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THE INTERIOR GRADIENT ESTIMATES OF SPECIAL LAGRANGIAN TYPE EQUATIONS WITH SUPERCRITICAL PHASE

LI Jia-hui
(*School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China*)

Abstract: In this paper, we study the special Lagrangian type equations $\arctan\{\Delta u\mathrm{I} - D^2u\} = \Theta(x)$ with supercritical phase, and establish the corresponding interior gradient estimates of solutions for the special Lagrangian type equations.

Keywords: the interior gradient estimates; special Lagrangian type equations; supercritical phase

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1 Introduction

The existence, regularity and uniqueness of solutions for elliptic partial differential equations are basic problems. The study of the interior priori estimates is the key to solve the interior regularity for solutions of elliptic partial differential equation. For nonlinear elliptic partial differential equations, it is difficult to prove the interior estimates. In [1], Chen established the interior gradient estimates of k-admissible solutions of Hessian quotient equations in Euclidean space. Later, Chen-Xu-Zhang [2] studied prescribed Hessian quotient curvature equations and derived the interior gradient estimates. For σ_2 Hessian equation, Guan-Qiu [3] studied the interior Hessian estimates for convex solution. In addition, Chen-Han-Ou [4] showed the interior C^2 estimates for the Monge-Ampère in dimension $n = 2$ by introducing a new auxiliary function. Recently, Chen-Tu-Xiang [5] gave Pogorelov type estimates of solutions to Hessian quotient type equations.

Naturally, in this thesis, we explore the solution of special Lagrangian type equations

$$\arctan\{\Delta u\mathrm{I} - D^2u\} = \Theta(x), \quad x \in B_r(0) \subset \mathbb{R}^n, \tag{1.1}$$

where

$$\arctan\{\Delta u\mathrm{I} - D^2u\} =: \arctan\eta_1 + \arctan\eta_2 + \cdots + \arctan\eta_n.$$

Here $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ are the eigenvalues of the matrix $\Delta u\mathrm{I} - D^2u$ in [6] with

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Biography: Li Jiahui(1997–), female, born at Fuyang, Anhui, postgraduate, major in partial differential equations. E-mail: lijiahui505@163.com.

$$\eta_i = \sum_{k \neq i} \lambda_k, \quad \forall i = 1, 2, \cdots, n,$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix (D^2u). In addition, Θ is called the phase of equation (1.1). For special Lagrangian type equations, $\Theta(x)$ is usually studied under three different types:

- Subcritical phase $\Theta(x) \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$.
- The critical phase $\Theta(x) = \frac{(n-2)\pi}{2}$.
- Supercritical phase $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$.

In this paper, we consider the special Lagrangian type equations (1.1) with supercritical phase, that is the third type. The special Lagrangian equation

$$\sum_i \arctan \lambda_i = \Theta(x), \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.2}$$

was introduced from special Lagrangian geometry in Harvey-Lawson [7] firstly. The graph $x \mapsto (x, Du(x))$ is called special when the phase is a constant. This means that u satisfies (1.2) is special when the graph is minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$. Thanks to the work of Harvey-Lawson, scholars have taken a great interest in special Lagrangian manifolds. Warren-Yuan [8, 9] successively studied a priori interior Hessian and gradient estimates for special Lagrangian equations in two dimensions and three dimensions. The former concerned on all phases, while the latter concerned on large phase. Later, Wang-Yuan [10] proved a priori interior Hessian estimate for the special Lagrangian equation with critical and supercritical phases in general higher dimensions.

To our best knowledge, the special Lagrangian type equations have not been studied before, therefore we want to understand it naturally. In this paper, we establish the interior gradient estimates of solutions for a special Lagrangian type equations with supercritical phase. We now show our result.

Theorem 1.1 Suppose $u \in C^3(B_r(0))$ is a solution to the real special Lagrangian type equation (1.1) with supercritical phase and $\Theta(x) \in C^1(\overline{B_r(0)})$, where $B_r(0) \subset \mathbb{R}^n$. Then we have

$$|Du(0)| \leq C_1, \tag{1.3}$$

where C_1 is a positive constant depending only on n, r, $\sup\limits_{B_r(0)} |u|$ and $|\Theta|_{C^1}$.

Remark 1.2 The conclusion can also be extended to a more general special Lagrangian type equation

$$\arctan\{\Delta u \mathrm{I} - D^2u\} = \Theta(x, u, Du), \quad x \in B_r(0) \subset \mathbb{R}^n, \tag{1.4}$$

which is complete by a similar proof of Theorem 1.1.

The paper is organized as follows. In Section 2, we present some results about the special Lagrangian type equation, which will be used in the proof of Theorem . In Section 3, we prove the interior gradient estimates for (1.1).

2 Preliminaries

In this section, we show some properties of the special Lagrangian type equations with supercritical phase, which could be found in [6, 10, 11].

Property 2.1 Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Suppose $u \in C^2(\Omega)$ is a solution to the special Lagrangian type equation (1.1), $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix D^2u with

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n, \tag{2.1}$$

and $\eta_i = \sum\limits_{k \neq i} \lambda_k$, $\forall i = 1, 2, \cdots, n$. Then we have some properties:

$$\eta_1 \leq \eta_2 \leq \cdots \leq \eta_n, \tag{2.2}$$

$$\eta_1 + \eta_2 + \cdots + \eta_n > 0, \tag{2.3}$$

$$|\eta_1| \leq \eta_2, \tag{2.4}$$

$$|\eta_1| < C_0, \tag{2.5}$$

where $C_0 = \max\{\tan\big(\frac{(n-1)\pi}{2} - \min\limits_{\overline{\Omega}} \Theta\big), \tan\big(\frac{\max_{\overline{\Omega}} \Theta}{n}\big)\}$.

Proof For any $i = 1, 2, \cdots, n$, we can know $\arctan \eta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\eta_i = \sum\limits_{k \neq i} \lambda_k$, $\forall i = 1, 2, \cdots, n$. Then we have

$$\arctan \eta_1 + \arctan \eta_2 = \Theta - \sum_{i=3}^{n} \arctan \eta_i > \Theta - \frac{(n-2)\pi}{2} > 0,$$

so $\eta_1 + \eta_2 > 0$, which implies (2.2), (2.3) and (2.4) hold.

Moreover,

$$\arctan \eta_1 = \Theta - \sum_{i=2}^{n} \arctan \eta_i > \min_{\overline{\Omega}} \Theta - \frac{(n-1)\pi}{2},$$

and

$$\arctan \eta_1 < \frac{\arctan \eta_1 + \arctan \eta_2 + \cdots + \arctan \eta_n}{n} \leq \frac{\max\limits_{\overline{\Omega}} \Theta}{n},$$

so we can obtain

$$|\eta_1| < \max\{\tan\Big(\frac{(n-1)\pi}{2} - \min_{\overline{\Omega}} \Theta\Big), \tan\Big(\frac{\max\limits_{\overline{\Omega}} \Theta}{n}\Big)\}.$$

Therefore, we complete the proof.

Property 2.2 Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Suppose $u \in C^2(\Omega)$ is a solution to special Lagrangian type equation (1.1) and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix D^2u. Denote

$$F^{ij} = \frac{\partial \arctan \eta}{\partial u_{ij}}.$$

For $x_0 \in \Omega$, we can assume $D^2u(x_0)$ is diagonal with $\lambda_i = u_{ii}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. By rotating the coordinate $(e_1, \cdots, e_n)$, then $F^{ij}(x_0)$ is diagonal. Then we have

$$F^{ij} =: \frac{\partial \arctan \eta}{\partial u_{ij}} = \begin{cases} \sum\limits_{p \neq i} \frac{1}{1+\eta_p^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.6}$$

From the special Lagrangian type equation (1.1), we know

$$\sum_{ij=1}^{n} F^{ij} u_{ijk} = \Theta_k. \tag{2.7}$$

Hence we can get from Property 2.1

$$F^{11} \leq F^{22} \leq \cdots \leq F^{nn}; \tag{2.8}$$

$$F^{ii} \geq c_0 \sum_{i=1}^{n} F^{ii}, \ \forall i \geq 2; \tag{2.9}$$

$$F^{ii} u_{ii} = \sum_{p=1}^{n} \frac{\eta_p}{1+\eta_p^2} \in (-\frac{1}{2}, \frac{n}{2}), \tag{2.10}$$

where c_0 is a positive constant depending on n.

3 Interior Gradient Estimates

In this section, we will give a proof of the interior gradient estimates for equation (1.1), i.e. Theorem 1.1.

Proof of Theorem 1.1 For any $x \in B_r(0) \subset \mathbb{R}^n$, let $v_y = \frac{1}{r^2} u(ry)$. Then, we know $D^2v(y) = D^2u(x)$, $\forall y \in B_1(0)$. Therefore, we can suppose $r = 1$. In $B_1(0)$, we choose the auxiliary function $\phi(x) = |Du| g(u) \rho(x)$, where $\rho(x) = 1 - |x|^2$, $g(u) = (2 \max\limits_{B_1(0)} |u| + 1 - u)^{-\frac{1}{6}}$. Naturally, $\phi(x)$ attains its maximum at some point $x_0 \in B_1(0)$. By rotating the coordinate $(e_1, \cdots, e_n)$, we assume $D^2u(x_0)$ is diagonal and

$$u_{i_0}^2(x_0) \geq \frac{1}{n} |Du(x_0)|^2 > 0, \ \text{ for any given index } i_0. \tag{3.1}$$

We now consider the function

$$\varphi(x) = \frac{1}{2} \log |Du|^2 + \log g(u) + \log \rho(x). \tag{3.2}$$

It is easy to know

$$\varphi(x_0) = \log \phi(x_0) \geq \log \phi(x) = \varphi(x),$$

for any x near x_0. Hence, $\varphi(x)$ attains its local maximum at $x_0 \in B_1(0)$. In the following, all the calculations are at x_0. Differentiate (3.2) at x_0 once to see that

$$0 = \varphi_i(x_0) = \frac{u_i u_{ii}}{|Du|^2} + \frac{g'}{g} u_i + \frac{\rho_i}{\rho}. \tag{3.3}$$

It means that

$$\frac{u_{i_0}u_{i_0i_0}}{|Du|^2}=-\frac{g'}{g}u_{i_0}-\frac{\rho_{i_0}}{\rho}=-\frac{g'}{2g}u_{i_0}-\frac{g'\rho u_{i_0}+2g\rho_{i_0}}{2g\rho}. \tag{3.4}$$

- If $g'\rho u_{i_0}+2g\rho_{i_0}<0$, the Theorem 1.1 is proved.
- If $g'\rho u_{i_0}+2g\rho_{i_0}\geq 0$, we can get $\frac{u_{i_0}u_{i_0i_0}}{|Du|^2}\leq -\frac{g'}{2g}u_{i_0}<0$, so $u_{i_0i_0}<0,\ u_{i_0i_0}\neq u_{11}$.

Moreover, differentiate (3.2) at x_0 twice to get that

$$\begin{aligned}
0\geq&\ \varphi_{ii}(x_0)\\
=&\ \frac{u_ku_{kii}+u_{ii}^2}{|Du|^2}-\frac{2u_i^2u_{ii}^2}{|Du|^4}+\frac{g'}{g}u_{ii}+\Big(\frac{g''}{g}-\frac{g'^2}{g^2}\Big)u_i^2+\frac{\rho_{ii}}{\rho}-\frac{\rho_i^2}{\rho^2}\\
=&\ \frac{u_ku_{kii}+u_{ii}^2}{|Du|^2}-2\Big(\frac{g'^2}{g^2}u_i^2+2\frac{g'}{g}\frac{\rho_i}{\rho}u_i+\frac{\rho_i^2}{\rho^2}\Big)+\frac{g'}{g}u_{ii}+\Big(\frac{g''}{g}-\frac{g'^2}{g^2}\Big)u_i^2+\frac{\rho_{ii}}{\rho}-\frac{\rho_i^2}{\rho^2}\\
\geq&\ \frac{u_ku_{kii}+u_{ii}^2}{|Du|^2}+\Big(\frac{g''}{g}-5\frac{g'^2}{g^2}\Big)u_i^2+\frac{g'}{g}u_{ii}-\frac{2}{\rho}-5\frac{\rho_i^2}{\rho^2},
\end{aligned}$$

where in the last inequality we use Cauchy-Schwartz inequality.

Recall $F^{ij}=\frac{\partial \arctan\eta}{\partial u_{ij}}$, and $\{F^{ij}\}$ is diagonal since $D^2u(x_0)$ is diagonal. Moreover, by calculation, we have

$$\begin{aligned}
&g'=\frac{1}{6}(2\max_{B_1(0)}|u|+1-u)^{-\frac{7}{6}},\ g''=\frac{7}{36}(2\max_{B_1(0)}|u|+1-u)^{-\frac{13}{6}},\\
&\frac{g''}{g}-5\frac{(g')^2}{g^2}=\frac{1}{18}(2\max_{B_1(0)}|u|+1-u)^{-2},\ \frac{g'}{g}=\frac{1}{6}(2\max_{B_1(0)}|u|+1-u)^{-1}>0.
\end{aligned}$$

Hence, we can get

$$\begin{aligned}
0\geq& F^{ii}\varphi_{ii}(x_0)\\
\geq&\frac{F^{ii}(u_ku_{kii}+u_{ii}^2)}{|Du|^2}+\Big(\frac{g''}{g}-5\frac{g'^2}{g^2}\Big)F^{ii}u_i^2+\frac{g'}{g}F^{ii}u_{ii}-\frac{2\sum_{i=1}^nF^{ii}}{\rho}-20\frac{\sum_{i=1}^nF^{ii}}{\rho^2}\\
\geq&\frac{F^{ii}u_ku_{kii}}{|Du|^2}+\frac{1}{18}(2\max_{B_1(0)}|u|+1-u)^{-2}F^{ii}u_i^2+\frac{g'}{g}F^{ii}u_{ii}-\frac{2n(n-1)}{\rho}-\frac{20n(n-1)}{\rho^2}\\
\geq&-\frac{|D\Theta|}{|Du|}+\frac{1}{18}(2\max_{B_1(0)}|u|+1-u)^{-2}F^{ii}u_i^2-\frac{1}{12}(2\max_{B_1(0)}|u|+1-u)^{-1}-\frac{22n(n-1)}{\rho^2},
\end{aligned}$$

by using (2.7)-(2.10). From (3.1), we have

$$\begin{aligned}
0\geq&-\frac{|D\Theta|}{|Du|}+\frac{1}{18}(2\max_{B_1(0)}|u|+1-u)^{-2}F^{i_0i_0}u_{i_0}^2-\frac{1}{12}(2\max_{B_1(0)}|u|+1-u)^{-1}-\frac{22n(n-1)}{\rho^2}\\
\geq&-\frac{|D\Theta|}{|Du|}+\frac{1}{18}(2\max_{B_1(0)}|u|+1-u)^{-2}u_{i_0}^2c_0\sum_{i=1}^nF^{ii}-\frac{1}{12}(2\max_{B_1(0)}|u|+1-u)^{-1}-\frac{22n(n-1)}{\rho^2}.
\end{aligned}$$

Thus, $u_{i_0}\rho\leq c_1$, where c_1 is a positive constant depending on n, $\sup\limits_{B_1(0)}|u|$, $|\Theta|_{C^1}$. Then, we get the following result

$$|Du(0)| = \rho(0)|Du(0)| \leq n^{\frac{1}{2}} u_{i_0} \rho(x_0) \leq C_1,$$

where C_1 is a positive constant depending on n, $\sup\limits_{B_1(0)} |u|$ and $|\Theta|_{C^1}$. The proof is complete.

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特殊拉格朗日型方程在超临界相位下的内部梯度估计

李佳慧

(宁波大学数学与统计学院, 浙江 宁波 315211)

摘 要: 本文研究了在超临界相位下的特殊拉格朗日型方程 $\arctan\{\Delta u \mathrm{I}_n - D^2 u\} = \Theta(x)$, 并建立了与特殊拉格朗日型方程相关的解的内部梯度估计.

关键词: 内部梯度估计; 特殊拉格朗日型方程; 超临界相位

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