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EXISTENCE THEOREM FOR A CLASS OF MEMS MODEL WITH A PERTURBATION TERM

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Abstract: In this paper, we study a class of MEMS models with a perturbation term. With the help of upper and lower solution method and variational approach, we obtain the existence and multiplicity results for these models. In particular, we find a curve which splits the area of parameters in the first quadrant into two parts according to the existence of solutions for a MEMS system. These are an extension on the existing literature in the aspect of MEMS models.

Keywords: MEMS model; a perturbation term; variational method; upper and lower solution

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1 Introduction

In this paper we discuss the existence and nonexistence of solutions to the following MEMS equation

$$\begin{cases} -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p} + P, & x \in \Omega, \\ 0 \le u < 1, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

and the following MEMS system

$$\begin{cases} -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-v)^p}, & x \in \Omega, \\ -\Delta v = \frac{\mu(1+\delta|\nabla v|^2)}{(1-u)^p}, & x \in \Omega, \\ 0 \le u, v < 1, & x \in \Omega, \\ u, v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \ge 3$, $p > 1$ and λ, μ, δ, P are all positive parameters. The relevant problems corresponding to (1.1), (1.2) have been studied in various areas such as mathematics, physics and so forth.

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In fact, problems like equation (1.1) originally came from the so-called MEMS model which MEMS means Micro-Electro-Mechanical Systems. This model describes the motion of an elastic membrane supported on a fixed ground plate and has been extensively studied for decades. Here we quote some of them and more details can be found in [1-8] and references therein.

MEMS device is composed of an elastic membrane supported on a fixed ground plate. When a voltage λ is applied, the elastic membrane deflects toward the ground plate. While the voltage exceeds a critical value λ^*, the two plates touch and no longer separate. This state is called unstable. MEMS device is no longer working properly. However, for the sake of research convenience, most researchers regard the parallel plates as infinite in length and ignore electrostatic field at the edges of the plates. After approximation, the authors in papers [6–8] introduced the following model

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $f(x) \in C(\overline{\Omega})$ is a non-negative function. In [6], the authors applied analytical and numerical techniques to establish upper and lower bounds for λ^* which is a critical value of (1.3). They also obtained some properties of the stable and semistable solution such as regularity, uniqueness, multiplicity and so on. In [7], the authors proved the existence of the second solution by variational approach and got the compactness along the branches of unstable solution. In [8], the authors applied the extend Pohozaev identity and addressed that when λ is a small voltage and the domain Ω is bounded and star-shaped, then stable solution is the unique solution.

In [9], Cassani, Marcos and Ghoussoub investigated the existence of biharmonic type as follows:

$$\begin{cases} -\Delta^2 u = \frac{\lambda f(x)}{(1-u)^2}, & \text{in } B_R, \\ 0 \le u < 1, & \text{in } B_R, \\ u = \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial B_R, \end{cases} \tag{1.4}$$

where B_R is a ball in $\mathbb{R}^N$ centered at the origin with radius R, $0 \le f(x) \le 1$ and η denotes the unit outward normal to ∂B_R. They proved that, there exists a $\lambda_* = \lambda_*(R, f) > 0$ such that for $0 < \lambda < \lambda_*$, problem (1.4) possesses a minimal positive and stable solution u_λ.

Since the approximation of (1.3) brings some errors in some cases, the models (1.3) need be corrected in several manners. In [10, 11], the authors started to consider the effect of edges of plates and added the corner-corrected term in (1.3). For instance, the authors in [11] studied the following equation

$$\begin{cases} -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p}, & \text{in } \Omega, \\ 0 \le u < 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\delta > 0$, $p > 1$ and $\lambda > 0$. They obtained the existence and nonexistence result depending on δ and some $\lambda_\delta^* > 0$.

If we correct the model (1.3) with external force or pressure, it can be reduced to

$$\begin{cases} \Delta u = \frac{\lambda f(x)}{u^2} + P, & \text{in } \Omega, \\ 0 \le u < 1, & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $P > 0$ is a parameter. In [12], Guo, Zhang and Zhou obtained the existence and nonexistence result of (1.6) which depends on λ and P.

Inspired by the researches in [11–13], we will study the problem (1.1) and get our first result.

Theorem 1.1 For any $\delta > 0$, we have

(i) There exists a $P^* > 0$ such that for any $P \ge P^*$, (1.1) has no solution in $H_0^1(\Omega)$.

(ii) For any $0 < P < P^*$, there exists a critical constant $\lambda_P^* > 0$ such that for $0 < \lambda < \lambda_P^*$, (1.1) has at least two positive solutions in $H_0^1(\Omega)$. Moreover, if $\lambda = \lambda_P^*$ then (1.1) has only one positive solution in $H_0^1(\Omega)$ and has no solution for $\lambda > \lambda_P^*$.

Since our equation (1.1) has both the corner-corrected term and external pressure term, we will use the upper and lower solution method to find the first solution. After showing that the first solution is exactly the local minimum of the corresponding energy functional of (1.1), we want to find the second solution with the help of Mountain Pass Lemma. However since the lack of Ambrosetti-Rabinowitz condition [14] (i.e., there exists an $a > 0$ such that for $|z| \ge a$, $G(x,z)+H(x,z) \le \theta(g(x,z)z+h(x,z)z)$ where $\theta \subset [0, \frac{1}{2})$ and $G(x,z) = \int_0^z g(x,t)dt$, $H(x,z) = \int_0^z h(x,t)dt$). Therefore we will use the monotonicity trick ([15]) to find a bounded $(PS)_c$ sequence so that we can get our result.

Recently, the following Lane-Emden system was considered by do Ó and Clemente ([16]):

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-v)^2}, & \text{in } \Omega, \\ -\Delta v = \frac{\mu g(x)}{(1-u)^2}, & \text{in } \Omega, \\ 0 \le u, v < 1, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

They obtained a curve Γ that separates the positive quadrant of the (λ,μ)-plane into two connected components O_1 and O_2. For $(\lambda,\mu) \in O_1$, problem (1.7) has a positive classical minimal solution (u_λ, v_λ). If $(\lambda,\mu) \in O_2$, there is no solution.

Motivated by the result in [16], we consider the system (1.2) in the second part. With the help of upper and lower solution approach we get the next result.

Theorem 1.2 There exists a curve Γ which splits the parameter area (λ,μ) of the first quadrant into two connected parts D_1 and D_2. When $(\lambda,\mu) \in D_1$, (1.2) has at least one solution. There is no solution if $(\lambda,\mu) \in D_2$.

The rest of our paper is organized as follows. In Section 2, we will introduce a function transformation to (1.1) and some auxilary results. In Section 3, we give the proof of Theorem 1.1. In Section 4, we show the proof of Theorem 1.2.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ for $N \geq 3$. Throughout the paper we use standard Sobolev space $H_0^1(\Omega)$ with the usual norm

$$\|u\| = (\int_\Omega |\nabla u|^2 \, dx)^{\frac{1}{2}},$$

and the usual Lebesgue space $L^p(\Omega)$ whose norms are denoted by $|u|_p$. Since it is hard to write the concrete form of energy functional of (1.1), we introduce a function transformation to overcome this difficulty. Set

$$v = \hat{f}(u) = \int_0^u e^{\frac{\lambda\delta}{(p-1)(1-t)^{p-1}}} \, dt$$

for $u \in (0,1)$. Then $v \in (0, \infty)$ and $\hat{f}'(u) = e^{\frac{\lambda\delta}{(p-1)(1-u)^{p-1}}} > 0$.

This shows that $\hat{f}$ is strictly increasing, and therefore has an inverse function f such that $u = f(v)$. Thus we have

$$\begin{aligned} -\Delta v = -\Delta \hat{f}(u) &= -\nabla(\hat{f}'(u)\nabla u) \\ &= -\nabla(e^{\frac{\lambda\delta}{(p-1)(1-u)^{p-1}}} \nabla u) \\ &= -e^{\frac{\lambda\delta}{(p-1)(1-u)^{p-1}}} \frac{\lambda\delta}{(1-u)^p} |\nabla u|^2 - e^{\frac{\lambda\delta}{(p-1)(1-u)^{p-1}}} \Delta u. \end{aligned}$$

Together with (1.1), we know that the existence of solution to (1.1) is equivalent to the existence of the following equation

$$\begin{cases} -\Delta v = g(v) + h(v), & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $g(v) = e^{\frac{\lambda\delta}{(p-1)(1-f(v))^{p-1}}} \dfrac{\lambda}{(1-f(v))^p}$ and $h(v) = Pe^{\frac{\lambda\delta}{(p-1)(1-f(v))^{p-1}}}$. Then it is easy to see that the energy functional associated to problem (2.1) can be denoted by

$$I(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega G(v) dx - \int_\Omega H(v) \, dx \tag{2.2}$$

for $v \in H_0^1(\Omega)$, where $G(v), H(v)$ are defined by

$$G(v) = \int_0^v g(s) \, ds = \int_0^v e^{\frac{\lambda\delta}{(p-1)(1-f(s))^{p-1}}} \frac{\lambda}{(1-f(s))^p} \, ds \tag{2.3}$$

and

$$H(v) = \int_0^v h(s) \, ds = \int_0^v Pe^{\frac{\lambda\delta}{(p-1)(1-f(s))^{p-1}}} \, ds. \tag{2.4}$$

We will give some properties satisfied by $g(s)$, $h(s)$ and $G(s)$, $H(s)$ defined by (2.1), (2.3) and (2.4). In the sequel, C, C', C'', C_i $(i \in N^+)$ represent different constants in the different circumstances.

Proposition 2.1 (i) We have $g(s) \in C^2(\mathbb{R})$ with $g(s) > 0, h(s) > 0, g'(s) > 0, h'(s) > 0$ and $g''(s) > 0$. Moreover, $g''(s)$ is bounded in $\mathbb{R}$.

(ii) There exist positive constants C_1, C_2 such that

$$C_1 s(\log s)^{\frac{2p}{p-1}} \le g(s) \le C_2 s(\log s)^{\frac{2p}{p-1}} \quad \text{for all } s \ge e.$$

(iii) There exist positive constants C_1, C_2 such that

$$C_1 s^2(\log s)^{\frac{2p}{p-1}} \le G(s) \le C_2 s^2(\log s)^{\frac{2p}{p-1}} \quad \text{for all } s \ge e.$$

(iv) There exist positive constants C_1, C_2 such that

$$C_1 s(\log s)^{\frac{p}{p-1}} \le h(s) \le C_2 s(\log s)^{\frac{p}{p-1}} \quad \text{for all } s \ge e.$$

(v) There exist positive constants C_1, C_2 such that

$$C_1 s^2(\log s)^{\frac{p}{p-1}} \le H(s) \le C_2 s^2(\log s)^{\frac{p}{p-1}} \quad \text{for all } s \ge e.$$

(vi) For any $s \in \mathbb{R}$, we have $|G(s)| \le C_1(1 + |s|^{q+1})$ and $|H(s)| \le C_2(1 + |s|^{\frac{q+1}{2}})$ for $q > 1$.

Proof Here we only prove (iv)–(vi) about $h(s), H(s)$ and the detailed proof of (i)–(iii) about $g(s), G(s)$ can be found in [11]. From the proof of Lemma 2.2 in [11] , we know

$$C_1 \frac{\lambda\delta s}{(1-f(s))^p} \le e^{\frac{\lambda\delta}{(p-1)(1-f(s))^{p-1}}} \le C_2 \frac{\lambda\delta s}{(1-f(s))^p}$$

and

$$C_1(\log s)^{\frac{p}{p-1}} \le \frac{1}{(1-f(s))^p} \le C_2(\log s)^{\frac{p}{p-1}} \text{ for some } C_1, C_2 > 0.$$

Therefore, we obtain

$$C_3 s(\log s)^{\frac{p}{p-1}} \le h(s) = Pe^{\frac{\lambda\delta}{(p-1)(1-f(s))^{p-1}}} \le C_4 s(\log s)^{\frac{p}{p-1}}.$$

Integrating by parts, we obtain $C_5 s^2(\log s)^{\frac{p}{p-1}} \le H(s) \le C_6 s^2(\log s)^{\frac{p}{p-1}}$. Since if s large enough, $(\log s)^{\frac{p}{p-1}} \le s^{\frac{q-1}{2}}$ for any $q > 1$, $H(s) \le C(1 + |s|^{\frac{q+1}{2}})$ for any $q > 1$.

Proposition 2.2 For any $m \in N$, there exists $s_m > 0$ such that

$$\begin{cases} g(s) = C_{1,m} s(\log s)^{\frac{2p}{p-1}} + C_{2,m} s, & s = s_m, \\ g(s) > C_{1,m} s(\log s)^{\frac{2p}{p-1}} + C_{2,m} s, & s > s_m, \end{cases} \tag{2.5}$$

and

$$\begin{cases} h(s) = C_{3,m} s(\log s)^{\frac{p}{p-1}} + C_{4,m} s, & s = s_m, \\ h(s) > C_{3,m} s(\log s)^{\frac{p}{p-1}} + C_{4,m} s, & s > s_m, \end{cases} \tag{2.6}$$

where $C_{i,m} > 0$ $(i = 1, 2, 3, 4)$ are positive constants depending on m.

Proof We know that when $s > e$, $C_1 s(\log s)^{\frac{2p}{p-1}} \le g(s) \le C_2 s(\log s)^{\frac{2p}{p-1}}$ and $C_3 s(\log s)^{\frac{p}{p-1}} \le h(s) \le C_4 s(\log s)^{\frac{p}{p-1}}$. We can choose $n \in N$ such that $C_1(m+n-1)^{\frac{2p}{p-1}} - C_2 m^{\frac{2p}{p-1}} + C_3(m+n-1)^{\frac{p}{p-1}} - C_4 m^{\frac{p}{p-1}} > 0$. Let

$$C_{1,m} = \frac{C_1(m+n-1)^{\frac{2p}{p-1}} - C_2 m^{\frac{2p}{p-1}}}{(m+n-1)^{\frac{2p}{p-1}} - m^{\frac{2p}{p-1}}},\ C_{3,m} = \frac{C_3(m+n-1)^{\frac{p}{p-1}} - C_4 m^{\frac{p}{p-1}}}{(m+n-1)^{\frac{p}{p-1}} - m^{\frac{p}{p-1}}},$$

and

$$C_{2,m} = \frac{m^{\frac{2p}{p-1}}(m+n-1)^{\frac{2p}{p-1}}(C_2 - C_1)}{(m+n-1)^{\frac{2p}{p-1}} - m^{\frac{2p}{p-1}}},\ C_{4,m} = \frac{m^{\frac{p}{p-1}}(m+n-1)^{\frac{p}{p-1}}(C_4 - C_3)}{(m+n-1)^{\frac{p}{p-1}} - m^{\frac{p}{p-1}}}.$$

Set $Y(s) := C_{1,m} s(\log s)^{\frac{2p}{p-1}} + C_{2,m} s + C_{3,m} s(\log s)^{\frac{p}{p-1}} + C_{4,m} s$. Then we get

$$\begin{aligned}
Y(e^m) =& C_{1,m} e^m m^{\frac{2p}{p-1}} + C_{2,m} e^m + C_{3,m} e^m m^{\frac{p}{p-1}} + C_{4,m} e^m \\
=& \frac{e^m m^{\frac{2p}{p-1}}[C_1(m+n-1)^{\frac{2p}{p-1}} - C_2 m^{\frac{2p}{p-1}}] + e^m m^{\frac{2p}{p-1}}[(m+n-1)^{\frac{2p}{p-1}}(C_2 - C_1)]}{(m+n-1)^{\frac{2p}{p-1}} - m^{\frac{2p}{p-1}}} \\
&+ \frac{e^m m^{\frac{p}{p-1}}[C_3(m+n-1)^{\frac{p}{p-1}} - C_4 m^{\frac{p}{p-1}}] + e^m m^{\frac{p}{p-1}}[(m+n-1)^{\frac{p}{p-1}}(C_4 - C_3)]}{(m+n-1)^{\frac{p}{p-1}} - m^{\frac{p}{p-1}}} \\
=& \frac{e^m m^{\frac{2p}{p-1}} C_2[(m+n-1)^{\frac{2p}{p-1}} - m^{\frac{2p}{p-1}}]}{(m+n-1)^{\frac{2p}{p-1}} - m^{\frac{2p}{p-1}}} + \frac{e^m m^{\frac{p}{p-1}} C_4[(m+n-1)^{\frac{p}{p-1}} - m^{\frac{p}{p-1}}]}{(m+n-1)^{\frac{p}{p-1}} - m^{\frac{p}{p-1}}} \\
=& C_2 e^m m^{\frac{2p}{p-1}} + C_4 e^m m^{\frac{p}{p-1}}.
\end{aligned}$$

This implies $Y(e^m) > g(e^m) + h(e^m)$, and we can see

$$\begin{aligned}
Y(e^{m+n-1}) &= [C_{1,m}(m+n-1)^{\frac{2p}{p-1}} + C_{2,m}]e^{m+n-1} + [C_{3,m}(m+n-1)^{\frac{p}{p-1}} + C_{4,m}]e^{m+n-1} \\
&= C_1 e^{m+n-1}(m+n-1)^{\frac{2p}{p-1}} + C_3 e^{m+n-1}(m+n-1)^{\frac{p}{p-1}} \\
&< g(e^{m+n-1}) + h(e^{m+n-1}).
\end{aligned}$$

Let $\tilde{Y}(s) = g(s) + h(s) - Y(s)$, and then $\tilde{Y}(e^m) < 0 < \tilde{Y}(e^{m+n-1})$. According to the continuity of function $\tilde{Y}(s)$, we can find $e^m < s_m < e^{m+n-1}$ satisfying (2.5) and (2.6) .

Since we will use upper and lower solution method to get the first solution of (1.1), we give the definition of upper and lower solution.

Definition 2.1 A function $\overline{v} \in H_0^1(\Omega)$ is a upper solution to problems (1.1) if the following

$$\begin{cases} -\Delta\overline{v} \ge \frac{\lambda(1+\delta|\nabla\overline{v}|^2)}{(1-\overline{v})^p} + P, & x \in \Omega, \\ 0 \le \overline{v} < 1, & x \in \Omega, \\ \overline{v} = 0, & x \in \partial\Omega \end{cases} \tag{2.7}$$

holds. Accordingly, if the first inequality in (2.7) is reversed for some $\underline{v}$, we call $\underline{v}$ a lower solution of problem (1.1). The upper and lower solution for (2.1) is defined in the same way.

In [17], T. Kusano established the existence of positive solutions in $C^2(\mathbb{R}^N)$ to the following problem

$$-\Delta u = f(x,u),\ x \in \mathbb{R}^N \tag{2.8}$$

via the upper and lower solution method.

Proposition 2.3 ([17]) If there exists a upper solution $\bar{u}$ and a lower solution $\underline{u}$ of (2.8), $\bar{u} \geq \underline{u}$ and $f(x,u)$ are locally Lipschitz continuous, then (2.8) has a solution u and $\underline{u} \leq u \leq \bar{u}$.

If we consider (2.8) on some bounded domain $\Omega \subset \mathbb{R}^N$ with $u = 0$ on $\partial\Omega$, then we quote the following result.

Proposition 2.4 ([18]) Assume that $\overline{u}$ is the upper solution and $\underline{u}$ is the lower solution of (2.8), $I(u)$ is the energy functional of (2.8). Let $U = \{u \in H_0^1(\Omega) | \underline{u} \leq u \leq \overline{u} \text{ a.e on } \Omega\}$, $a(u) \in L^1_{loc}(0,\infty)$ for any $u \in U$, $b(x) \in L^1(\Omega)$ and $f(x,u) \leq a(u)b(x)$, then there exists a solution u of (2.8) in set U which is the minimum point of $I(u)$ in U.

Proposition 2.5 ([19]) If $u \in H_0^1(\Omega)$ is the solution of the following equation

$$\begin{cases} Lu = f(x,u) + tJ, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.9}$$

where L is a linear elliptic operator and $t \geq 0$ is a constant. J is the first eigenfunction of L. The function $f(x,u)$ is continous and nonnegative function defined on $\overline{\Omega} \times [0,\infty)$. Moreover, if we assume

(i) $\underline{\lim}_{u\to\infty} \frac{f(x,u)}{u} > \lambda_1$, for any $x \in \overline{\Omega}$, where λ_1 is the first eigenvalue of L;

(ii) $\lim_{u\to\infty} \frac{f(x,u)}{u^\beta} = 0$, for any $x \in \overline{\Omega}$ and $\beta = \frac{N+1}{N-1}$;

then there is a constant K such that $|u|_{L^\infty} \leq K$.

Lemma 2.1 For any solution v of (2.1), there exists a constant $m_0 > 0$ such that $|v|_{L^\infty} \leq m_0$.

Proof We prove this by Proposition 2.5. Let $L = -\Delta$, $t = 0$, $f(x,u) = g(v) + h(v)$. We only need to verify whether the conditions in Proposition 2.5 are satisfied. In fact, we have

$$\underline{\lim}_{v\to\infty} \frac{g(v)+h(v)}{v} \geq \underline{\lim}_{v\to\infty} \frac{C_1 v(\log v)^{\frac{2p}{p-1}} + C_1 v(\log v)^{\frac{p}{p-1}}}{v} = \infty > \lambda_1,$$

and then using n times of L'Hopital rule we get

$$\begin{aligned} 0 \leq \lim_{v\to\infty} \frac{g(v)+h(v)}{v^\beta} &\leq \lim_{v\to\infty} \frac{C_2 v(\log v)^{\frac{2p}{p-1}} + C_2 v(\log v)^{\frac{p}{p-1}}}{v^\beta} \\ &= \lim_{v\to\infty} \frac{C_2 v(\log v)^{\frac{2p-n(p-1)}{p-1}} + C_2 v(\log v)^{\frac{p-n(p-1)}{p-1}}}{(\beta-1)^n v^{\beta-1}} \\ &= 0. \end{aligned}$$

Hence $\lim_{v\to\infty} \frac{g(v)+h(v)}{v^\beta} = 0$, and by Proposition 2.5, we know that there exists a constant m_0 such that $|v|_{L^\infty} \leq m_0$.

We will use the variational method to find the second solution of (1.1), and therefore we recall some basic results about this method.

Definition 2.2 ([20]) Given a real Banach space X, we say a functional $I : X \to \mathbb{R}$ of class C^1 satisfying the mountain pass geometry if there exists $u_0, u_1 \in X$ and $0 < r < \|u_1 - u_0\|$ such that

$$\inf_{\|u-u_0\|=r} I(u) > \max\{I(u_0), I(u_1)\}.$$

We define the Palais-Smale sequence at level c ($(PS)_c$ sequence for short) and $(PS)_c$ conditions in X for I as follows.

Definition 2.3

(i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_c$ sequence in X for I if $I(u_n) \to c$, $I'(u_n) \to 0$ as $n \to \infty$.

(ii) I satisfies the $(PS)_c$ condition in X if every $(PS)_c$ sequence in X for I contains a convergent subsequence.

Now we quote the Mountain Pass Lemma.

Proposition 2.6 ([20]) If the functional I satisfies mountain pass geometry, then it has a $(PS)_c$ sequence $\{u_n\}$ in X, where $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{t\in[0,1]} I(\gamma(t))$, and $\Gamma = \{\gamma \in C([0,1], X); \gamma(0) = u_0, \gamma(1) = u_1\}$. Moreover, if I satisfies $(PS)_c$ conditions in X, then it admits a critical point u such that $I(u) = c,\ I'(u) = 0$.

In order to find a bounded $(PS)_c$ sequence for the functional I in $H_0^1(\Omega)$, we recall the following monotonicity trick.

Proposition 2.7 ([21]) Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{I_\mu\}_{\mu\in J}$ of C^1-functionals on X of the form

$$I_\mu(u) = A(u) - \mu B(u),\ \forall \mu \in J,$$

such that $A(u) \to \infty$ as $\|u\|_X \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(v_1), I_\mu(v_2)\},$$

where $\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = v_1, \gamma(1) = v_2\}$. Then, for almost every $\mu \in J$, there is a sequence $\{v_n\} \subset X$ such that $\{v_n\}$ is bounded in X, $I_\mu(v_n) \to c_\mu$ and $I'_\mu(v_n) \to 0$. Moreover, the map $\mu \to c_\mu$ is continuous from the left.

In the sequel we will take $J = [\frac{1}{2}, 1]$ and prove that our functional I_μ satisfies the conditions in Proposition 2.7, and then I_μ has a bounded $(PS)_{c_\mu}$ sequence $\{u_{\mu,n}\}$ at level c_μ as $\mu \to 1$.

3 The proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. First we will show the nonexistence result for (1.1).

Proof of Theorem 1.1 (i) We argue by contradiction. Let $\varphi \in H_0^1(\Omega)$ is the positive solution of

$$\begin{cases} -\Delta\varphi = 1, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

By Lax-Milgram theorem we know the existence and uniqueness of solution to (3.1) and $\varphi, \nabla\varphi \in L^\infty(\Omega)$ follows by the elliptic regularity results. Then we define $P^* = \frac{1}{|\varphi|_\infty}$. Suppose that (1.1) has a solution u when $P \geq P^*$, and then for any $\lambda > 0$ we get u satisfying the following equation

$$\begin{cases} -\Delta(u - P\varphi) = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p} > 0, & x \in \Omega, \\ u - P\varphi = 0, & x \in \partial\Omega. \end{cases}$$

By the maximum principle of elliptic partial differential equation we obtain $P\varphi < u < 1$. However, the definition of P^* shows that $P|\varphi|_\infty = \frac{P}{P^*} \geq 1$. This contradicts with $P|\varphi|_\infty \leq u < 1$. Therefore when $P \geq P^*$, (1.1) has no solution.

In the following we show the relationship between the existence of solutions of (1.1) and the parameter λ.

Lemma 3.1 For any $\delta > 0$, if (1.1) has a solution for $\lambda = \lambda_1$ with any fixed $P > 0$, then (1.1) has a solution for $0 < \lambda_2 < \lambda_1$.

Proof Suppose that (1.1) has a solution u_{λ_1} for $\lambda = \lambda_1$, and then for any $0 < \lambda_2 < \lambda_1$ we have

$$-\Delta u_{\lambda_1} = \frac{\lambda_1(1+\delta|\nabla u_{\lambda_1}|^2)}{(1-u_{\lambda_1})^p} + P > \frac{\lambda_2(1+\delta|\nabla u_{\lambda_1}|^2)}{(1-u_{\lambda_1})^p} + P.$$

This implies u_{λ_1} is a upper solution and also 0 is a lower solution of (1.1) with $\lambda = \lambda_2$. By Proposition 2.3, we know (1.1) has a solution for any $0 < \lambda_2 < \lambda_1$.

Now we discuss the relationship between λ and P.

Lemma 3.2 There exists a $\lambda_P > 0$ such that (1.1) has at least one solution when $0 < \lambda < \lambda_P$ for any fixed $0 < P < P^*$.

Proof For any fixed $0 < P < P^*$, we choose $\frac{P}{P^*} < s < 1$ such that $\overline{u} = sP^*\varphi$. We can verify easily $0 < \overline{u} < 1$ and $\overline{u}$ satisfying

$$\begin{aligned} -\Delta\overline{u} &= P + \frac{1+\delta|\nabla\overline{u}|^2}{(1-s)^p}(1-s)^p P^*(s - \frac{P}{P^*})\frac{1}{1+\delta|\nabla\overline{u}|^2} \\ &\geq P + \frac{P^*}{1+\delta {P^*}^2|\nabla\varphi|_\infty^2}(1-s)^p(s - \frac{P}{P^*})\frac{1+\delta|\nabla\overline{u}|^2}{(1-s)^p} \\ &\geq P + \frac{P^*}{1+\delta {P^*}^2|\nabla\varphi|_\infty^2}(1-s)^p(s - \frac{P}{P^*})\frac{1+\delta|\nabla\overline{u}|^2}{(1-\overline{u})^p}. \end{aligned}$$

Let $s = \frac{P^*+P}{2P^*}$ and $\lambda_P = \frac{P^*}{1+\delta {P^*}^2|\nabla\varphi|_\infty^2}(1 - \frac{P^*+P}{2P^*})^p(\frac{P^*+P}{2P^*} - \frac{P}{P^*})$. Then we know $\overline{u} = sP^*\varphi$ is a upper solution and 0 is a lower solution of (1.1) with $\lambda = \lambda_P$. By Proposition 2.3 we get (1.1) has at least one solution in $H_0^1(\Omega)$. By Lemma 3.1 we know (1.1) has at least one solution for any $0 < \lambda < \lambda_P$. We get the assertion.

In order to prove Theorem 1.1 (ii), we define λ_P^* as follows.

$$\lambda_P^* = \sup\{\lambda_P > 0 \,|\, (1.1) \text{ has at least one solution for any fixed } 0 < P < P^*\}. \tag{3.2}$$

Now we show that λ_P^* is well-defined.

Lemma 3.3 λ_P^* defined in (3.2) is bounded.

Proof Suppose that $0 < u < 1$ is the solution of (1.1) for $\lambda_P > 0$ and any fixed $0 < P < P^*$. Multiplying (1.1) by φ and integrating on Ω on both sides of equation (1.1), we have

$$|\Omega| \ge \int_\Omega -\Delta u\varphi \, dx = \int_\Omega \Big(\frac{\lambda_P(1+\delta|\nabla u|^2)}{(1-u)^p} + P\Big)\varphi \, dx \ge \int_\Omega \lambda_P\varphi \, dx + \int_\Omega P\varphi \, dx,$$

where $|\Omega|$ denotes the measure of Ω. Therefore we get $\lambda_P \le \frac{|\Omega| - P\int_\Omega \varphi \, dx}{\int_\Omega \varphi \, dx}$ and $\frac{|\Omega| - P\int_\Omega \varphi \, dx}{\int_\Omega \varphi \, dx}$ is positive and finite. This implies λ_P^* is bounded.

Lemma 3.4 For any $\delta > 0$, $0 < P < P^*$ and $0 < \lambda < \lambda_P^*$, the equation (1.1) has at least one upper solution and then at least one solution in $H_0^1(\Omega)$.

Proof According to Lemma 3.2 and Lemma 3.3, we know that when $0 < \lambda_1 < \lambda_2 < \lambda_P^*$, the equation (1.1) has at least one solution $u_{\lambda_1,P}$ for $\lambda = \lambda_1$ and $u_{\lambda_2,P}$ for $\lambda = \lambda_2$, respectively, for any fixed $0 < P < P^*$. Moreover, we can regard $u_{\lambda_2,P}$ as a upper solution of equation (1.1) with $\lambda = \lambda_1$. Therefore for any fixed $0 < P < P^*$, when $0 < \lambda < \lambda_P^*$, the equation (1.1) has a solution u_λ and an upper solution $\overline{u}_\lambda$ and a lower solution $\underline{u}_\lambda = 0$.

There is a corresponding solution $v = \hat{f}(u_\lambda)$, an upper solution $\overline{v} = \hat{f}(\overline{u}_\lambda)$ and a lower solution $\underline{v} = \hat{f}(\underline{u}_\lambda) = 0$ to the equation (2.1). By Lemma 2.1 we know $\overline{v} = \hat{f}(\overline{u}_\lambda) \le C$. Define $U = \{v \in H_0^1(\Omega) \,|\, 0 = \underline{v} < v < \overline{v} = \hat{f}(\overline{u}_\lambda) \le C \ a.e. \text{ on } \Omega\}$. If we set $a(x,v) := g(v) + h(v) = e^{\frac{\lambda\delta}{(p-1)(1-f(v))^{(p-1)}}} \frac{\lambda}{(1-f(v))^p} + Pe^{\frac{\lambda\delta}{(p-1)(1-f(v))^{(p-1)}}}$, $b(x) = 1$, in Proposition 2.4, where $f(v)$ is bounded owing to the boundedness of v, then we have $a(x,v) \in L^1_{loc}(\Omega)$, and v is the minimum point of the functional I in U, where I is defined in (2.2). In the following, we will show that v is the local minimum point of I in $H_0^1(\Omega)$.

Lemma 3.5 The solution v is the local minimum point of the functional I in $H_0^1(\Omega)$.

Proof We follow the idea in [22] and argue by contradiction. Suppose that v is not the local minimum point of I on $H_0^1(\Omega)$, and then there exists a sequence $\{v_n\} \subset H_0^1(\Omega)$ such that $\|v_n - v\| \to 0$ and $I(v_n) < I(v)$ as $n \to \infty$.

Let $v_{n,0} = \max\{\underline{v}, \min\{\overline{v}, v_n\}\}$, $v_{n,+} = \max\{v_n - \overline{v}, 0\}$, $v_{n,-} = \max\{\underline{v} - v_n, 0\}$, where $\overline{v}$ is the upper solution and $\underline{v}$ is the lower solution of (2.1). This implies $v_n = v_{n,0} + v_{n,+} - v_{n,-}$. Define $\Omega_n^0 = \{x \in \Omega \,|\, \underline{v} \le v_n \le \overline{v}\}$, $\Omega_n^+ = \text{supp}\{v_{n,+}\}$, $\Omega_n^- = \text{supp}\{v_{n,-}\}$, and $F(v_n) := G(v_n) + H(v_n)$, $F'(v_n) = g(v_n) + h(v_n)$, where "supp" means the support of functional in Ω. Then

$$\begin{aligned} I(v_n) = &\int_{\Omega_n^0} \frac{1}{2}|\nabla v_n|^2 \, dx - \int_{\Omega_n^0} F(v_n) \, dx + \int_{\Omega_n^+} \frac{1}{2}|\nabla v_n|^2 \, dx - \int_{\Omega_n^+} F(v_n) \, dx \\ &+ \int_{\Omega_n^-} \frac{1}{2}|\nabla v_n|^2 \, dx - \int_{\Omega_n^-} F(v_n) \, dx. \end{aligned}$$

In Ω_n^+, we have

$$\int_{\Omega_n^+} \frac{1}{2}|\nabla v_n|^2\,dx - \int_{\Omega_n^+} F(v_n)\,dx = \int_{\Omega_n^+} \frac{1}{2}|\nabla(\overline{v}+v_{n,+})|^2\,dx - \int_{\Omega_n^+} F(\overline{v}+v_{n,+})\,dx.$$

Similarly we obtain

$$\int_{\Omega_n^-} \frac{1}{2}|\nabla v_n|^2\,dx - \int_{\Omega_n^-} F(v_n)\,dx = \int_{\Omega_n^-} \frac{1}{2}|\nabla(\underline{v}-v_{n,-})|^2\,dx - \int_{\Omega_n^-} F(\underline{v}-v_n^-)\,dx.$$

Since $\underline{v} \le v_n = v_{n,0} \le \overline{v}$ in Ω_n^0, we obtain

$$\int_{\Omega_n^0} \frac{1}{2}|\nabla v_n|^2\,dx - \int_{\Omega_n^0} F(v_n)\,dx = I(v_{n,0}) - \int_{\Omega_n^+} [\frac{1}{2}|\nabla\overline{v}|^2 - F(\overline{v})]\,dx - \int_{\Omega_n^-} [\frac{1}{2}|\nabla\underline{v}|^2 - F(\underline{v})]\,dx.$$

Therefore we conclude

$$\begin{aligned} I(v_n) =& I(v_{n,0}) + \int_{\Omega_n^+} \frac{1}{2}(|\nabla(\overline{v}+v_{n,+})|^2 - |\nabla\overline{v}|^2)\,dx - \int_{\Omega_n^+} [F(\overline{v}+v_{n,+}) - F(\overline{v})]\,dx \\ &+ \int_{\Omega_n^-} \frac{1}{2}(|\nabla(\underline{v}-v_{n,-})|^2 - |\nabla\underline{v}|^2)\,dx - \int_{\Omega_n^-} [F(\underline{v}-v_{n,-}) - F(\underline{v})]\,dx. \end{aligned} \tag{3.3}$$

Because $\underline{v}$ is a lower solution, we know $-\Delta\underline{v} \le g(\underline{v}) + h(\underline{v})$ and

$$\int_\Omega \nabla\underline{v}\nabla(-v_{n,-})\,dx \ge \int_\Omega [g(\underline{v}) + h(\underline{v})](-v_{n,-})\,dx.$$

Similarly we have

$$\int_\Omega \nabla\overline{v}\nabla v_{n,+}\,dx \ge \int_\Omega [g(\overline{v}) + h(\overline{v})]v_{n,+}\,dx.$$

Summing up, we get

$$\begin{aligned} I(v_n) \ge& I(v_{n,0}) + \int_\Omega [\frac{1}{2}|\nabla v_{n,+}|^2 + \frac{1}{2}|\nabla v_{n,-}|^2]\,dx - \int_{\Omega_n^+} [F(\overline{v}+v_{n,+}) - F(\overline{v}) - f(\overline{v})v_{n,+}]\,dx \\ &- \int_{\Omega_n^-} [F(\underline{v}-v_{n,-}) - F(\underline{v}) - f(\underline{v})(-v_{n,-})]\,dx. \end{aligned} \tag{3.4}$$

Since $F(\overline{v}+v_{n,+}) - F(\overline{v}) - f(\overline{v})v_{n,+} = G(\overline{v}+v_{n,+}) + H(\overline{v}+v_{n,+}) - G(\overline{v}) - H(\overline{v}) - g(\overline{v})v_{n,+} - h(\overline{v})v_{n,+}$, by Sobolev embedding theorem, Proposition 2.1 (vi) and Hölder inequality, it follows that

$$\begin{aligned} \int_{\Omega_n^+} [G(\overline{v}+v_{n,+}) - G(\overline{v}) - g(\overline{v})v_{n,+}]\,dx &\le \int_{\Omega_n^+} C_1(1 + |\overline{v}+v_{n,+}|^{q+1})\,dx \\ &\le C_1|\Omega_n^+| + C_1|\Omega_n^+|^{\frac{1}{\beta}}|\overline{v}+v_{n,+}|_{2^*}^{q+1} \\ &\le C_1|\Omega_n^+| + C_3|\Omega_n^+|^{\frac{1}{\beta}}. \end{aligned}$$

Here β satisfies $\frac{q+1}{2^*} + \frac{1}{\beta} = 1$ for any $q > 1$. Due to $v < \overline{v}$, then for every $\varepsilon > 0$ there exists $\theta > 0$, such that $\text{meas}\{x|v(x) + \theta > \overline{v}\} < \varepsilon$, where "meas" means the measure of set.

Since $\Omega_n^+ \subset \{x|v(x)+\theta > \overline{v}\} \cup \{x|v(x)+\theta \le \overline{v} \le v_n(x)\}$ and $v_n \to v$ in H_0^1 as $n \to \infty$, by Poincaré inequality, we obtain

$$\begin{aligned}\varepsilon\theta^2 \ge \int_\Omega |\nabla(v_n - v)|^2\,dx \ge C\int_\Omega (v_n - v)^2\,dx \\ &\ge C\int_{v(x)+\theta\le v_n(x)} (v_n - v)^2\,dx \\ &\ge C\theta^2 \text{meas}\{x|v(x)+\theta \le v_n(x)\}.\end{aligned}$$

This implies $\lim\limits_{n\to\infty} |\Omega_n^+| = 0$. Then for n large enough, we have

$$\int_{\Omega_n^+} [G(\overline{v}+v_{n,+}) - G(\overline{v}) - g(\overline{v})v_{n,+}]\,dx \le C_1|\Omega_n^+| + C_3|\Omega_n^+|^{\frac{1}{\beta}} < C_4\varepsilon.$$

By Proposition 2.1 (vi), Sobolev embedding theorem and Hölder inequality, we have

$$\begin{aligned}\int_{\Omega_n^+} [H(\overline{v}+v_{n,+}) - H(\overline{v}) - h(\overline{v})v_{n,+}]\,dx &\le \int_{\Omega_n^+} C_2(1+|\overline{v}+v_{n,+}|^{\frac{q+1}{2}})\,dx \\ &\le C_2|\Omega_n^+| + C_2|\Omega_n^+|^{\frac{1}{2}}|\overline{v}+v_{n,+}|_{q+1}^{2(q+1)} \\ &\le C_2|\Omega_n^+| + C_5|\Omega_n^+|^{\frac{1}{2}} \\ &< C_6\varepsilon\end{aligned}$$

for n large enough. Therefore we conclude $\int_{\Omega_n^+}[F(\overline{v}+v_{n,+}) - F(\overline{v}) - f(\overline{v})v_{n,+}]\,dx < C_5\varepsilon + C_6\varepsilon < C'\varepsilon$. By an analogous argument, we obtain

$$\int_{\Omega_n^-} [F(\underline{v}-v_{n,-}) - F(\underline{v}) - f(\underline{v})(-v_{n,-})]\,dx < C''\varepsilon.$$

With $I(v_n) \le I(v_{n,0})$ and (3.4), we have

$$\int_\Omega (\frac{1}{2}|\nabla v_{n,+}|^2 + \frac{1}{2}|\nabla v_{n,-}|^2)\,dx \le I(v_n) - I(v_{n,0}) + C'\varepsilon + C''\varepsilon \le C\varepsilon. \tag{3.5}$$

However, since $\Omega_n^+ \bigcap \Omega_n^- = \emptyset$, we conclude

$$C(\int_\Omega v_{n,+}^{2^*}\,dx)^{\frac{N-2}{2N}} \le \frac{1}{2}\int_\Omega |\nabla v_{n,+}|^2\,dx \le C\varepsilon,$$

and

$$C(\int_\Omega v_{n,-}^{2^*}\,dx)^{\frac{N-2}{2N}} \le \frac{1}{2}\int_\Omega |\nabla v_{n,-}|^2\,dx \le C\varepsilon$$

for any $\varepsilon > 0$. These happen only if $v_{n,+}(x) = v_{n,-}(x) = 0$ *a.e.* $x \in \Omega$. This implies $v_n = v_{n,0} \in U$ *a.e.* on Ω and $I(v) \le I(v_n)$, which conflicts with the hypothesis $I(v) > I(v_n)$. So v is a local minimum point of I on $H_0^1(\Omega)$.

Lemma 3.6 The energy functional I in (2.2) has a mountain pass geometry.

Proof Firstly we take x_0 in Ω arbitrarily, and choose a proper R such that $B_R(x_0) \subset \Omega$. Let $\psi \in C_0^\infty(\Omega)$ be a cut-off function satisfying

$$\begin{cases} 0 \le \psi \le 1, & \text{in } \Omega, \\ \psi \equiv 1, & \text{in } B_R(x_0). \end{cases}$$

Then by Proposition 2.1 (iii), we know

$$\begin{aligned} I(t\psi) &= \frac{t^2}{2}\int_\Omega |\nabla\psi|^2\,dx - \int_\Omega G(t\psi)dx - \int_\Omega H(t\psi)\,dx \\ &\le \frac{t^2}{2}\int_\Omega |\nabla\psi|^2\,dx - \int_{B_R(x_0)} G(t\psi)\,dx - \int_{\Omega\backslash B_R(x_0)} G(t\psi)\,dx \\ &\le \frac{t^2}{2}\int_\Omega |\nabla\psi|^2\,dx - C_1 t^2(\log t)^{\frac{2p}{p-1}}|B_R(x_0)| \to -\infty \quad \textit{as } t \to +\infty. \end{aligned}$$

Therefore we can take a large enough t such that $w = t\psi$ satisfies $\|w\| > \|v\|$, but $I(w) < I(v)$. From Lemma 3.5 we know v is a local minimum point of I, and so we can choose a proper $0 < r < \|v - w\|$ such that

$$\inf_{\|\tilde v - v\| = r} I(\tilde v) > \max\{I(v), I(w)\},$$

which implies the functional I has the mountain pass geometry as in Definition 2.2.

Define $c = \inf\limits_{\gamma\in\Gamma}\max\limits_{t\in[0,1]} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) \,|\, \gamma(0) = v, \gamma(1) = w\}$. According to Proposition 2.6, there exists a $(PS)_c$ sequence $\{v_n\} \subset H_0^1(\Omega)$ such that

$$I(v_n) \to c,\ I'(v_n) \to 0 \quad \text{as } n \to \infty.$$

Now we show that if $\{v_n\}$ is bounded, then $\{v_n\}$ satisfies $(PS)_c$ condition.

Lemma 3.7 If the $(PS)_c$ sequence $\{v_n\}$ of the functional I is bounded in $H_0^1(\Omega)$, then it has a convergent subsequence in $H_0^1(\Omega)$.

Proof Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, it has a weakly convergent subsequence (still denoted as $\{v_n\}$). We may assume that as $n \to \infty$,

$$\begin{cases} v_n \rightharpoonup v \text{ in } \ H_0^1(\Omega), \\ v_n \to v \text{ in } \ L^{q+1}(\Omega) \text{ for } q \in [1, \dfrac{N+2}{N-2}), \\ v_n \to v \quad a.e.\ x \in \Omega \end{cases} \tag{3.6}$$

for some $v \in H_0^1(\Omega)$. Moreover, there also exists a $m(x) \in L^{q+1}(\Omega)$ for $q \in [1, \frac{N+2}{N-2})$ such that $|v(x)| \le m(x)$ $a.e.$ $x \in \Omega$. From $I'(v_n) \to 0$ in $H_0^1(\Omega)$ as $n \to \infty$, we obtain

$$\langle I'(v_n), \varphi\rangle = \int_\Omega \nabla u_n \nabla\varphi\,dx - \int_\Omega g(v_n)\varphi\,dx - \int_\Omega h(v_n)\varphi\,dx \to 0 \tag{3.7}$$

for any $\varphi \in H_0^1(\Omega)$. Since $g(s), h(s) \in C^1(\mathbb{R})$, it follows that

$$g(v_n)\varphi \to g(v)\varphi,\ \ h(v_n)\varphi \to h(v)\varphi \quad a.e.\ x \in \Omega.$$

By Proposition 2.1 (ii), (iv), we have

$$\begin{aligned}
&|g(v_n)\varphi| \le |C(1+|v_n|^q)\varphi| \le C|\varphi| + C|m(x)|^q|\varphi| \in L^1(\Omega),\\
&|h(v_n)\varphi| \le |C(1+|u_n|^{\frac{q-1}{2}})\varphi| \le C|\varphi| + C|m(x)|^{\frac{q-1}{2}}|\varphi| \in L^1(\Omega).
\end{aligned}$$

Then by Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\Omega g(v_n)\varphi\,dx \to \int_\Omega g(v)\varphi\,dx,\ \int_\Omega h(v_n)\varphi\,dx \to \int_\Omega h(v)\varphi\,dx \quad \text{as } n \to \infty.$$

Moreover, $v_n \rightharpoonup v\ in\ H_0^1(\Omega)$ implies $\int_\Omega \nabla v_n \nabla\varphi\,dx \to \int_\Omega \nabla v \nabla \varphi\,dx$ and

$$\begin{aligned}
\int_\Omega \nabla v_n \nabla\varphi\,dx - \int_\Omega g(v_n)\varphi\,dx - \int_\Omega h(v_n)\varphi\,dx \to \int_\Omega \nabla v\nabla\varphi\,dx - \int_\Omega g(v)\varphi\,dx&\\
- \int_\Omega h(v)\varphi\,dx.&
\end{aligned}\tag{3.8}$$

Then from (3.7) and (3.8) we get $\displaystyle\int_\Omega \nabla v\nabla\varphi\,dx - \int_\Omega g(v)\varphi\,dx - \int_\Omega h(v)\varphi\,dx = 0$. Set $\varphi = v$ we have

$$\int_\Omega \nabla v_n \nabla v\,dx \to \int_\Omega |\nabla v|^2\,dx$$

and

$$\int_\Omega |\nabla v|^2\,dx - \int_\Omega g(v)v\,dx - \int_\Omega h(v)v\,dx = 0. \tag{3.9}$$

Now by (3.7) and (3.9), it yields that

$$\begin{aligned}
\int_\Omega |\nabla(v_n - v)|^2\,dx &= \int_\Omega |\nabla v_n|^2\,dx - \int_\Omega |\nabla v|^2\,dx - 2\int_\Omega (\nabla v_n\nabla v - |\nabla v|^2)\,dx\\
&= \int_\Omega g(v_n)v_n\,dx + \int_\Omega h(v_n)v_n\,dx - (\int_\Omega g(v)v\,dx + \int_\Omega h(v)v\,dx) + o(1)\\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This means $v_n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$ and $I'(v) = 0$. We get the assertion.

In order to obtain the existence of a mountain pass solution to (1.1), it suffices to show the boundedness of the $(PS)_c$ sequence $\{v_n\}$. However it is difficult to prove it directly since the functions $g(s)$ and $h(s)$ do not satisfy any Ambrosetti-Rabinowitz type conditions. Therefore we will apply the monotonicity trick as in [15]. First, we modify the nonlinear terms g and h as follows:

$$\tilde{g}(s) = \begin{cases} g(s), & s < s_m,\\ C_{1,m}s(\log s)^{\frac{2p}{p-1}} + C_{2,m}s, & s \ge s_m, \end{cases} \tag{3.10}$$

and

$$\tilde{h}(s)=\begin{cases}h(s), & s<s_m,\\ C_{3,m}s(\log s)^{\frac{p}{p-1}}+C_{4,m}s, & s\geq s_m,\end{cases}\tag{3.11}$$

where $m_0<e^m<s_m<e^{m+n-1}$ $(m,n\in N^+)$ such that $\tilde{g}(s)+\tilde{h}(s)=g(s)+h(s)$ for $s=s_m$ and $\tilde{g}(s)+\tilde{h}(s)<g(s)+h(s)$ for $s>s_m$. Here s_m and $C_{i,m}$ $(i=1,2,3,4)$ are defined in Proposition 2.2.

We consider the equation

$$\begin{cases}-\Delta v=\mu\tilde{g}(v)+\mu\tilde{h}(v), & x\in\Omega,\\ v=0, & x\in\partial\Omega,\end{cases}\tag{3.12}$$

where $\mu\in[\frac{1}{2},1]$. The energy functional of (3.12) is $\tilde{I}_\mu:H_0^1(\Omega)\to\mathbb{R}$ defined by

$$\tilde{I}_\mu(v)=\frac{1}{2}\int_\Omega|\nabla v|^2\,dx-\mu\int_\Omega\tilde{G}(v)dx-\mu\int_\Omega\tilde{H}(v)\,dx,\tag{3.13}$$

where $\tilde{G}(v),\tilde{H}(v)$ are defined by

$$\tilde{G}(v)=\int_0^v\tilde{g}(s)\,ds,\ \tilde{H}(v)=\int_0^v\tilde{h}(s)\,ds.$$

Lemma 3.8 There exists an interval $J\subset[\frac{1}{2},1]$ such that the family of functionals $\{\tilde{I}_\mu\}_{\mu\in J}$ has a mountain pass geometry.

Proof From Lemma 3.6, we know that v is the local minimum point of I and $I(v)>I(t\psi)$ for t large enough. Here v is the solution of (2.1). By Lemma 2.1, we know $|v|_{L^\infty}\leq m_0$. This implies when $\mu=1$, we have $\tilde{I}_\mu(v)=I(v)$. So we obtain for $\mu=1$, $\tilde{I}_\mu(v)$ has a mountain pass geometry by Lemma 3.6. Furthermore, when $\frac{1}{2}<\mu<1$, we have $\tilde{I}_\mu(v)>I(v)$ for any positive $v\in H_0^1(\Omega)$. Because $\tilde{I}_\mu(v)$ is continuous with respect to μ, there exists $\frac{1}{2}<\mu_0<1$ such that $\tilde{I}_\mu(v)>\tilde{I}_\mu(t\psi)$ for any $\mu\in[\mu_0,1]$.

From Lemma 3.5, we may assume that there exists $\phi_\mu\in H_0^1(\Omega)\bigcap C^1(\overline{\Omega})$ such that $I(v+\phi_\mu)>I(v)$. According to the continuity of $\tilde{I}_\mu$ with respect to μ, we can choose μ_0 close to 1 properly such that $\tilde{I}_\mu(v)<I(v+\phi_\mu)$ for $\mu\in[\mu_0,1]$. Since $I(v+\phi_\mu)<\tilde{I}_\mu(v+\phi_\mu)$, we get $\tilde{I}_\mu(v+\phi_\mu)>\tilde{I}_\mu(v)$. Again by Lemma 3.5, v is also the local minimum point of $\tilde{I}_\mu(v)$ on $H_0^1(\Omega)$ for any $\mu\in[\mu_0,1]$.

Define $c_\mu=\inf\limits_{\gamma\in\Gamma}\max\limits_{t\in[0,1]}\tilde{I}_\mu(\gamma(t))$, where $\Gamma=\{\gamma\in C([0,1],H_0^1(\Omega))\,|\,\gamma(0)=v,\gamma(1)=t\psi\}$. According to Proposition 2.6, for almost every $\mu\in[\mu_0,1]$, we can find a $(PS)_{c_\mu}$ sequence $\{v_{\mu,n}\}\subset H_0^1(\Omega)$ such that

$$\tilde{I}_\mu(v_{\mu,n})\to c_\mu,\ \tilde{I}'_\mu(v_{\mu,n})\to 0\quad\text{as } n\to\infty.$$

Lemma 3.9 For almost every $\mu\in[\mu_0,1]$, the $(PS)_{c_\mu}$ sequence $\{v_{\mu,n}\}$ of functional $\tilde{I}_\mu$ satisfies $(PS)_{c_\mu}$ condition.

Proof Set $A(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$, $B(u) = \int_\Omega \tilde{G}(u) + \int_\Omega \tilde{H}(u)\,dx$. We see $A(u) \to \infty$ as $\|u\| \to \infty$ in $H_0^1(\Omega)$. Therefore by Proposition 2.7, we obtain a $(PS)_{c_\mu}$ sequence $\{v_{\mu,n}\}$ which is bounded in $H_0^1(\Omega)$. Then with the help of Lemma 3.7, we get that there exists a function $v_\mu \in H_0^1(\Omega)$ such that $v_{\mu,n} \to v_\mu$ in $H_0^1(\Omega)$ and $\tilde{I}'(v_\mu) = 0$, which implies $(PS)_{c_\mu}$ condition holds.

Now we choose a sequence $\{v_{\mu_j}\}_{j\in N}$ such that $\tilde{I}_{\mu_j}(v_{\mu_j}) = c_{\mu_j}$, $\tilde{I}'_{\mu_j}(v_{\mu_j}) = 0$ as $\mu_j \nearrow 1$. For simplicity, we denote $\{v_{\mu_j}\}$ as $\{v_j\}$. Now we show that$\{v_j\}_{j\in N}$ is bounded.

Lemma 3.10 The sequence $\{v_j\}_{j\in N}$ is uniformly bounded in $H_0^1(\Omega)$.

Proof We argue by contradiction. Suppose that $\|v_j\| \to \infty$ as $n \to \infty$. Let $w_j = \frac{v_j}{\|v_j\|}$, and then $\|w_j\| = 1$. Up to a subsequence $\{w_j\}$ if necessary, there exists a $w_0 \in H_0^1(\Omega)$ such that

$$\begin{cases} w_j \rightharpoonup w_0 \text{ in } H_0^1(\Omega), \\ w_j \to w_0 \text{ in } L^{q+1}(\Omega) \text{ for } q \in [1, \dfrac{N+2}{N-2}), \\ w_j \to w_0 \quad a.e.\ x \in \Omega. \end{cases} \tag{3.14}$$

We can also find a $d(x) \in L^{q+1}(\Omega)$ for $q \in [1, \frac{N+2}{N-2})$ such that $|w_0| \le d(x)$ $a.e.\ x \in \Omega$. Now we seperate our proof in four steps.

Step 1 We claim $w_0 \equiv 0$. Define $\Omega_0 = \{x \in \Omega : w_0 \ne 0\}$. In fact, if Ω_0 is not empty, then $v_j(x) = w_j(x)\|v_j\| \to \infty$ for $x \in \Omega_0$. Therefore by Proposition 2.1 (iii), (iv), we have

$$\begin{aligned} \int_\Omega \frac{\tilde{G}(v_j) + \tilde{H}(v_j)}{v_j^2} w_j^2\,dx &\ge \int_{\Omega_0} \frac{C_1 v_j^2 (\log v_j)^{\frac{2p}{p-1}} + C_2 v_j^2 (\log v_j)^{\frac{p}{p-1}}}{v_j^2} w_j^2\,dx \\ &\ge \int_\Omega (C_1 (\log v_j)^{\frac{2p}{p-1}} + C_2 (\log v_j)^{\frac{p}{p-1}}) w_j^2\,dx \to +\infty \text{ as } j \to \infty. \end{aligned} \tag{3.15}$$

However since $\tilde{I}_{\mu_j}(v_j) = c_{\mu_j}$, we know

$$\begin{aligned} \lim_{j\to\infty} \int_\Omega \frac{\tilde{G}(v_j) + \tilde{H}(v_j)}{v_j^2} w_j^2\,dx &= \lim_{j\to\infty} \int_\Omega \frac{\tilde{G}(v_j) + \tilde{H}(v_j)}{v_j^2} \frac{|v_j|^2}{\|v_j\|^2}\,dx \\ &= \lim_{j\to\infty} \int_\Omega \frac{\tilde{G}(v_j) + \tilde{H}(v_j)}{\|v_j\|^2}\,dx \\ &= \lim_{j\to\infty} \frac{1}{\|v_j\|^2} (\frac{1}{2}\|v_j\|^2 - \tilde{I}_{\mu_j}(v_j)) \\ &= \frac{1}{2}, \end{aligned} \tag{3.16}$$

which contradicts with (3.15). Therefore $w_0 \equiv 0$.

Step 2 Define $\tilde{F}(s) = s\tilde{g}(s) - 2\tilde{G}(s) + s\tilde{h}(s) - 2\tilde{H}(s)$. We claim that there exists a constant $C > 0$ such that $\tilde{F}(t) < \tilde{F}(s) + C$ for any $0 < t < s$. Actually, for $s > s_m > e$, we have

$$\tilde{F}'(s) = s\tilde{g}'(s) - \tilde{g}(s) + s\tilde{h}'(s) - \tilde{h}(s) = \frac{2p}{p-1} C_{1,m} s (\log s)^{\frac{p+1}{p-1}} + \frac{p}{p-1} C_{3,m} s (\log s)^{\frac{1}{p-1}} > 0.$$

This implies that $\tilde{F}(s)$ is increasing when $s > s_m$. Therefore we have

(i) If $s_m \le t < s$, then $\tilde{F}(t) < \tilde{F}(s) + C_1$ for any $C_1 \ge 0$.

(ii) If $0 < t < s_m \le s$ and denote $C_2 = \max_{s\in[0,s_m]} |\tilde{F}(s)|$, then $\tilde{F}(t) \le \tilde{F}(s) + C_2$.

(iii) If $0 < t < s < s_m$, then $\tilde{F}(t) \le \tilde{F}(s) + 2C_2$.

We take $C \ge \max\{C_1, C_2\}$ large enough such that $C + c_{\frac{1}{2}} > 0$. Then we get $\tilde{F}(t) < \tilde{F}(s) + C$ for any $0 < t < s$.

Step 3 Let $t_j \in [0,1]$ such that $\tilde{I}_{\mu_j}(t_j v_j) = \max_{t\in[0,1]} \tilde{I}_{\mu_j}(tv_j)$. We claim $2\tilde{I}_{\mu_j}(tv_j) \le 2c_{\frac{1}{2}} + C$. We will have the following cases.

(i) If $t_j = 0$, then $2\tilde{I}_{\mu_j}(t_j v_j) = 0 < 2c_{\frac{1}{2}} + C$.

(ii) If $t_j = 1$, then $2\tilde{I}_{\mu_j}(t_j v_j) = 2\tilde{I}_{\mu_j}(v_j) = 2c_{\mu_j} < 2c_{\frac{1}{2}} < 2c_{\frac{1}{2}} + C$.

(iii) If $0 < t_j < 1$, then $\tilde{I}'_{\mu_j}(t_j v_j)t_j v_j = t_j \tilde{I}'_{\mu_j}|_{t=t_j} = 0$ since $t_j v_j$ is the maximum point of $\tilde{I}_{\mu_j}(tv_j)$.

Therefore we get

$$\begin{aligned} 2\tilde{I}_{\mu_j}(tv_j) &\le 2\tilde{I}_{\mu_j}(t_j v_j) - \tilde{I}'_{\mu_j}(t_j v_j)t_j v_j \\ &= \mu_j \int_\Omega [t_j v_j \tilde{g}(t_j v_j) - 2\tilde{G}(t_j v_j) + t_j v_j \tilde{h}(t_j v_j) - 2\tilde{H}(t_j v_j)]\, dx \\ &\le \int_\Omega \mu_j(\tilde{F}(v_j) + C)\, dx = -\tilde{I}'_{\mu_j}(v_j) + 2\tilde{I}_{\mu_j}(v_j) + C\mu_j \\ &\le 2c_{\mu_j} + C \le 2c_{\frac{1}{2}} + C. \end{aligned}$$

The last inequality is deduced by the monotonicity of $\mu \to c_\mu$.

Step 4 We show that if $\|v_j\| \to \infty$, a contradiction occurs. For any constant $T > 0$, by Proposition 2.1 (i) we know $\tilde{G}(Tw_j) + \tilde{H}(Tw_j) \to \tilde{G}(Tw_0) + \tilde{H}(Tw_0)$ *a.e.* $x \in \Omega$. By Proposition 2.1 (vi), we have

$$\begin{aligned} \tilde{G}(Tw_j) + \tilde{H}(Tw_j) &\le C(1 + |Tw_j|^{q+1} + |Tw_j|^{\frac{q+1}{2}}) \\ &\le C(1 + |Td(x)|^{q+1} + |Td(x)|^{\frac{q+1}{2}}), \end{aligned}$$

where $q \in [1, \frac{N+2}{N-2})$. Thus $C(1+|Td(x)|^{q+1}+|Td(x)|^{\frac{q+1}{2}}) \in L^1(\Omega)$. Since $w_0 \equiv 0$, by Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\Omega \tilde{G}(Tw_j) + \tilde{H}(Tw_j)\, dx \to \int_\Omega \tilde{G}(Tw_0) + \tilde{H}(Tw_0)\, dx = 0 \quad \text{as } j \to \infty.$$

Therefore

$$2\tilde{I}_{\mu_j}(Tw_j) = T^2 \int_\Omega |\nabla w_j|^2\, dx - 2\mu_j \int_\Omega (\tilde{G}(Tw_j) + \tilde{H}(Tw_j))\, dx \to T^2 \quad \text{as } j \to \infty. \tag{3.17}$$

Denote $t = \frac{T}{\|v_j\|}$. We may assume that $0 < t < 1$ for j large due to the hypothesis $\|v_j\| \to \infty$. Since $T > 0$ is finite and arbitrary, we can choose a suitable T such that

$$2\tilde{I}_{\mu_j}(Tw_j) = 2\tilde{I}_{\mu_j}(\frac{Tv_j}{\|v_j\|}) = 2\tilde{I}_{\mu_j}(tv_j) \le 2c_{\frac{1}{2}} + C < \frac{T^2}{2} \quad as\ j \to \infty. \tag{3.18}$$

Obviously (3.17) and (3.18) contradict each other. Hence the hypothesis is not true and we show the sequence $\{v_j\}$ is uniformly bounded in $H_0^1(\Omega)$.

By an analogous argument as in Lemma 3.7, we conclude that $(PS)_{c_{\mu_j}}$ sequence $\{v_j\}_{j\in N}$ has a strong convergent subsequence in $H_0^1(\Omega)$ which is still denoted as $\{v_j\}$ and $v_j \to v_1$ as $j \to \infty$ for some $v_1 \in H_0^1(\Omega)$. In fact, we can show that v_1 is the second solution of (2.1).

Proof of Theorem 1.1 (ii) By Lemma 3.4, we get the first solution v of (2.1) so that $u = \hat{f}(v)$ is the first solution of (1.1). Now we claim $u_1 = \hat{f}(v_1)$ is the second solution of equation (1.1) for any $0 < P < P^*$ and $0 < \lambda < \lambda_P^*$. In fact, if $|v_1|_{L^\infty} \le m_0$, then v_1 is the solution of (2.1). We argue by contradiction. Suppose that there exists a $x_0 \in \Omega$ such that $v_1(x_0) > m_0$. Then it is easy to see that v_1 is a lower solution of (2.1) due to $-\Delta v_1 = \tilde{g}(v_1) + \tilde{h}(v_1) \le g(v_1) + h(v_1)$. By Lemma 3.4 and Proposition 2.3 we get that (2.1) has a solution v_2 such that $v_2 \ge v_1$ satisfying $v_2(x_0) > m_0$. But this is contradicted with the result $|v_2|_{L^\infty} \le m_0$ by Lemma 2.1. Therefore we obtain $|v_1|_{L^\infty} \le m_0$ and v_1 is a solution of (2.1) so that $u_1 = \hat{f}(v_1)$ is the second solution of (1.1).

Now we show the second assertion of Theorem 1.1 (ii). For this we define the set

$$W = \{v_{\lambda_P} \mid v_{\lambda_P} \text{ is a solution of (2.1) for any } P \in (0, P^*), \lambda \in (0, \lambda_P^*)\}.$$

Then it is clear that W is bounded in $H_0^1(\Omega)$ since

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla v_{\lambda_P}|^2\,dx &= \int_\Omega G(v_{\lambda_P})\,dx + \int_\Omega H(v_{\lambda_P})\,dx \\ &\le \int_\Omega C(1 + |v_{\lambda_P}|^{q+1} + |v_{\lambda_P}|^{\frac{q+1}{2}})\,dx \\ &\le C|\Omega| + C_1 m_0^{q+1} \le C_2, \end{aligned} \tag{3.19}$$

where $q > 1$. Here we use the fact that v_{λ_P} is a solution of (2.1) and Proposition 2.1 (vi). Then $v_{\lambda_P^*} := \lim_{\lambda\to\lambda_P^*} v_{\lambda_P}$ is a solution of (2.1) for $\lambda = \lambda_P^*$. Since $h(s) = Pe^{\frac{\lambda\delta}{(p-1)(1-f(s))^{(p-1)}}} > 0$, and $h'(s) = h(s)\frac{\lambda\delta}{(1-f(s))^p} f'(s)$, we get

$$\begin{aligned} h''(s) &= h'(s)\frac{\lambda\delta}{(1-f(s))^p} f'(s) + h(s)\frac{p\lambda\delta(f'(s))^2}{(1-f(s))^{p+1}} + h(s)\frac{\lambda\delta}{1-f(s))^p} f''(s) \\ &= h(s)\frac{p\lambda\delta(f'(s))^2}{(1-f(s))^{p+1}} \\ &> 0. \end{aligned}$$

This implies $h(s)$ is convex. By Proposition 2.3 (i), we know $g(s)$ is also convex. Therefore by the convexity of $g(s) + h(s)$, it is classical (see [23] or [11]) to guarantee that $v_{\lambda_P^*}$ is the unique solution of equation (2.1). So $u_{\lambda_P^*} = \hat{f}(v_{\lambda_P^*})$ is the unique solution of (1.1). From the definition of λ_P^*, it is clear that (1.1) has no solution for $\lambda > \lambda_P^*$.

4 The proof of Theorem 1.2

In this section, we focus on the equation (1.2), i.e.,

$$\begin{cases} -\Delta u = \frac{\lambda(1+\delta|\nabla u|^2)}{(1-v)^p}, & x \in \Omega, \\ -\Delta v = \frac{\mu(1+\delta|\nabla v|^2)}{(1-u)^p}, & x \in \Omega, \\ 0 \le u, v < 1, & x \in \Omega, \\ u, v = 0, & x \in \partial\Omega, \end{cases}$$

where λ, μ, δ are positive parameters and $p > 1$. It is interesting to find that the existence of solution of (1.2) depends on the parameter area (λ, μ) in the first quadrant. We will again apply the upper and lower solution method to get the proof of Theorem 1.2.

Definition 4.1 If a pair $(\overline{u}, \overline{v})$ satisfies

$$\begin{cases} -\Delta \overline{u} \ge \frac{\lambda(1+\delta|\nabla \overline{u}|^2)}{(1-\overline{v})^p}, & x \in \Omega, \\ -\Delta \overline{v} \ge \frac{\mu(1+\delta|\nabla \overline{v}|^2)}{(1-\overline{u})^p}, & x \in \Omega, \\ 0 \le \overline{u}, \overline{v} < 1, & x \in \Omega, \\ \overline{u}, \overline{v} = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

then we say the pair $(\overline{u}, \overline{v})$ is a upper solution of (1.2). If the first two inequalities in (4.1) are reversed for some $(\underline{u}, \underline{v})$, we call $(\underline{u}, \underline{v})$ a lower solution of (1.2).

The following result is important for the existence of solution of (1.2). For the proof we refer to Section 3.3 in [24].

Proposition 4.1 ([24]) Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \ge 2$. Consider the following equation

$$\begin{cases} -\Delta u = f(x, u, \nabla u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Assume $\overline{u}$ and $\underline{u}$ are the upper and lower solution which is defined in Definition 2.7. Denote $\underline{c} = \min \underline{u}$ and $\overline{c} = \max \overline{u}$ in $\overline{\Omega}$. If there exists a continuous function ψ such that $|f(x, u, \nabla u)| \le \psi(u)(1 + |\nabla u|^2)$ for any $x \in \overline{\Omega}$, and $u \in [\underline{c}, \overline{c}]$, then this equation has a classical solution u with $\underline{u} \le u \le \overline{u}$.

Lemma 4.1 If the system (1.2) has a upper solution $(\overline{u}, \overline{v})$, then (1.2) must have a classical solution (u, v) with $0 < u \le \overline{u} < 1, 0 < v \le \overline{v} < 1$.

Proof We prove the assertion with iterative method. Let $(u_1, v_1) = (\overline{u}, \overline{v})$. For $n \ge 2$, by Proposition 4.1, we construct two sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$\begin{cases} -\Delta u_n = \frac{\lambda(1+\delta|\nabla u_n|^2)}{(1-v_{n-1})^p}, & \text{in } \Omega, \\ -\Delta v_n = \frac{\mu(1+\delta|\nabla v_n|^2)}{(1-u_{n-1})^p}, & \text{in } \Omega, \\ 0 \le u_n, v_n < 1, & \text{in } \Omega, \\ u_n, v_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

For $n = 2$, since $\overline{u}$ ia a upper solution of the following equation

$$-\Delta u_2 = \frac{\lambda(1 + \delta|\nabla u_2|^2)}{(1 - v_1)^p} = \frac{\lambda(1 + \delta|\nabla u_2|^2)}{(1 - \overline{v})^p}, \tag{4.3}$$

and 0 is the lower solution of (4.3), by Proposition 4.1, we get a solution u_2 to (4.3) with $0 < u_2 \le u_1$. Analogously we obtain a solution v_2 to $-\Delta v_2 = \dfrac{\lambda(1+\delta|\nabla v_2|^2)}{(1-u_1)^p}$ such that $0 < v_2 \le v_1$. By induction, we suppose $0 < u_n \le u_{n-1}, 0 < v_n \le v_{n-1}$ and $u_{n-1} = u_n = v_{n-1} = v_n = 0$ on $\partial\Omega$. Now we claim that $0 < u_{n+1}(x) \le u_n(x)$ and $0 < v_{n+1}(x) \le v_n(x)$ for $x \in \Omega$. We argue by contradiction. Suppose that there exists a point $x' \in \Omega$ such that $u_n(x') - u_{n+1}(x') < 0$. By the maximum principle, $u_n - u_{n+1}$ have a minimum point $x_0 \in \Omega$ such that $\nabla(u_n - u_{n+1})(x_0) = 0$ and $\Delta(u_n - u_{n+1})(x_0) > 0$. However we have

$$
\begin{aligned}
-\Delta(u_n - u_{n+1})(x_0) &= \frac{\lambda(1+\delta|\nabla u_n(x_0)|^2)}{(1-v_{n-1}(x_0))^p} - \frac{\lambda(1+\delta|\nabla u_{n+1}(x_0)|^2)}{(1-v_n(x_0))^p} \\
&\ge \frac{\lambda(1+\delta|\nabla u_n(x_0)|^2)}{(1-v_n(x_0))^p} - \frac{\lambda(1+\delta|\nabla u_{n+1}(x_0)|^2)}{(1-v_n(x_0))^p} \\
&= \frac{\lambda\delta\nabla(u_n - u_{n+1})(x_0)\nabla(u_n + u_{n+1})(x_0)}{(1-v_n(x_0))^p} \\
&= 0.
\end{aligned}
\tag{4.4}
$$

This imlplies $\Delta(u_n - u_{n+1})(x_0) \le 0$ which contradicts with $\Delta(u_n - u_{n+1})(x_0) > 0$. Hence we conclude $0 < u_{n+1} \le u_n \le \overline{u}$. In the same way we can get $0 < v_{n+1} \le v_n \le \overline{v}$. This means the sequences $\{u_n\}$ and $\{v_n\}$ are decreasing and bounded in Ω. Therefore $\{u_n\}$ and $\{v_n\}$ have the limit functions u, v such that $\lim\limits_{n\to\infty} u_n = u$, $\lim\limits_{n\to\infty} v_n = v$. By a standard compactness argument we know (u, v) are the classic solution of (1.2) and in particular different from zero.

Define the set $D_1 = \{(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid (1.2)$ has a classical solution $(u, v)\}$. We need to illustrate this set is not empty. It suffices to show that (1.2) has a upper solution $(\overline{u}, \overline{v})$.

Lemma 4.2 The set D_1 is not empty when λ and μ are properly small.

Proof Let B_R be a ball of radius R centered at 0 such that $\Omega \subset B_R$. Denote $\beta_1 > 0$ the first eigenvalue of Laplace operator in B_R with Dirichlet boundary condition and ϕ the corresponding eigenfunction which satisfies $0 < \phi \le 1$ in B_R. We can choose a constant $\theta \in (0, 1)$ such that $0 < \psi = \theta\phi < 1$ in B_R. Now we wish (ψ, ψ) is a upper solution of (1.2), which means

$$
\begin{cases}
-\Delta\psi = \beta_1\theta\phi \ge \frac{\lambda(1+\delta|\nabla\psi|^2)}{(1-\theta\phi)^p} \ge \frac{\lambda}{(1-\theta\phi)^p}, & \text{in } \Omega, \\
-\Delta\psi = \beta_1\theta\phi \ge \frac{\mu(1+\delta|\nabla\psi|^2)}{(1-\theta\phi)^p} \ge \frac{\mu}{(1-\theta\phi)^p}, & \text{in } \Omega.
\end{cases}
\tag{4.5}
$$

This happens if $\lambda \le \beta_1\theta\phi(1-\theta\phi)^p$ and $\mu \le \beta_1\theta\phi(1-\theta\phi)^p$. Since $0 < \theta\phi < 1$ in Ω, we can choose $\lambda, \mu > 0$ small enough such that (4.5) holds and (ψ, ψ) is a upper solution of (1.2). Then due to Lemma 4.1, the system (1.2) must have a solution.

Lemma 4.3 The set D_1 is contained in a bounded region.

Proof If the system (1.2) has a solution (u, v), then

$$
\begin{cases}
\beta_1|\Omega| \ge \int_\Omega -\Delta u\phi\, dx = \int_\Omega \frac{\lambda(1+\delta|\nabla u|^2)}{(1-v)^p}\phi\, dx \ge \lambda|\Omega|, & \text{in } \Omega, \\
\beta_1|\Omega| \ge \int_\Omega -\Delta v\phi\, dx = \int_\Omega \frac{\mu(1+\delta|\nabla v|^2)}{(1-u)^p}\phi\, dx \ge \mu|\Omega|, & \text{in } \Omega.
\end{cases}
$$

This implies $\lambda \le \beta_1$ and $\mu \le \beta_1$. Hence $D_1 \subset (0, \beta_1] \times (0, \beta_1]$ is bounded.

Lemma 4.4 If the system (1.2) has a solution with the parameter pair $(\lambda, \mu) \in D_1$, then (λ', μ') is still in D_1 for any $\lambda' \le \lambda$, $\mu' \le \mu$.

Proof It is easy to verify that the solution (u, v) with the parameter pair (λ, μ) is a upper solution of the system (1.2) with the pair (λ', μ'). Then by Lemma 4.1 we know the system (1.2) must have at least one solution.

Based on the above-mentioned argument, we can find a curve Γ in the first quadrant of (λ, μ)-plane such that the existence of (1.2) depends on the region divided by Γ. More precisely, for any $\sigma > 0$, we define

$$\lambda^*(\sigma) = \sup\{\lambda > 0 \mid (\lambda, \sigma\lambda) \in D_1\}.$$

It is obvious that $\{(\lambda, \sigma\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 < \lambda \le \lambda^*(\sigma)\} \subset D_1$ and $\{(\lambda, \sigma\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \lambda > \lambda^*(\sigma)\} \cap D_1 = \varnothing$. We also can define $\mu^*(\sigma) = \sigma\lambda^*(\sigma)$.

Lemma 4.5 The curve $\Gamma(\sigma) = (\lambda^*(\sigma), \mu^*(\sigma))$ is continuous.

Proof We prove this by contradiction. Suppose that $\Gamma(\sigma)$ is not continuous at some $\sigma_0 > 0$. Then there exists $\varepsilon_0 > 0$ such that for any $\eta > 0$, when $0 < |\sigma - \sigma_0| < \eta$ we have $|\Gamma(\sigma) - \Gamma(\sigma_0)| > \varepsilon_0$. This implies either the case $\lambda^*(\sigma) > \lambda^*(\sigma_0)$, $\mu^*(\sigma) > \mu^*(\sigma_0)$ or the case $\lambda^*(\sigma) < \lambda^*(\sigma_0)$, $\mu^*(\sigma) < \mu^*(\sigma_0)$ appears. Without loss of generality, we just discuss the first case. Let $\lambda_1, \lambda_2 > 0$ such that $\lambda^*(\sigma) > \lambda_2 > \lambda_1 > \lambda^*(\sigma_0)$, $\mu^*(\sigma) > \sigma\lambda_2 > \sigma_0\lambda_1 > \mu^*(\sigma_0)$. By the definition of $\lambda^*(\sigma)$, we obtain

$$\begin{cases} -\Delta u = \frac{\lambda_2(1+\delta|\nabla u|^2)}{(1-v)^p}, & \text{in } \Omega, \\ -\Delta v = \frac{\sigma\lambda_2(1+\delta|\nabla v|^2)}{(1-u)^p}, & \text{in } \Omega, \\ 0 \le u, v < 1, & \text{in } \Omega, \\ u, v = 0, & \text{on } \partial\Omega, \end{cases}$$

then we have a solution $(u_{\lambda_2}, v_{\lambda_2})$. Obviously it is a upper solution of the system (1.2) when parameter pair equals to $(\lambda_1, \sigma_0\lambda_1)$. This implies $\lambda_1 \le \lambda^*(\sigma_0)$ which contradicts with the assumption $\lambda_1 > \lambda^*(\sigma_0)$.

Lemma 4.6 $\lambda^*(\sigma)$ is decreasing and $\mu^*(\sigma)$ is increasing with respect to σ.

Proof (i) We first show that $\lambda^*(\sigma)$ is decreasing. We argue by contradiction. Suppose that $\lambda^*(\sigma_1) < \lambda^*(\sigma_2)$ for $\sigma_1 < \sigma_2$. Then $\mu^*(\sigma_1) = \sigma_1\lambda^*(\sigma_1) < \sigma_2\lambda^*(\sigma_2) = \mu^*(\sigma_2)$. We can choose two constants λ_1, λ_2 such that $\lambda^*(\sigma_1) < \lambda_1 < \lambda_2 < \lambda^*(\sigma_2)$ and $\sigma_1\lambda^*(\sigma_1) < \sigma_1\lambda_1 < \sigma_2\lambda_2 < \sigma_2\lambda^*(\sigma_2)$. Similar to the proof process as in Lemma 4.5, we can obtain $\lambda^*(\sigma_1) \ge \lambda_1$ which is contradicted with $\lambda^*(\sigma_1) < \lambda_1$. Hence the hypothesis is not valid and $\lambda^*(\sigma)$ is decreasing.

(ii) We next show that $\mu^*(\sigma)$ is increasing. We argue again by contradiction. Suppose that $\mu^*(\sigma_1) = \sigma_1\lambda^*(\sigma_1) > \mu^*(\sigma_2) = \sigma_2\lambda^*(\sigma_2)$ for $\sigma_1 < \sigma_2$. By (i), we know $\lambda^*(\sigma_1) > \lambda^*(\sigma_2)$. Therefore we can choose two proper constants $\lambda_1, \lambda_2 > 0$ such that $\lambda^*(\sigma_1) > \lambda_1 > \lambda_2 > \lambda^*(\sigma_2)$ and $\sigma_1\lambda^*(\sigma_1) > \sigma_2\lambda_1 > \sigma_2\lambda_2 > \sigma_2\lambda^*(\sigma_2)$. By Lemma 4.4, we conclude the system

(1.2) has one solution with parameter pair $(\lambda_2, \sigma_2\lambda_2)$. This implies $\lambda_2 \leq \lambda^*(\sigma_2)$ which contradicts with $\lambda_2 > \lambda^*(\sigma_2)$. We finish the proof.

Now we can give the proof of Theorem 1.2.

Proof of Theorem 1.2 By Lemma 4.2, 4.3, 4.5 and 4.6, if we take $\mu^*(\sigma)$ as horizontal axis and $\lambda^*(\sigma)$ as vertical axis, then the curve $\Gamma(\sigma) = (\lambda^*(\sigma), \mu^*(\sigma))$ splits the first quadrant of $(\mu^*(\sigma), \lambda^*(\sigma))$-plane into two connected parts. When the parameter pair is above the curve Γ, there is no solution of (1.2). While the parameter pair is below the curve, there exists at least one solution of (1.2).

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一类带扰动项的MEMS模型的存在性定理

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摘要: 本文研究了一类带扰动项的MEMS模型. 利用上下解和变分的方法, 获得了该模型的存在性和多解的结果. 特别地, 对一个MEMS系统, 根据解的存在性, 我们找到了一条曲线将第一象限的参数区域划分成两部分. 推广了已有文献在MEMS模型方面的研究

关键词: MEMS模型; 扰动项; 变分法; 上下解

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