \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \phi_p({}_0D_t^\alpha u(t))\, {}_0D_t^\alpha x(t) dt \\
=& \sum_{j=1}^n [{}_tD_T^{\alpha-1} \phi_p({}_0D_t^\alpha u(t_j^+)) x(t_j) - {}_tD_T^{\alpha-1} \phi_p({}_0D_t^\alpha u(t_j^-)) x(t_j)] + \int_0^T \phi_p({}_0D_t^\alpha u(t))\, {}_0D_t^\alpha x(t) dt \\
=& \mu \sum_{j=1}^n I_j(u(t_j)) x(t_j) + \int_0^T \phi_p({}_0D_t^\alpha u(t))\, {}_0D_t^\alpha x(t) dt.
\end{aligned}$$

Thus, we get the definition of the weak solution of problem (1.1).

Definition 2.2 Let $(u, v) \in X$ be a weak solution of problem (1.1), if

$$\begin{aligned}
&\int_0^T \left(\phi_p({}_0D_t^\alpha u(t))\, {}_0D_t^\alpha x(t) + a(t)\phi_p(u(t))x(t)\right) dt + \int_0^T \left(\phi_p({}_0D_t^\beta v(t))\, {}_0D_t^\beta y(t) + b(t)\phi_p(v(t))y(t)\right) dt \\
&+ \mu\Big(\sum_{j=1}^m I_j(u(t_j))x(t_j) + \sum_{i=1}^n S_i(v({t'}_i))y({t'}_i)\Big) - \chi \int_0^T (f_u(t, u(t), v(t))x(t) + f_v(t, u(t), v(t))y(t)) dt = 0
\end{aligned}$$

holds for any $\forall (x, y) \in X$.

Define functional $\varphi : X \to \mathbb{R}$ as follows

$$\begin{aligned}
\varphi(u, v) =& \frac{1}{p}\left(\|u\|_\alpha^p + \|v\|_\beta^p\right) + \mu\Big(\sum_{j=1}^m \int_0^{u(t_j)} I_j(s) ds \\
&+ \sum_{i=1}^n \int_0^{v({t'}_i)} S_i(z) dz\Big) - \chi \int_0^T f(t, u(t), v(t)) dt, \forall (u, v) \in X.
\end{aligned} \tag{2.15}$$

By the continuity of functions I_j and S_i and $f(t, \cdot, \cdot)$ is a C^1 function in $\mathbb{R}^2$ for any $t \in [0, T]$, it is easy to prove $\varphi \in C^1(X, \mathbb{R})$. In addition, for $\forall\,(x, y) \in X$, one has

$$\begin{aligned}\varphi'(u,v)\,(x,y) = & \int_0^T \left(\phi_p({}_0D_t^{\alpha}u(t))_0D_t^{\alpha}x(t) + a(t)\phi_p(u(t))x(t)\right)dt \\ & + \int_0^T \left(\phi_p({}_0D_t^{\beta}v(t))_0D_t^{\beta}y(t) + b(t)\phi_p(v(t))y(t)\right)dt \\ & + \mu(\sum_{j=1}^{m} I_j(u(t_j))x(t_j) + \sum_{i=1}^{n} S_i(v(t'_i))y(t'_i)) \\ & - \chi\int_0^T (f_u(t,u(t),v(t))x(t) + f_v(t,u(t),v(t))y(t))dt.\end{aligned} \tag{2.16}$$

Therefore, the critical point of functional φ corresponds to the weak solution of (1.1).

3 Main result

The three critical point theorems used in this article are first introduced.

Lemma 3.1 ([20]) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semi continuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that

$$\inf_{x\in X} \Phi\,(x) = \Phi\,(0) = \Psi\,(0) = 0.$$

Assume that there exist $r > 0$, $\overline{x} \in X$ with $r < \Phi\,(\overline{x})$ such that

(i) $\sup\{\Psi\,(x) : \Phi\,(x) \le r\} < r\frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) for each $\lambda \in \Lambda_r = \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup\{\Psi(x):\Phi(x)\le r\}}\right)$, the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Next, we first consider three solutions of problem (1.1) in the case of parameter $\mu \ge 0$, and get the following results.

Theorem 3.1 Let $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, u, v)$ is continuous in $[0, T]$ for every $(u, v) \in \mathbb{R}^2$ and $f(t, \cdot, \cdot)$ is a C^1 function in $\mathbb{R}^2$ for any $t \in [0, T]$, and $f(t, 0, 0) = 0$, $\forall t \in [0, T]$. Assume that all of the following conditions are true

(H_1) $a(t)$, $b(t) \in C([0, T], \mathbb{R})$, and $\mathrm{essinf}_{t\in[0,T]}a(t) > -\lambda_1$, $\mathrm{essinf}_{t\in[0,T]}b(t) > -{\lambda_1}'$, where λ_1, ${\lambda_1}'$ are defined in Lemmas 2.5, 2.7, respectively;

(H_2) There exist L, L_i, $D_j > 0$, $0 < q \le p$, $0 < d_j < p$, $0 < l_i < p$, $j = 1, 2, \cdots, m$, $i = 1, 2, \cdots, n$, so that for $\forall\,(t, u, v) \in [0, T] \times \mathbb{R}^2$, we have

$$f\,(t,u,v) \le L\,(1 + |u|^q + |v|^q)\,, \tag{3.1}$$

$$-J_j\,(u) \le D_j\left(1 + |u|^{d_j}\right), -W_i\,(v) \le L_i\left(1 + |v|^{l_i}\right), \tag{3.2}$$

where $J_j(u) = \int_0^u I_j(t)dt$, $W_i\,(v) = \int_0^v S_i(t)dt$;

(H_3) There are $r > 0$, $\omega = (\omega_1, \omega_2) \in X$, such that $\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p > pr$,

$$\int_0^T f(t, \omega_1(t), \omega_2(t))dt > 0,\ \sum_{j=1}^m J_j(\omega_1(t_j)) > 0, \sum_{i=1}^n W_i\left(\omega_2(t'_i)\right) > 0$$

and the following inequality holds:

$$B_l := \frac{\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p}{p\int_0^T f(t, \omega_1(t), \omega_2(t))dt} < B_r := \frac{r}{\int_0^T \sup\limits_{(u,v)\in\Omega(M^p r)} f\left(t, u, v\right)dt}, \tag{3.3}$$

where the definition of $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, X and M are shown in (2.5), (2.8), (2.11), (2.13) and

$$\Omega(M^p r) = \left\{(u, v) \in \mathbb{R}^2 : \frac{1}{p}\left(|u|^p + |v|^p\right) \le M^p r\right\}.$$

Then, for every $\chi \in \Lambda_B = (B_l, B_r)$, there exists

$$\gamma := \min\left\{\frac{r - \chi\int_0^T \sup\limits_{(u,v)\in\Omega(M^p r)} f\left(t, u, v\right)dt}{\max\limits_{(u,v)\in\Omega(M^p r)}\left(\sum\limits_{j=1}^m\left(-J_j(u)\right) + \sum\limits_{i=1}^n\left(-W_i\left(v\right)\right)\right)},\right.$$

$$\left.\frac{\chi p\int_0^T f(t, \omega_1(t), \omega_2(t))dt - \left(\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p\right)}{p\left(\sum\limits_{j=1}^m J_j(\omega_1(t_j)) + \sum\limits_{i=1}^n W_i\left(\omega_2(t'_i)\right)\right)}\right\},$$

so that for every $\mu \in [0, \gamma)$, (1.1) has at least three weak solutions.

Proof Define the functionals $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ as below:

$$\Phi\left(u, v\right) = \frac{1}{p}\left(\|u\|_\alpha^p + \|v\|_\beta^p\right), \tag{3.4}$$

$$\Psi\left(u, v\right) = \int_0^T f(t, u(t), v\left(t\right))dt - \frac{\mu}{\chi}\left(\sum_{j=1}^m J_j(u(t_j)) + \sum_{i=1}^n\left(W_i\left(v(t'_i)\right)\right)\right), \tag{3.5}$$

then $\varphi(u, v) = \Phi\left(u, v\right) - \chi\Psi\left(u, v\right)$. Through the simple calculation, we can gain

$$\inf_{(u,v)\in X}\Phi\left(u, v\right) = \Phi\left(0, 0\right) = 0,$$

$$\Psi\left(0, 0\right) = \int_0^T f(t, 0, 0)dt - \frac{\mu}{\chi}\left(\sum_{j=1}^m J_j(0) + \sum_{i=1}^n W_i\left(0\right)\right) = 0.$$

Furthermore, Φ and Ψ are continuous Gâteaux differential and for $\forall\left(x, y\right) \in X$, one has

$$\begin{aligned}\Phi'(u, v)\left(x, y\right) = &\int_0^T \left(\phi_p({}_0D_t^\alpha u(t)){}_0D_t^\alpha x(t) + a(t)\phi_p(u(t))x(t)\right)dt \\ &+ \int_0^T \left(\phi_p({}_0D_t^\beta v(t)){}_0D_t^\beta y(t) + b(t)\phi_p(v(t))y(t)\right)dt,\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
\Psi'(u,v)\,(x,y) = &\int_0^T (f_u(t,u(t),v(t))x(t) + f_v(t,u(t),v(t))y(t))dt \\
&- \frac{\mu}{\chi}\left(\sum_{j=1}^{m} I_j(u(t_j))x(t_j) + \sum_{i=1}^{n} S_i(v(t'_i))y(t'_i)\right).
\end{aligned} \tag{3.7}
$$

In addition, $\Phi' : X \to X^*$ is continuous. Next, we prove that $\Psi' : X \to X^*$ is continuous compact. Assuming that $\{(u_n, v_n)\} \subset X$, then there exists $(u, v) \in X$, such that $(u_n, v_n) \rightharpoonup (u, v)$, $n \to +\infty$, so $(u_n, v_n) \to (u, v)$ on $[0, T]$. Because $f(t, \cdot, \cdot)$ is a C^1 function in $\mathbb{R}^2$ for any $t \in [0, T]$, so f is continuous in $\mathbb{R}^2$ for any $t \in [0, T]$. Thus $f\,(t, u_n, v_n) \to f\,(t, u, v)$ as $n \to +\infty$. Since $I_j, S_i \in C(\mathbb{R}, \mathbb{R})$, $I_j(u_n(t_j)) \to I_j(u(t_j))$, $S_i(v_n(t'_i)) \to S_i(v(t'_i))$ as $n \to +\infty$. By Lebesgue control convergence theorem, we can get that $\Psi'(u_n, v_n) \to \Psi'\,(u, v)$, $n \to +\infty$. Thus, Ψ' is strongly continuous. From Proposition 26.2 in [21], Ψ' is compact. Thus, $\Phi : X \to \mathbb{R}$ is weakly semi-continuous, coercive and Φ' has a continuous inverse operator on X^*.

The following is to verify the condition (i) in Lemma 3.1. Choose $(u_0, v_0) = (0, 0)$, $(u_1, v_1) = (\omega_1, \omega_2)$. If $(\xi, \eta) \in X$ satisfies $\Phi\,(\xi, \eta) = \frac{1}{p}\left(\|\xi\|_\alpha^p + \|\eta\|_\beta^p\right) \le r$, then, by (2.7), (2.10), we have $\Phi\,(\xi, \eta) \ge \frac{1}{p}\left(\frac{1}{\Lambda_\infty^p}\|\xi\|_\infty^p + \frac{1}{{\Lambda'_\infty}^p}\|\eta\|_\infty^p\right)$, and

$$
\begin{aligned}
\{(\xi, \eta) \in X : \Phi\,(\xi, \eta) \le r\} \quad &\subseteq \left\{(\xi, \eta) \in X : \frac{1}{p}\left(\frac{1}{\Lambda_\infty^p}\|\xi\|_\infty^p + \frac{1}{{\Lambda'_\infty}^p}\|\eta\|_\infty^p\right) \le r\right\} \\
&\subseteq \left\{(\xi, \eta) \in X : \frac{1}{p}\left(\|\xi\|_\infty^p + \|\eta\|_\infty^p\right) \le M^p r\right\}.
\end{aligned}
$$

Thus, by $\chi > 0$, $\mu \ge 0$, we get

$$
\begin{aligned}
&\sup\{\Psi\,(\xi, \eta) : \Phi\,(\xi, \eta) \le r\} \\
&= \sup\left\{\int_0^T f(t, \xi(t), \eta\,(t))dt - \frac{\mu}{\chi}\left(\sum_{j=1}^{m} J_j(\xi(t_j)) + \sum_{i=1}^{n}\left(W_i\,(\eta(t'_i))\right)\right) : \Phi\,(\xi, \eta) \le r\right\} \\
&\le \int_0^T \sup_{(\xi,\eta)\in\Omega(M^p r)} f\,(t, \xi, \eta)\,dt + \frac{\mu}{\chi}\max_{(\xi,\eta)\in\Omega(M^p r)}\left(\sum_{j=1}^{m}(-J_j(\xi)) + \sum_{i=1}^{n}(-W_i\,(\eta))\right).
\end{aligned}
$$

If $\max\limits_{(\xi,\eta)\in\Omega(M^p r)}\left(\sum\limits_{j=1}^{m}(-J_j(\xi)) + \sum\limits_{i=1}^{n}(-W_i\,(\eta))\right) = 0$, by $\chi < B_r$, we obtain

$$
\sup\{\Psi\,(\xi, \eta) : \Phi\,(\xi, \eta) \le r\} < \frac{r}{\chi}. \tag{3.8}
$$

If $\max\limits_{(\xi,\eta)\in\Omega(M^p r)}\left(\sum\limits_{j=1}^{m}(-J_j(\xi)) + \sum\limits_{i=1}^{n}(-W_i\,(\eta))\right) > 0$, (3.8) is also correct for $\mu \in [0, \gamma)$. Besides, for $\mu < \gamma$, we have

$$\begin{aligned}\Psi(\omega_1,\omega_2) &= \int_0^T f(t,\omega_1(t),\omega_2(t))dt - \frac{\mu}{\chi}\left(\sum_{j=1}^m J_j(\omega_1(t_j)) + \sum_{i=1}^n W_i(\omega_2(t'_i))\right)\\ &> \int_0^T f(t,\omega_1(t),\omega_2(t))dt - \frac{\chi p\int_0^T f(t,\omega_1(t),\omega_2(t))dt - \left(\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p\right)}{p\left(\sum\limits_{j=1}^m J_j(\omega_1(t_j)) + \sum\limits_{i=1}^n W_i(\omega_2(t'_i))\right)}\\ &\quad\times\frac{1}{\chi}\left(\sum_{j=1}^m J_j(\omega_1(t_j)) + \sum_{i=1}^n W_i(\omega_2(t'_i))\right)\\ &> \int_0^T f(t,\omega_1(t),\omega_2(t))dt - \int_0^T f(t,\omega_1(t),\omega_2(t))dt + \frac{\left(\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p\right)}{\chi p}\\ &> \frac{\Phi(\omega_1,\omega_2)}{\chi}.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we obtain $\frac{\Psi(\omega_1,\omega_2)}{\Phi(\omega_1,\omega_2)} > \frac{1}{\chi} > \frac{\sup\{\Psi(\xi,\eta):\Phi(\xi,\eta)\le r\}}{r}$, which implies the condition (i) of Lemma 3.1 holds.

Last, we will verify that for any $\forall\chi \in \Lambda_B$, the functional $\Phi - \chi\Psi$ is coercive. For $\forall(\xi,\eta) \in X$, by (2.7), (2.10), (2.13) and (H_2), one has

$$\begin{aligned}&\int_0^T f(t,\xi(t),\eta(t))dt \le L\int_0^T (1+|\xi|^q+|\eta|^q)dt \le LT + LT\|\xi\|_\infty^q + LT\|\eta\|_\infty^q\\ \le& LT + LT\Lambda_\infty^q\|\xi\|_\alpha^q + LT{\Lambda'_\infty}^q\|\eta\|_\beta^q \le LT + LTM^q\left(\|\xi\|_\alpha^q + \|\eta\|_\beta^q\right)\end{aligned} \tag{3.10}$$

and

$$-J_j(\xi(t_j)) \le D_j\left(1+|\xi(t_j)|^{d_j}\right) \le D_j\left(1+\|\xi\|_\infty^{d_j}\right) \le D_j\left(1+\Lambda_\infty^{d_j}\|\xi\|_\alpha^{d_j}\right). \tag{3.11}$$

So

$$\sum_{j=1}^m(-J_j(\xi(t_j))) \le \sum_{j=1}^m D_j\left(1+\Lambda_\infty^{d_j}\|\xi\|_\alpha^{d_j}\right). \tag{3.12}$$

Similarly, we can get

$$\sum_{i=1}^n(-W_i(\eta(t'_i))) \le \sum_{i=1}^n L_i\left(1+{\Lambda'_\infty}^{l_i}\|\eta\|_\beta^{l_i}\right). \tag{3.13}$$

Thus, for $(\xi,\eta) \in X$, since $\frac{\mu}{\chi} \ge 0$, by (3.10), (3.12), (3.13), we have

$$\begin{aligned}\Phi(\xi,\eta) - \chi\Psi(\xi,\eta) \ge& \frac{1}{p}\left(\|\xi\|_\alpha^p + \|\eta\|_\beta^p\right) - \chi LT - \chi LTM^q\left(\|\xi\|_\alpha^q + \|\eta\|_\beta^q\right)\\ &-\mu\left(\sum_{j=1}^m D_j\left(1+\Lambda_\infty^{d_j}\|\xi\|_\alpha^{d_j}\right) + \sum_{i=1}^n L_i\left(1+{\Lambda'_\infty}^{l_i}\|\eta\|_\beta^{l_i}\right)\right).\end{aligned}$$

If $0 < q, d_j, l_i < p$, for $\chi > 0$, one has $\lim\limits_{\|(\xi,\eta)\|_X \to +\infty} (\Phi(\xi,\eta) - \chi\Psi(\xi,\eta)) = +\infty$. Obviously, the functional $\Phi - \chi\Psi$ is coercive. If $q = p$, then

$$\Phi(\xi,\eta) - \chi\Psi(\xi,\eta) \geq \left(\frac{1}{p} - \chi LTM^p\right)\left(\|\xi\|_\alpha^p + \|\eta\|_\beta^p\right) - \chi LT$$
$$-\mu\left(\sum_{j=1}^m D_j\left(1+\Lambda_\infty^{d_j}\|\xi\|_\alpha^{d_j}\right) + \sum_{i=1}^n L_i\left(1+\Lambda'^{\,l_i}_{\infty}\|\eta\|_\beta^{l_i}\right)\right).$$

Choose $L < \frac{\int_0^T \sup\limits_{(\xi,\eta)\in\Omega(M^p r)} f(t,\xi,\eta)dt}{prTM^p}$. For $\chi < B_r$, one has $\frac{1}{p} - \chi LTM^p > 0$. If $0 < d_j, l_i < p$, for $\forall \chi \in \Lambda_B$, one has $\lim\limits_{\|(\xi,\eta)\|_X \to +\infty} (\Phi(\xi,\eta) - \chi\Psi(\xi,\eta)) = +\infty$. Obviously, the functional $\Phi - \chi\Psi$ is coercive. Therefore, the conditions in Lemma 3.1 are all true. By Lemma 3.1, we get that, for each $\chi \in \Lambda_B$, the functional $\varphi = \Phi - \chi\Psi$ has at least three different critical points in X.

Remark 3.1 The assumption (H_2) studies both $0 < q < p$ and $q = p$. Obviously when $p = 2$, the assumption (H_2) contains the condition $0 < q < 2$ in [14-15]. In addition, the assumption (H_1) allows $a(t)$ can have a negative lower bound, satisfying $\operatorname{essinf}_{t\in[0,T]} a(t) > -\lambda_1$, where $\lambda_1 = \inf\limits_{u \in E_0^{\alpha,p}\setminus\{0\}} \frac{\int_0^T |{}_0D_t^\alpha u(t)|^p dt}{\int_0^T |u(t)|^p dt} > 0$, but $a(t)$ in [14-15] has a positive lower bound, satisfying $0 < a_1 \leq a(t) \leq a_2$. Thus, our conclusions extend the existing results.

In Theorem 3.1, we consider the case of the parameter $\mu \geq 0$, and we will consider the three solutions of problem (1.1) in the case of the parameter $\mu < 0$, and get the following result.

Theorem 3.2 Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, u, v)$ is continuous in $[0,T]$ for every $(u,v) \in \mathbb{R}^2$ and $f(t,\cdot,\cdot)$ is a C^1 function in $\mathbb{R}^2$ for any $t \in [0,T]$, and $f(t,0,0) = 0$, $\forall t \in [0,T]$. Assume that the condition (H_1) and the following conditions hold

(H_4) There exist L, L_i, $D_j > 0$, $0 < q \leq p$, $0 < d_j < p$, $0 < l_i < p$, $j = 1,2,\cdots,m$, $i = 1,2,\cdots,n$ so that for $\forall (t,\xi,\eta) \in [0,T] \times \mathbb{R}^2$, we have

$$f(t,\xi,\eta) \leq L(1+|\xi|^q + |\eta|^q), J_j(\xi) \leq D_j\left(1+|\xi|^{d_j}\right), W_i(\eta) \leq L_i\left(1+|\eta|^{l_i}\right);$$

(H_5) There are $r > 0$, $\omega = (\omega_1, \omega_2) \in X$, such that $\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p > pr$,

$$\int_0^T f(t,\omega_1(t),\omega_2(t))dt > 0, \sum_{j=1}^m J_j(\omega_1(t_j)) < 0, \sum_{i=1}^n W_i(\omega_2(t'_i)) < 0$$

and (3.3) holds. Then, for every $\chi \in \Lambda_B = (B_l, B_r)$, there exists

$$\gamma^* := \max\left\{\frac{\chi\int_0^T \sup\limits_{(\xi,\eta)\in\Omega(M^p r)} f(t,\xi,\eta)dt - r}{\max\limits_{(\xi,\eta)\in\Omega(M^p r)}\left(\sum\limits_{j=1}^m J_j(\xi) + \sum\limits_{i=1}^n W_i(\eta)\right)},\right.$$
$$\left.\frac{\chi p\int_0^T f(t,\omega_1(t),\omega_2(t))dt - \left(\|\omega_1\|_\alpha^p + \|\omega_2\|_\beta^p\right)}{p\left(\sum\limits_{j=1}^m J_j(\omega_1(t_j)) + \sum\limits_{i=1}^n W_i(\omega_2(t'_i))\right)}\right\},$$

so that for every $\mu \in (\gamma^*, 0]$, (1.1) has at least three weak solutions.

Proof The verification process is analogue to Theorem 3.1, which is omitted here.

Remark 3.2 The assumption (H_4) studies both $0 < q < p$ and $q = p$. Obviously when $p = 2$, the assumption (H_4) contains the condition $0 < q < 2$ in [14-15]. In addition, the assumption (H_1) allows $a(t)$ can have a negative lower bound, satisfying $\mathrm{essinf}_{t\in[0,T]}a(t) > -\lambda_1$, $\lambda_1 > 0$, but $a(t)$ in [14-15] has a positive lower bound, satisfying $0 < a_1 \le a(t) \le a_2$. Thus, our conclusions extend the existing results.

This method is also applicable to fractional impulsive equations, such as the following impulsive Dirichlet problems

$$\begin{cases} {}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) + a(t)\phi_p(u(t)) = \chi f(t, u(t)),\ t \ne t_j,\ \text{a.e. } t \in [0, T], \\ \Delta({}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)),\ j = 1, 2, \cdots, n,\ n \in \mathbb{N}, \\ u(0) = u(T) = 0, \end{cases} \tag{3.14}$$

where $p > 1$, $\alpha \in (1/p, 1]$, $\chi > 0$, $\mu \in \mathbb{R}$, $a(t) \in C([0, T], \mathbb{R})$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $T > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$, $I_j \in C(\mathbb{R}, \mathbb{R})$, and

$$\begin{aligned} &\Delta({}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u))(t_j) = {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^-), \\ &{}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^+) = \lim_{t\to t_j^+} {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t), \\ &{}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t_j^-) = \lim_{t\to t_j^-} {}_tD_T^{\alpha-1}\phi_p({}_0^CD_t^{\alpha}u)(t). \end{aligned}$$

In the case of parameter $\mu \ge 0$, the following result is obtained.

Corollary 3.1 Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \cdots, n$ be continuous functions. Assume that all of the following conditions are true

(G_1) $a(t) \in C([0, T], \mathbb{R})$ and $\mathrm{essinf}_{t\in[0,T]}a(t) > -\lambda_1$, where λ_1 is defined in Lemma 2.5;

(G_2) There exist $L, L_1, \cdots, L_n > 0$, $0 < \beta \le p$, $0 < d_j < p$, $j = 1, \cdots, n$, so that for $\forall\,(t, x) \in [0, T] \times \mathbb{R}$, we have

$$F(t, x) \le L\left(1 + |x|^{\beta}\right), -J_j(x) \le L_j\left(1 + |x|^{d_j}\right),$$

where $F(t, u) = \int_0^u f(t, s)ds$, $J_j(u) = \int_0^u I_j(t)dt$.

Suppose that there are $r > 0$, $\omega \in E_0^{\alpha,p}$, such that $\frac{1}{p}\|\omega\|_{\alpha}^p > r$, $\int_0^T F(t, \omega(t))dt > 0$, $\sum\limits_{j=1}^{n} J_j(\omega(t_j)) > 0$, and

$$A_l := \frac{\frac{1}{p}\|\omega\|_{\alpha}^p}{\int_0^T F(t, \omega(t))dt} < A_r := \frac{r}{\int_0^T \max\limits_{|x|\le\Lambda_\infty(pr)^{1/p}} F(t, x)\,dt}. \tag{3.15}$$

Then, for every $\chi \in \Lambda_r = (A_l, A_r)$, there exists

$$\gamma := \min\left\{\frac{r - \chi\int_0^T \max\limits_{|x|\le\Lambda_\infty(pr)^{1/p}} F(t, x)\,dt}{\max\limits_{|x|\le\Lambda_\infty(pr)^{\frac{1}{p}}} \sum\limits_{j=1}^{n}(-J_j(x))}, \frac{\chi\int_0^T F(t, \omega)dt - \frac{1}{p}\|\omega\|_{\alpha}{}^p}{\sum\limits_{j=1}^{n} J_j(\omega(t_j))}\right\}$$

so that for $\forall\mu \in [0,\gamma)$, (3.14) has at least three weak solutions in $E_0^{\alpha,p}$.

When the parameter $\mu < 0$, there is another conclusion, specifically as follows:

Corollary 3.2 Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\cdots,n$ be continuous. Assuming (G_1) and the following conditions are met.

(G_3) There are $L, L_1,\cdots,L_n > 0$, $0 < \beta \le p$, $0 < d_j < p$, $j = 1,\cdots,n$, so that for $\forall\,(t,x) \in [0,T]\times\mathbb{R}$, we have $F(t,x) \le L\left(1+|x|^{\beta}\right)$, $J_j(x) \le L_j\left(1+|x|^{d_j}\right)$. Suppose there is $r > 0$, $\omega \in E_0^{\alpha,p}$, so that $\frac{1}{p}\|\omega\|_\alpha^p > r$, $\int_0^T F(t,\omega(t))dt > 0$, $\sum\limits_{j=1}^{n} J_j(\omega(t_j)) < 0$ and (3.15) holds. Then, for every $\chi \in \Lambda_r = (A_l, A_r)$, there exists

$$\gamma^* := \max\left\{\frac{\chi\int_0^T \max\limits_{|x|\le\Lambda_\infty(pr)^{1/p}} F(t,x)\,dt - r}{\max\limits_{|x|\le\Lambda_\infty(pr)^{\frac{1}{p}}} \sum\limits_{j=1}^{n} J_j(x)}, \frac{\chi\int_0^T F(t,\omega)dt - \frac{1}{p}\|\omega\|_\alpha{}^p}{\sum\limits_{j=1}^{n} J_j(\omega(t_j))}\right\}$$

so that for every $\mu \in (\gamma^*, 0]$, (3.14) has at least three weak solutions in $E_0^{\alpha,p}$.

4 Conclusion

In this paper, we discuss the multiplicity of solutions for a class of coupled systems of fractional p-Laplacian differential equation with impulsive effects. By using the three critical points theorem, the multiplicity results of weak solutions are obtained under the conditions of p-sublinear growth. Compared with the existing related work, our results weaken the existing related conditions and improve and enrich the related results to a certain extent.

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具有脉冲效应的分数阶p-Laplacian方程耦合系统的可解性

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摘要: 本文研究了一类具有脉冲效应的分数阶p-Laplacian方程耦合系统的问题. 利用变分方法，获得了该系统解存在的一些新结果. 在证明过程中, 弱化了系统中变系数和非线性项的条件, 推广了已有结果.

关键词: 分数阶微分方程; p-Laplacian 算子; 脉冲; 弱解

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