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# ESTIMATES FOR EIGENVALUES OF THE ELLIPTIC OPERATOR IN WEIGHTED DIVERGENCE FORM ON THE CIGAR SOLITON

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**Abstract:** In this paper, we investigate the Dirichlet eigenvalue problem of the elliptic operator in weighted divergence form $\mathfrak{L}_{A,f}$ on the cigar soliton $(\mathbb{R}^2, g, f)$ as follows

$$\begin{cases} \mathfrak{L}_{A,f}u + Vu = \lambda\rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $V$ is a non-negative continuous function and $\rho$ is a positive continuous function on $\Omega$. We establish some inequalities for eigenvalues of this problem.

**Keywords:** cigar soliton; elliptic operator in weighted divergence form; eigenvalue

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## 1 Introduction

Let $M$ be an $n$-dimensional complete Riemannian manifold. The Dirichlet eigenvalue problem of the Laplacian $\Delta$ on a bounded domain $\Omega$ of $M$ is described by

$$\begin{cases} \Delta u = -\lambda u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{1.1}$$

Many mathematicians have obtained some universal inequalities for eigenvalues of problem (1.1) (cf. [1–4]).

Let $A : \Omega \to \mathrm{End}\,(\mathrm{T}\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of the tangent bundle $\mathrm{T}\Omega$ of $M$. Define the following elliptic operator in divergence form

$$L_A = -\operatorname{div}(A\nabla), \tag{1.2}$$

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where div is the divergence operator and $\nabla$ is the gradient operator. It is easy to see that the operator $L_A$ defined in (1.2) includes the Laplacian operator as a special case. In 2010, Do Carmo, Wang and Xia [5] considered the eigenvalue problem of $L_A$ as follows

$$\begin{cases} L_A u + Vu = \lambda\rho u, & \text{in } M, \\ u = 0, & \text{on } \partial M, \end{cases} \tag{1.3}$$

where $V$ is a non-negative continuous function and $\rho$ is a positive continuous function on $M$. They obtained the following Yang-type inequality

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4\xi_2\rho_2^2}{n\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{1}{\xi_1}\left(\lambda_i - \frac{V_0}{\rho_2}\right)+\frac{n^2H_0^2}{4\rho_1}\right], \tag{1.4}$$

where $\xi_1$, $\xi_2$, $\rho_1$, $\rho_2$ are positive constants, $H_0 = \max\limits_{x\in M}|\mathbf{H}(x)|$, $V_0 = \min\limits_{x\in M} V(x)$ and $\mathbf{H}$ is the mean curvature vector of $M$ in $\mathbb{R}^m$. For more reference about $L_A$, we refer to [6].

In recent years, metric measure spaces have received a lot of attention in geometry and analysis. For some significant results about metric measure spaces, we refer to [7, 8] and the references therein. A smooth metric measure space is actually a Riemannian manifold equipped with some measures which is absolutely continuous with respect to the usual Riemannian measure. More precisely, for a given $n$-dimensional complete Riemannian manifold $(M, g)$ with a smooth metric $g$, we say that the triple $(M, g, d\mu)$ is a smooth metric measure space, where $d\mu = e^{-f}dv$, $f$ is a smooth real-valued function on $M$ and $dv$ is the Riemannian volume element related to $g$.

Let $\Omega$ be a bounded domain in a smooth metric measure space $(M, g, e^{-f}dv)$. Define the elliptic operator in weighted divergence form $\mathfrak{L}_{A,f}$ as

$$\mathfrak{L}_{A,f} = -\operatorname{div}_f A\nabla, \tag{1.5}$$

where $\operatorname{div}_f X = e^f \operatorname{div}\left(e^{-f}X\right)$ is the weighted divergence of the vector field $X$ on $M$. When $A$ is an identity map, $-\mathfrak{L}_{A,f}$ becomes the drifting Laplacian $\Delta_f = \operatorname{div}_f \nabla$. Moreover, when $f$ is a constant, $\mathfrak{L}_{A,f}$ becomes $L_A$ defined in (1.2). There have been some interesting results for $\mathfrak{L}_{A,f}$ (see [9–11]).

As an important example of complete metric measure spaces, we consider Ricci solitons introduced by Hamilton [12, 13]. They are corresponding to self-similar solutions of Hamilton's Ricci flow. We say that $(M, g, f)$ is a gradient Ricci soliton if there is a constant $K$, such that

$$\mathrm{Ric} + \mathrm{Hess} f = Kg. \tag{1.6}$$

The function $f$ is called a potential function of the gradient Ricci soliton. For $K > 0$, $K = 0$ and $K < 0$, the Ricci soliton is called shrinking, steady or expanding respectively. When the dimension is two, Hamilton discovered the first complete non-compact example of a steady Ricci soliton on $\mathbb{R}^2$, called the cigar soliton. The metric and potential function of the cigar soliton $(\mathbb{R}^2, g, f)$ is given by

$$g = \frac{d(x^1)^2 + d(x^2)^2}{1+|x|^2}$$

and

$$f = -\log(1 + |x|^2),$$

where $|x|^2 = (x^1)^2 + (x^2)^2$. In physics, the cigar soliton $(\mathbb{R}^2, g, f)$ is regarded as the Euclidean-Witten black hole under first-order Ricci flow of the world-sheet sigma model. As an important tractable model for understanding black hole physics (cf. [14]), it is of great significance in both geometry and physics. In 2018, Zeng [15] considered the following Dirichlet eigenvalue problem of the drifting Laplacian $\Delta_f$ on a bounded domain $\Omega$ in the cigar soliton $(\mathbb{R}^2, g, f)$

$$\begin{cases} \Delta_f u = -\lambda u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega \end{cases} \tag{1.7}$$

and derived

$$\begin{aligned} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big\{&\Big[2\Big(1+\max_{x\in\Omega}|x|^2\Big)+\min_{x\in\Omega}|x|^2\Big]\lambda_i \\ &-\min_{x\in\Omega}|x|^2\lambda_{k+1}-2\Big(2+3\min_{x\in\Omega}|x|^2\Big)\Big\}. \end{aligned} \tag{1.8}$$

In this paper, on a bounded domain $\Omega$ of the cigar soliton $(\mathbb{R}^2, g, f)$, we consider the Dirichlet eigenvalue problem of $\mathfrak{L}_{A,f}$ as follows

$$\begin{cases} \mathfrak{L}_{A,f} u + Vu = \lambda \rho u, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{1.9}$$

where $V$ is a non-negative continuous function and $\rho$ is a positive continuous function on $M$. We obtain the following results.

**Theorem 1.1** Let $\Omega$ be a bounded domain in the cigar soliton $(\mathbb{R}^2, g, f)$. Let $\lambda_i$ be the $i$-th eigenvalue of problem (1.9). Assume that $\xi_1 I \le A \le \xi_2 I$ throughout $\Omega$, and $\rho_1 \le \rho(x) \le \rho_2$, $\forall x \in \Omega$, where $I$ is the identity map, $\xi_1$, $\xi_2$, $\rho_1$, $\rho_2$ are positive constants. Then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{2\rho_2\xi_2^2}{\rho_1\xi_1(1+C_1)}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{(1+C_0)(\lambda_i-\rho_2^{-1}V_0)}{\xi_1}-\frac{2+3C_1}{\rho_2}\right], \tag{1.10}$$

where $C_0 = \max\limits_{x\in\Omega}\{|x|^2\}$, $C_1 = \min\limits_{x\in\Omega}\{|x|^2\}$ and $V_0 = \min\limits_{x\in\Omega}\{V(x)\}$.

**Remark 1.1** If $A$ is an identity map, $\rho(x) \equiv 1$ and $V(x) \equiv 0$, then $\xi_1 = \xi_2 = 1$, $\rho_1 = \rho_2 = 1$ and $V_0 = 0$. Thus (1.10) becomes (1.8). Therefore, our result generalizes (1.8) of [15].

Moreover, we derive the following result for lower order eigenvalues of problem (1.9).

**Theorem 1.2** Let $\Omega$ be a bounded domain in the cigar soliton $(\mathbb{R}^2, g, f)$. Let $\lambda_i$ be the $i$-th eigenvalue of problem (1.9). Assume that $\xi_1 I \le A \le \xi_2 I$ throughout $\Omega$, and

$\rho_1 \le \rho(x) \le \rho_2$, $\forall x \in \Omega$, where $I$ is the identity map, $\xi_1$, $\xi_2$, $\rho_1$, $\rho_2$ are positive constants. Then we have

$$\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}} \le \frac{2\rho_2}{\rho_1(1+C_1)}\left\{2\xi_2(1+C_0)\left[\frac{(1+C_0)\left(\lambda_1-\rho_2^{-1}V_0\right)}{\xi_1}-\frac{2+3C_1}{\rho_2}\right]\right\}^{\frac{1}{2}}, \tag{1.11}$$

where $C_0=\max\limits_{x\in\Omega}\{|x|^2\}$, $C_1=\min\limits_{x\in\Omega}\{|x|^2\}$ and $V_0=\min\limits_{x\in\Omega}\{V(x)\}$.

**Corollary 1.1** Let $\Omega$ be a bounded domain in the cigar soliton $(\mathbb{R}^2,g,f)$. Denote by $\lambda_i$ the $i$-th eigenvalue of problem (1.7). Then we have

$$\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}} \le \frac{2}{1+C_1}\left\{2\left(1+C_0\right)\left[\left(1+C_0\right)\lambda_1-\left(2+3C_1\right)\right]\right\}^{\frac{1}{2}}, \tag{1.12}$$

where $C_0=\max\limits_{x\in\Omega}\{|x|^2\}$ and $C_1=\min\limits_{x\in\Omega}\{|x|^2\}$.

## 2 Proofs of the Main results

In this section, we give the proofs of the main results.

**Proof of Theorem 1.1** Suppose that $x^p$ is the $p$-th local coordinate of $x_0\in\Omega\subset\mathbb{R}^2$, where $p=1,2$. Consider the test functions

$$\varphi_i = x^p u_i - \sum_{j=1}^{k} a_{ij}u_j,\ \text{for } i=1,\dots,k. \tag{2.1}$$

where

$$a_{ij}=\int_\Omega \rho x^p u_i u_j d\mu.$$

It is easy to find that

$$\varphi_i|_{\partial\Omega}=0,\ \int_\Omega \rho\varphi_i u_j d\mu=0,\ \forall i,j=1,\dots,k. \tag{2.2}$$

Hence the Rayleigh-Ritz inequality reads as

$$\lambda_{k+1}\int_\Omega \rho\varphi_i^2 d\mu \le \int_\Omega \varphi_i\big(\mathfrak{L}_{A,f}+V\big)\varphi_i d\mu. \tag{2.3}$$

According to the definition of $\mathfrak{L}_{A,f}$, we have

$$\begin{aligned}\mathfrak{L}_{A,f}(x^pu_i) &= -\operatorname{div}_f\left(A\nabla(x^pu_i)\right)\\ &= -e^f\operatorname{div}\left(e^{-f}\left(A\left(x^p\nabla u_i+u_i\nabla x^p\right)\right)\right)\\ &= -\operatorname{div}\left(A\left(x^p\nabla u_i+u_i\nabla x^p\right)\right)-\left\langle\nabla f, A\left(x^p\nabla u_i+u_i\nabla x^p\right)\right\rangle\\ &= -x^p\operatorname{div}_f\left(A\nabla u_i\right)-\langle\nabla x^p, A\nabla u_i\rangle-u_i\operatorname{div}_f(A\nabla x^p)-\langle\nabla u_i, A\nabla x^p\rangle\\ &= x^p\mathfrak{L}_{A,f}u_i+u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle.\end{aligned} \tag{2.4}$$

Hence, we derive

$$\begin{aligned}\int_\Omega \varphi_i\big(\mathfrak{L}_{A,f}+V\big)\varphi_i d\mu &= \int_\Omega \varphi_i\left[\left(\mathfrak{L}_{A,f}+V\right)(x^p u_i)-\rho\sum_{j=1}^k a_{ij}\lambda_j u_j\right]d\mu \\ &= \int_\Omega \varphi_i\left(x^p\lambda_i\rho u_i + u_i\mathfrak{L}_{A,f}x^p - 2\left\langle\nabla x^p, A\nabla u_i\right\rangle\right)d\mu \\ &= \lambda_i\int_\Omega \rho\varphi_i^2 d\mu + P_i,\end{aligned} \tag{2.5}$$

where $P_i=\int_\Omega \varphi_i\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle\right)d\mu$. Substituting (2.5) into (2.3), we can get

$$(\lambda_{k+1}-\lambda_i)\int_\Omega \rho\varphi_i^2 d\mu \le P_i. \tag{2.6}$$

Set

$$b_{ij}=\int_\Omega \left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle\right)u_j d\mu,\ \forall i,j=1,\ldots,k.$$

Then we have

$$\begin{aligned} b_{ij} &= \int_\Omega 2x^p\left\langle\nabla u_j, A\nabla u_i\right\rangle d\mu-\int_\Omega 2x^p u_j\mathfrak{L}_{A,f}u_i d\mu+\int_\Omega x^p\mathfrak{L}_{A,f}\left(u_iu_j\right)d\mu \\ &= \int_\Omega x^p u_i\mathfrak{L}_{A,f}u_j d\mu-\int_\Omega x^p u_j\mathfrak{L}_{A,f}u_i d\mu \\ &= \int_\Omega x^p u_i\left(\mathfrak{L}_{A,f}+V\right)u_j d\mu-\int_\Omega x^p u_j\left(\mathfrak{L}_{A,f}+V\right)u_i d\mu \\ &= (\lambda_j-\lambda_i)a_{ij}.\end{aligned} \tag{2.7}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} P_i^2 &= \left[\int_\Omega \varphi_i\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle-\rho\sum_{j=1}^k b_{ij}u_j\right)d\mu\right]^2 \\ &\le \int_\Omega \rho\varphi_i^2 d\mu\cdot\int_\Omega \frac{1}{\rho}\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle-\rho\sum_{j=1}^k b_{ij}u_j\right)^2 d\mu.\end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8), we infer that

$$\begin{aligned}(\lambda_{k+1}-\lambda_i)P_i^2 &\le (\lambda_{k+1}-\lambda_i)\int_\Omega \rho\varphi_i^2 d\mu\cdot\int_\Omega \frac{1}{\rho}\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle-\rho\sum_{j=1}^k b_{ij}u_j\right)^2 d\mu \\ &\le P_i\cdot\int_\Omega \frac{1}{\rho}\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle-\rho\sum_{j=1}^k b_{ij}u_j\right)^2 d\mu.\end{aligned} \tag{2.9}$$

It implies that

$$(\lambda_{k+1}-\lambda_i)P_i \le \int_\Omega \frac{1}{\rho}\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p, A\nabla u_i\right\rangle-\rho\sum_{j=1}^k b_{ij}u_j\right)^2 d\mu. \tag{2.10}$$

Multiplying both sides of (2.10) by $(\lambda_{k+1}-\lambda_i)$, taking the sum over $i$ from 1 to $k$ and $p$ from 1 to 2, we get

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{p=1}^{2}P_i\\ \leq&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{p=1}^{2}\int_\Omega\frac{1}{\rho}\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p,A\nabla u_i\right\rangle-\rho\sum_{j=1}^{k}b_{ij}u_j\right)^2d\mu.\end{aligned}\tag{2.11}$$

Using (2.7), we deduce

$$\begin{aligned}&\sum_{p=1}^{2}\int_\Omega\frac{1}{\rho}\left(u_i\mathfrak{L}_{A,f}x^p-2\left\langle\nabla x^p,A\nabla u_i\right\rangle-\rho\sum_{j=1}^{k}b_{ij}u_j\right)^2d\mu\\ =&\sum_{p=1}^{2}\int_\Omega\frac{1}{\rho}\left[(u_i\mathfrak{L}_{A,f}x^p)^2+4\left\langle\nabla x^p,A\nabla u_i\right\rangle^2-4u_i\mathfrak{L}_{A,f}x^p\left\langle\nabla x^p,A\nabla u_i\right\rangle\right]d\mu-2\sum_{j=1}^{k}b_{ij}^2\\ =&\sum_{p=1}^{2}\int_\Omega\frac{1}{\rho}\left[(u_i\mathfrak{L}_{A,f}x^p)^2+4\left\langle\nabla x^p,A\nabla u_i\right\rangle^2-4u_i\mathfrak{L}_{A,f}x^p\left\langle\nabla x^p,A\nabla u_i\right\rangle\right]d\mu-2\sum_{j=1}^{k}(\lambda_i-\lambda_j)^2a_{ij}^2\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}\sum_{p=1}^{2}P_i=&\sum_{p=1}^{2}\int_\Omega\left(x^pu_i^2\mathfrak{L}_{A,f}x^p-2x^pu_i\left\langle\nabla x^p,A\nabla u_i\right\rangle\right)d\mu-2\sum_{j=1}^{k}a_{ij}b_{ij}\\ =&\sum_{p=1}^{2}\int_\Omega u_i^2\left\langle\nabla x^p,A\nabla x^p\right\rangle d\mu+2\sum_{j=1}^{k}(\lambda_i-\lambda_j)a_{ij}^2.\end{aligned}\tag{2.13}$$

Using (2.12) and (2.13), we obtain

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{p=1}^{2}\int_\Omega u_i^2\left\langle\nabla x^p,A\nabla x^p\right\rangle d\mu+2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)a_{ij}^2\\ \leq&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{p=1}^{2}\int_\Omega\frac{1}{\rho}\left[(u_i\mathfrak{L}_{A,f}x^p)^2+4\left\langle\nabla x^p,A\nabla u_i\right\rangle^2-4u_i\mathfrak{L}_{A,f}x^p\left\langle\nabla x^p,A\nabla u_i\right\rangle\right]d\mu\\ &-2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{ij}^2.\end{aligned}\tag{2.14}$$

Moreover, observe that

$$\begin{aligned}&\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)a_{ij}^2\\ =&\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_{k+1}-\lambda_j)(\lambda_i-\lambda_j)a_{ij}^2-\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{ij}^2\\ =&-\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2a_{ij}^2.\end{aligned}\tag{2.15}$$

Therefore, combining (2.14) with (2.15), we have

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{p=1}^{2}\int_\Omega u_i^2\left\langle\nabla x^p,A\nabla x^p\right\rangle d\mu\\ \le&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\sum_{p=1}^{2}\int_\Omega\frac{1}{\rho}\left[(u_i\mathfrak{L}_{A,f}x^p)^2+4\left\langle\nabla x^p,A\nabla u_i\right\rangle^2-4u_i\mathfrak{L}_{A,f}x^p\left\langle\nabla x^p,A\nabla u_i\right\rangle\right]d\mu.\end{aligned}\tag{2.16}$$

Now it is necessary to calculate and estimate some terms in (2.16). It is not difficult to obtain

$$\langle\nabla x^1,\nabla x^1\rangle=\langle\nabla x^2,\nabla x^2\rangle=1+|x|^2,\tag{2.17}$$

$$\langle\nabla x^1,\nabla x^2\rangle=0\tag{2.18}$$

and

$$\Delta x^1=\Delta x^2=0.\tag{2.19}$$

Using (2.17) and (2.18), and noticing

$$\nabla(-\log(1+|x|^2))=-\frac{1}{1+|x|^2}\nabla|x|^2=-\frac{2}{1+|x|^2}\sum_{q=1}^{2}x^q\nabla x^q,\tag{2.20}$$

we infer that

$$\langle\nabla f,\nabla x^p\rangle=-\frac{2}{1+|x|^2}\sum_{q=1}^{2}x^q\langle\nabla x^q,\nabla x^p\rangle=-2x^p.\tag{2.21}$$

It implies

$$\langle\nabla f,\nabla|x|^2\rangle=-\frac{1}{1+|x|^2}\sum_{p,q=1}^{2}4x^px^q\langle\nabla x^p,\nabla x^q\rangle=-4|x|^2.\tag{2.22}$$

Hence, (2.19) and (2.21) yield

$$\Delta_f x^p=\Delta x^p-\langle\nabla f,\nabla x^p\rangle=2x^p.\tag{2.23}$$

Furthermore, utilizing (2.17) and (2.19), we have

$$\Delta|x|^2=\sum_{p=1}^{2}\Delta(x^p)^2=\sum_{p=1}^{2}(2x^p\Delta x^p+2\langle\nabla x^p,\nabla x^p\rangle)=4(1+|x|^2).\tag{2.24}$$

Combining (2.22) and (2.24), we derive

$$\Delta_f |x|^2 = \Delta |x|^2 - \langle \nabla f, \nabla |x|^2 \rangle = 4(1 + 2|x|^2). \tag{2.25}$$

According to the assumptions of the theorem, we have

$$\rho_2^{-1} \le \int_\Omega u_i^2 d\mu = \int_\Omega \frac{1}{\rho} \rho u_i^2 d\mu \le \rho_1^{-1}. \tag{2.26}$$

Now we calculate the righthand side of (2.16). Since $A \le \xi_2 I$, we can infer from (2.23) and (2.25) that

$$\sum_{p=1}^{2} \int_\Omega (u_i \mathfrak{L}_{A,f} x^p)^2 d\mu \le \xi_2^2 \sum_{p=1}^{2} \int_\Omega u_i^2 (\Delta_f x^p)^2 d\mu = 4\xi_2^2 \int_\Omega u_i^2 |x|^2 d\mu \tag{2.27}$$

and

$$\sum_{p=1}^{2} \int_\Omega \langle \nabla x^p, A \nabla u_i \rangle^2 d\mu \le \xi_2^2 \sum_{p=1}^{2} \int_\Omega \langle \nabla x^p, \nabla u_i \rangle^2 d\mu = \xi_2^2 \int_\Omega \left(1 + |x|^2\right) |\nabla u_i|^2 d\mu. \tag{2.28}$$

Moreover, from

$$\sum_{p=1}^{2} \int_\Omega u_i \Delta_f x^p \langle \nabla x^p, \nabla u_i \rangle d\mu = \frac{1}{2} \int_\Omega \left\langle \nabla |x|^2, \nabla u_i^2 \right\rangle d\mu = -\frac{1}{2} \int_\Omega u_i^2 \Delta_f |x|^2 d\mu,$$

we obtain

$$\sum_{p=1}^{2} \int_\Omega \left(-u_i \mathfrak{L}_{A,f} x^p \langle \nabla x^p, A \nabla u_i \rangle\right) d\mu \le -2\xi_2^2 \int_\Omega u_i^2 \left(1 + 2|x|^2\right) d\mu. \tag{2.29}$$

Using (2.26) and noticing that $A \ge \xi_1 I$, we have

$$\lambda_i = \int_\Omega u_i \left(\mathfrak{L}_{A,f} + V\right) u_i d\mu = \int_\Omega \langle \nabla u_i, A \nabla u_i \rangle d\mu + \int_\Omega V u_i^2 d\mu \ge \xi_1 \int_\Omega |\nabla u_i|^2 d\mu + \rho_2^{-1} V_0.$$

It implies that

$$\int_\Omega |\nabla u_i|^2 d\mu \le \frac{\lambda_i - \rho_2^{-1} V_0}{\xi_1}. \tag{2.30}$$

Then it follows from (2.27–2.30) that

$$\begin{aligned} &\sum_{p=1}^{2} \int_\Omega \frac{1}{\rho} \left[ (u_i \mathfrak{L}_{A,f} x^p)^2 + 4 \langle \nabla x^p, A \nabla u_i \rangle^2 - 4 u_i \mathfrak{L}_{A,f} x^p \langle \nabla x^p, A \nabla u_i \rangle \right] d\mu \\ \le & 4\xi_2^2 \int_\Omega \frac{1}{\rho} \left[ \left(1 + |x|^2\right) |\nabla u_i|^2 - \left(2 + 3|x|^2\right) u_i^2 \right] d\mu \\ \le & \frac{4\xi_2^2}{\rho_1} \left[ \frac{(1 + C_0)(\lambda_i - \rho_2^{-1} V_0)}{\xi_1} - \frac{2 + 3C_1}{\rho_2} \right]. \end{aligned} \tag{2.31}$$

Moreover, we acquire

$$\sum_{p=1}^{2}\int_\Omega u_i^2\left\langle \nabla x^p, A\nabla x^p\right\rangle d\mu \geq 2\xi_1\int_\Omega u_i^2\left(1+|x|^2\right)d\mu \geq 2\xi_1\frac{1+C_1}{\rho_2}. \tag{2.32}$$

It follows from (2.16), (2.31) and (2.32) that

$$\begin{aligned}&2\xi_1\frac{1+C_1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\ \leq&\frac{4\xi_2^2}{\rho_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{(1+C_0)(\lambda_i-\rho_2^{-1}V_0)}{\xi_1}-\frac{2+3C_1}{\rho_2}\right].\end{aligned} \tag{2.33}$$

Hence we can know that (1.10) holds. This completes the proof of Theorem 1.1.

Now we give the proof of Theorem 1.2.

**Proof of Theorem 1.2** Define a $2\times 2$ matrix $C=(C_{ps})$, where $C_{ps}=\int_\Omega \rho x^p u_1 u_{s+1}d\mu$. Using the orthogonalization of Gram-Schmidt, we know that there exist an upper triangle matrix $R=(R_{ps})$ and an orthogonal matrix $T=(T_{ps})$ such that $R=TC$. That is to say, for $1\leq s<p\leq 2$, we have

$$R_{ps}=\sum_{k=1}^{2}T_{pk}C_{ks}=\int_\Omega\sum_{k=1}^{2}T_{pk}\rho x^k u_1 u_{s+1}d\mu=0.$$

Setting $y^p=\sum\limits_{k=1}^{2}T_{pk}x^k$, we get

$$\int_\Omega \rho y^p u_1 u_{s+1}d\mu=0,\quad \text{for } 1\leq s<p\leq 2. \tag{2.34}$$

For $p=1,2$, define the test functions $\varphi_p$ by

$$\varphi_p=y^p u_1-a_p u_1, \tag{2.35}$$

where

$$a_p=\int_\Omega \rho y^p u_1^2 d\mu.$$

Since (2.34) holds, it yields

$$\int_\Omega \rho\varphi_p u_{s+1}d\mu=0,\quad \text{for } 0\leq s<p\leq 2. \tag{2.36}$$

According to the Rayleigh-Ritz inequality, we have

$$\lambda_{p+1}\int_\Omega \rho\varphi_p^2 d\mu\leq\int_\Omega \varphi_p\big(\mathfrak{L}_{A,f}+V\big)\varphi_p d\mu. \tag{2.37}$$

It follows from (2.36) that

$$\int_\Omega\rho\varphi_p^2 d\mu=\int_\Omega\rho\varphi_p y^p u_1 d\mu-a_p\int_\Omega\rho\varphi_p u_1 d\mu=\int_\Omega\rho\varphi_p y^p u_1 d\mu. \tag{2.38}$$

Similar to the proof of (2.4), we acquire

$$\mathfrak{L}_{A,f}(y^p u_1) = y^p \mathfrak{L}_{A,f} u_1 + u_1 \mathfrak{L}_{A,f} y^p - 2\left\langle \nabla y^p, A\nabla u_1 \right\rangle. \tag{2.39}$$

Combining (2.38) and (2.39), we have

$$\int_\Omega \varphi_p \big(\mathfrak{L}_{A,f} + V\big)\varphi_p d\mu = \lambda_1 \int_\Omega \rho\varphi_p^2 d\mu + \int_\Omega y^p u_1 \left(u_1 \mathfrak{L}_{A,f} y^p - 2\left\langle \nabla y^p, A\nabla u_1 \right\rangle\right) d\mu. \tag{2.40}$$

At the same time, using

$$-2\int_\Omega y^p u_1 \left\langle \nabla y^p, A\nabla u_1 \right\rangle d\mu = \int_\Omega u_1^2 \left\langle \nabla y^p, A\nabla y^p \right\rangle d\mu - \int_\Omega y^p u_1^2 \mathfrak{L}_{A,f} y^p d\mu,$$

we obtain

$$\int_\Omega y^p u_1 \left(u_1 \mathfrak{L}_{A,f} y^p - 2\left\langle \nabla y^p, A\nabla u_1 \right\rangle\right) d\mu = \int_\Omega u_1^2 \left\langle \nabla y^p, A\nabla y^p \right\rangle d\mu. \tag{2.41}$$

Substituting (2.40) and (2.41) into (2.37), we deduce

$$(\lambda_{p+1} - \lambda_1)\int_\Omega \rho\varphi_p^2 d\mu \le \int_\Omega u_1^2 \left\langle \nabla y^p, A\nabla y^p \right\rangle d\mu. \tag{2.42}$$

Observing that

$$\int_\Omega u_1 \left(\langle \nabla u_1, \nabla y^p \rangle + \frac{1}{2}u_1 \Delta_f y^p\right) d\mu = 0,$$

we infer

$$\begin{aligned}-2\int_\Omega \varphi_p \left(\langle \nabla u_1, \nabla y^p \rangle + \frac{1}{2}u_1 \Delta_f y^p\right) d\mu =& -2\int_\Omega y^p u_1 \left(\langle \nabla u_1, \nabla y^p \rangle + \frac{1}{2}u_1 \Delta_f y^p\right) d\mu \\ =& \int_\Omega u_1^2 |\nabla y^p|^2 d\mu.\end{aligned} \tag{2.43}$$

Therefore, using (2.42) and (2.43), and summing over $p$ from 1 to 2, we derive

$$\begin{aligned}&\sum_{p=1}^{2}(\lambda_{p+1} - \lambda_1)^{\frac{1}{2}} \int_\Omega u_1^2 |\nabla y^p|^2 d\mu \\ =& -2\sum_{p=1}^{2}(\lambda_{p+1} - \lambda_1)^{\frac{1}{2}} \int_\Omega \varphi_p \left(\langle \nabla u_1, \nabla y^p \rangle + \frac{1}{2}u_1 \Delta_f y^p\right) d\mu \\ \le& \delta\sum_{p=1}^{2}(\lambda_{p+1} - \lambda_1)\int_\Omega \rho\varphi_p^2 d\mu + \frac{1}{\delta}\sum_{p=1}^{2}\int_\Omega \frac{1}{\rho}\left(\langle \nabla u_1, \nabla y^p \rangle + \frac{1}{2}u_1 \Delta_f y^p\right)^2 d\mu \\ \le& \delta\sum_{p=1}^{2}\int_\Omega u_1^2 \left\langle \nabla y^p, A\nabla y^p \right\rangle d\mu + \frac{1}{\delta}\sum_{p=1}^{2}\int_\Omega \frac{1}{\rho}\left(\langle \nabla u_1, \nabla y^p \rangle + \frac{1}{2}u_1 \Delta_f y^p\right)^2 d\mu,\end{aligned} \tag{2.44}$$

where $\delta$ is any positive constant.

Since $y^p = \sum\limits_{k=1}^{2} T_{pk}x^k$ and $T$ is an orthogonal matrix, we know that $y^1$ and $y^2$ are the standard coordinate functions of $\mathbb{R}^2$. It is not difficult to check that

$$|y|^2 = |x|^2,$$

$$|\nabla y^p|^2 = 1 + |x|^2, \tag{2.45}$$

$$\Delta_f y^p = 2y^p \tag{2.46}$$

and

$$\Delta_f |y|^2 = 4(1 + 2|x|^2). \tag{2.47}$$

Noticing that $\rho_2^{-1} \le \int_\Omega u_1^2 d\mu \le \rho_1^{-1}$, and using (2.45), we obtain

$$\begin{aligned}\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2|\nabla y^p|^2 d\mu &= \sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}}\int_\Omega u_1^2\left(1+|x|^2\right)d\mu \\ &\ge \frac{1+C_1}{\rho_2}\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}}\end{aligned} \tag{2.48}$$

and

$$\sum_{p=1}^{2}\int_\Omega u_1^2\left\langle \nabla y^p, A\nabla y^p\right\rangle d\mu \le 2\xi_2\int_\Omega u_1^2\left(1+|x|^2\right)d\mu \le 2\xi_2\frac{1+C_0}{\rho_1}. \tag{2.49}$$

Similar to the proof of (2.30), we have

$$\int_\Omega |\nabla u_1|^2 d\mu \le \frac{\lambda_1 - \rho_2^{-1}V_0}{\xi_1}. \tag{2.50}$$

Then it follows from (2.46), (2.47) and (2.50) that

$$\begin{aligned}&\sum_{p=1}^{2}\int_\Omega \frac{1}{\rho}\left(\langle \nabla u_1, \nabla y^p\rangle + \frac{1}{2}u_1\Delta_f y^p\right)^2 d\mu \\ =&\int_\Omega \frac{1}{\rho}\left[\left(1+|x|^2\right)|\nabla u_1|^2 - \frac{1}{2}u_1^2\Delta_f|y|^2 + u_1^2|y|^2\right]d\mu \\ =&\int_\Omega \frac{1}{\rho}\left[\left(1+|x|^2\right)|\nabla u_1|^2 - \left(2+3|x|^2\right)u_1^2\right]d\mu \\ \le&\frac{1}{\rho_1}\left[\frac{(1+C_0)(\lambda_1-\rho_2^{-1}V_0)}{\xi_1} - \frac{2+3C_1}{\rho_2}\right].\end{aligned} \tag{2.51}$$

Substituting (2.48), (2.49) and (2.51) into (2.44), we get

$$\frac{1+C_1}{\rho_2}\sum_{p=1}^{2}(\lambda_{p+1}-\lambda_1)^{\frac{1}{2}} \le \delta\left[\frac{2\xi_2\left(1+C_0\right)}{\rho_1}\right] + \frac{1}{\delta}\left\{\frac{1}{\rho_1}\left[\frac{(1+C_0)(\lambda_1-\rho_2^{-1}V_0)}{\xi_1} - \frac{2+3C_1}{\rho_2}\right]\right\}. \tag{2.52}$$

Taking

$$\delta=\frac{\left\{\dfrac{1}{\rho_1}\left[\dfrac{(1+C_0)(\lambda_1-\rho_2^{-1}V_0)}{\xi_1}-\dfrac{2+3C_1}{\rho_2}\right]\right\}^{\frac{1}{2}}}{\left[\dfrac{2\xi_2\,(1+C_0)}{\rho_1}\right]^{\frac{1}{2}}}$$

in (2.52), we obtain (1.11). This finishes the proof of Theorem 1.2.

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# Cigar孤立子上加权散度型椭圆算子的特征值估计

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**摘要**: 本文研究了cigar孤立子$(\mathbb{R}^2, g, f)$上加权散度型椭圆算子$\mathfrak{L}_{A,f}$的如下Dirichlet特征值问题:

$$\begin{cases} \mathfrak{L}_{A,f}u + Vu = \lambda\rho u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

其中$V$和$\rho$分别是$\Omega$上的非负连续函数和正连续函数. 我们建立了该问题的一些特征值不等式.

**关键词**: cigar孤立子; 加权散度型椭圆算子; 特征值

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