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PERIODIC AND CONTINUOUS SOLUTIONS FOR POLYNOMIAL-LIKE ITERATIVE FUNCTIONAL EQUATION WITH VARIABLE COEFFICIENTS

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Abstract: Schauder's fixed point theorem and the Banach contraction principle are used to study the polynomial-like iterative functional equation with variable coefficients $\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \ldots + \lambda_n(x)f^n(x) = F(x)$. We give sufficient conditions for the existence, uniqueness, and stability of the periodic and continuous solutions. Finally, some examples were considered by our results. The results enrich and extend the theory about polynomial-like iterative functional equation.

Keywords: Iterative functional equation; periodic solutions; fixed point theorem

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1 Introduction

Let S be a non-empty set and $f : S \to S$ be a self-map. For each positive integer n, f^n, the n-fold composition of f with itself, also known as the n-th iterate of f, is defined recursively by $f^1 = f$, and $f^n = f \circ f^{n-1}$.

Let S be a non-empty subset of the real line $\mathbb{R}$ and $F : S \to \mathbb{R}$ be given. The polynomial-like iterative equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \ldots + \lambda_n f^n(x) = F(x), \quad \forall x \in S, \tag{1.1}$$

where $\lambda_i, i = 1, 2, \ldots, n$ are constants, has been discussed in many papers under various settings. Si [1] obtained results on C^2 solutions with $S = [a, b]$, a finite closed interval, and F, a $\mathcal{C}^2$ self-map on [a,b], $F(a) = a$, $F(b) = b$. His work is based on Zhang's paper [2] in which the results cover the existence, the uniqueness, and the stability of the differentiable solutions. In [3] invertible solutions are obtained in some local neighbourhoods of the fixed points of the functions. Monotonic solutions and convex solutions are discussed by Nikodem, Xu, and

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Zhang in [4, 5]. Recently, Ng and Zhao [6] studied the periodic and continuous solutions of Eq. (1.1). For some properties of solutions for polynomial-like iterative functional equations, we refer the interested readers to [7]–[10].

In 2000, Zhang and Baker [11] studied the continuous solutions of

$$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \ldots + \lambda_n(x)f^n(x) = F(x). \tag{1.2}$$

Later, Xu [12] considered the analytic solutions of Eq. (1.2). In this paper, we continue to consider the continuous and periodic solutions of Eq. (1.2). In fact, if $\lambda_i(x), i = 1, 2, \ldots, n$ are constants, then the conclusions in [6] can be derived by our results.

Notations, preliminaries, and the current setting.

$C(\mathbb{R}, \mathbb{R})$ - the linear space of all continuous self-maps on $\mathbb{R}$.

$\mathcal{P}_T = \{f \in C(\mathbb{R}, \mathbb{R}) : f(x + T) = f(x), \ \forall x \in \mathbb{R}\}\,(T > 0)$

- a Banach space with the norm $\|f\| = \max\limits_{x\in[0,T]} |f(x)| = \max\limits_{x\in\mathbb{R}} |f(x)|$.

$\mathcal{P}_T(L, M) = \{f \in \mathcal{P}_T : \|f\| \le L, \ |f(y) - f(x)| \le M|y - x|, \ \forall x, y \in \mathbb{R}\}$.

- a closed convex nonempty subset of $\mathcal{P}_T$ (constants $L \ge 0, M \ge 0$).

Assume that $\lambda_i \in \mathcal{P}_T(L_i, M_i), |\lambda_1(x)| \ge k_1 > 0, \forall x \in [0, T]$ and $F \in \mathcal{P}_T(L', M')$, and we seek solutions $f \in \mathcal{P}_T(L, M)$.

2 Sufficient Conditions for the Existence of Solutions

We shall find sufficient conditions on the constants L', M', L and M under which the existence of a solution is assured. The Schauder fixed point theorem being a main tool, its statement is included, and will be applied with $\Omega := \mathcal{P}_T(L, M)$ in the Banach space $\mathcal{P}_T$.

Theorem 2.1 (Schauder([13])) Let Ω be a closed convex and nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A maps Ω into Ω and is compact and continuous. Then there exists $z \in \Omega$ with $z = Az$.

Let $A : \mathcal{P}_T(L, M) \to \mathcal{P}_T$ be defined by

$$Af(x) = \frac{1}{\lambda_1(x)}F(x) - \frac{1}{\lambda_1(x)}\sum_{i=2}^{n}\lambda_i(x)f^i(x), \ \forall f \in \mathcal{P}_T(L, M). \tag{2.1}$$

The space $\mathcal{P}_T$ is closed under composition. The range of A is clearly contained in $\mathcal{P}_T$. Fixed points of A correspond to the solutions of (1.2). Hence we seek conditions under which the assumptions in Schauder's theorem are met.

Lemma 2.2 (see [2]) For any $f, g \in \mathcal{P}_T(L, M), x, y \in \mathbb{R}$, the following inequalities hold for every positive integer n.

$$|f^n(y) - f^n(x)| \le M^n|y - x|, \tag{2.2}$$

$$||f^n - g^n|| \le \sum_{j=0}^{n-1} M^j\|f - g\|. \tag{2.3}$$

Lemma 2.3 Suppose $\lambda_i \in \mathcal{P}_T(L_i, M_i), |\lambda_1(x)| \geq k_1 > 0, \forall x \in [0, T]$, then operator A is continuous and compact on $\mathcal{P}_T(L, M)$.

Proof For any $f, g \in \mathcal{P}_T(L, M), x \in \mathbb{R}$, by (2.3) we get

$$\begin{aligned} |(Af)(x) - (Ag)(x)| &\leq \frac{1}{|\lambda_1(x)|} \sum_{i=2}^{n} |\lambda_i(x)| |f^i(x) - g^i(x)| \\ &\leq \frac{1}{k_1} \sum_{i=2}^{n} \sum_{j=0}^{i-1} L_i M^j \|f - g\|. \end{aligned}$$

Thus

$$\|Af - Ag\| \leq \frac{1}{k_1} \sum_{i=2}^{n} \sum_{j=0}^{i-1} L_i M^j \|f - g\|. \tag{2.4}$$

This proves that A is continuous. $\mathcal{P}_T(L, M)$ is closed, uniformly bounded and equicontinuous on $\mathbb{R}$. By the Arzelà-Ascoli theorem ([14], page 28), applied to a sufficiently large closed interval of $\mathbb{R}$ and taking the periodicity of the functions into account, it is compact. Since continuous functions map compact sets to compact sets, A is a compact map.

Theorem 2.4 Suppose that $F \in \mathcal{P}_T(L', M'), \lambda_i \in \mathcal{P}_T(L_i, M_i), i = 1, 2, \ldots, n$ and $|\lambda_1(x)| \geq k_1 > 0$, $\forall x \in [0, T]$. If the constants L', M', L, M satisfy the conditions

$$L' \leq \Big(k_1 - \sum_{i=2}^{n} L_i\Big) L, \ \ M' \leq k_1 M - \frac{L'}{k_1} M_1 - \sum_{i=2}^{n} \Big(L_i M^i + L M_i + \frac{L L_i}{k_1} M_1\Big), \tag{2.5}$$

then (1.2) has a solution in $\mathcal{P}_T(L, M)$.

Proof By (2.1), for all $x \in \mathbb{R}$, we have

$$\begin{aligned} |(Af)(x)| &\leq \frac{1}{|\lambda_1(x)|} |F(x)| + \frac{1}{|\lambda_1(x)|} \sum_{i=2}^{n} |\lambda_i(x)| |f^i(x)| \\ &\leq \frac{1}{|\lambda_1(x)|} L' + \frac{1}{|\lambda_1(x)|} \sum_{i=2}^{n} |\lambda_i(x)| L \\ &\leq \frac{1}{k_1} L' + \frac{L}{k_1} \sum_{i=2}^{n} L_i. \end{aligned}$$

Thus the first condition of (2.5) assures

$$\|Af\| \leq L.$$

For all $x, y \in \mathbb{R}$, by (2.1), (2.2) , we have

$$\begin{aligned}
& |(Af)(y) - (Af)(x)| \\
= \ & \Big|\frac{1}{\lambda_1(y)}F(y) - \frac{1}{\lambda_1(y)}\sum_{i=2}^{n}\lambda_i(y)f^i(y) - \frac{1}{\lambda_1(x)}F(x) + \frac{1}{\lambda_1(x)}\sum_{i=2}^{n}\lambda_i(x)f^i(x)\Big| \\
\le \ & \Big|\frac{1}{\lambda_1(y)}F(y) - \frac{1}{\lambda_1(x)}F(x)\Big| + \sum_{i=2}^{n}\Big|\frac{\lambda_i(y)}{\lambda_1(y)}f^i(y) - \frac{\lambda_i(x)}{\lambda_1(x)}f^i(x)\Big| \\
\le \ & \frac{1}{|\lambda_1(y)|}|F(y) - F(x)| + \Big|\frac{1}{\lambda_1(y)} - \frac{1}{\lambda_1(x)}\Big||F(x)| \\
& + \sum_{i=2}^{n}\frac{|\lambda_i(y)|}{|\lambda_1(y)|}|f^i(y) - f^i(x)| + \sum_{i=2}^{n}\Big|\frac{\lambda_i(y)}{\lambda_1(y)} - \frac{\lambda_i(x)}{\lambda_1(x)}\Big||f^i(x)| \\
\le \ & \frac{1}{k_1}M'|y-x| + \Big|\frac{\lambda_1(x) - \lambda_1(y)}{\lambda_1(y)\lambda_1(x)}\Big||F(x)| \\
& + \sum_{i=2}^{n}\frac{L_i}{k_1}M^i|y-x| + \sum_{i=2}^{n}\Big|\frac{\lambda_i(y)\lambda_1(x) - \lambda_i(x)\lambda_1(y)}{\lambda_1(y)\lambda_1(x)}\Big||f^i(x)| \\
\le \ & \frac{1}{k_1}M'|y-x| + \frac{M_1L'}{k_1^2}|y-x| + \frac{1}{k_1}\sum_{i=2}^{n}L_iM^i|y-x| \\
& + \sum_{i=2}^{n}\frac{|\lambda_1(x)||\lambda_i(y) - \lambda_i(x)| + |\lambda_i(x)||\lambda_1(x) - \lambda_1(y)|}{|\lambda_1(y)\lambda_1(x)|}L \\
\le \ & \frac{1}{k_1}M'|y-x| + \frac{M_1L'}{k_1^2}|y-x| + \frac{1}{k_1}\sum_{i=2}^{n}L_iM^i|y-x| \\
& + \frac{L}{k_1}\sum_{i=2}^{n}M_i|y-x| + \frac{LM_1}{k_1^2}\sum_{i=2}^{n}L_i|y-x| \\
= \ & \Big(\frac{1}{k_1}M' + \frac{L'M_1}{k_1^2} + \frac{1}{k_1}\sum_{i=2}^{n}L_iM^i + \frac{L}{k_1}\sum_{i=2}^{n}M_i + \frac{LM_1}{k_1^2}\sum_{i=2}^{n}L_i\Big)|y-x|
\end{aligned}$$

The second condition of (2.5) assures

$$|(Af)(y) - (Af)(x)| \le M|y-x|.$$

Therefore $Af \in \mathcal{P}_T(L, M)$. This proves that A maps $\mathcal{P}_T(L, M)$ into itself. All conditions of Schauder's fixed point theorem are satisfied. Thus there exists an f in $\mathcal{P}_T(L, M)$ such that $f = Af$. This is equivalent to that f is a solution of (1.2) in $\mathcal{P}_T(L, M)$.

3 Uniqueness and Stability

In this section, uniqueness and stability of (1.2) will be proved.

Theorem 3.1 (i) Suppose that A is defined by (2.1) and

$$\alpha := \frac{1}{k_1}\sum_{i=2}^{n}\sum_{j=0}^{i-1}L_iM^j < 1. \tag{3.1}$$

Then A is contractive with contraction constant $\alpha < 1$. It has at most one fixed point and (1.2) has at most one solution in $\mathcal{P}_T(L, M)$.

(ii) Suppose that (2.5) and (3.1) are satisfied. Then (1.2) has a unique solution in $\mathcal{P}_T(L, M)$.

Proof (i) According to (2.4), $\|Af - Ag\| \le \alpha\|f - g\|$ and so α is a contraction constant for A. Let $f, g \in \mathcal{P}_T(L, M)$ be fixed points of A. Then $\|f - g\| = \|Af - Ag\| \le \alpha\|f - g\|$. Hence $\alpha < 1$ yields $\|f - g\| = 0$, resulting in $f = g$. This proves that there is at most one fixed point. (ii) The existence is cared for by Theorem 2.4.

Theorem 3.2 Let $\mathcal{P}_T(L, M)$ and $\mathcal{P}_T(L', M')$ be fixed and allow $F(x)$ and $\lambda_i(x)$ in (1.2) to vary. The unique solution obtained in Theorem 3.1, part (ii), depends continuously on $F(x)$ and $\lambda_i(x)$ $(i = 1, 2, \ldots, n)$.

Proof Under the assumptions of Theorem 3.1, part (ii), we consider any two functions F, G in $\mathcal{P}_T(L', M')$ and a parallel pair of unique f, g in $\mathcal{P}_T(L, M)$ satisfying

$$\begin{aligned}
&\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \ldots + \lambda_n(x)f^n(x) = F(x),\\
&\mu_1(x)g(x) + \mu_2(x)g^2(x) + \ldots + \mu_n(x)g^n(x) = G(x).
\end{aligned}$$

Then

$$\begin{aligned}
&\|f - g\|\\
= {}& \max_{x\in[0,T]} \Big|\frac{1}{\lambda_1(x)}F(x) - \frac{1}{\lambda_1(x)}\sum_{i=2}^{n}\lambda_i(x)f^i(x) - \frac{1}{\mu_1(x)}G(x) + \frac{1}{\mu_1(x)}\sum_{i=2}^{n}\mu_i(x)g^i(x)\Big|\\
\le {}& \max_{x\in[0,T]} \Big|\frac{1}{\lambda_1(x)}F(x) - \frac{1}{\mu_1(x)}G(x)\Big| + \sum_{i=2}^{n}\max_{x\in[0,T]}\Big|\frac{\lambda_i(x)}{\lambda_1(x)}f^i(x) - \frac{\mu_i(x)}{\mu_1(x)}g^i(x)\Big|\\
\le {}& \max_{x\in[0,T]} \frac{1}{|\lambda_1(x)|}|F(x) - G(x)| + \max_{x\in[0,T]}\Big|\frac{1}{\lambda_1(x)} - \frac{1}{\mu_1(x)}\Big||G(x)|\\
&+ \sum_{i=2}^{n}\max_{x\in[0,T]}\Big|\frac{\lambda_i(x)}{\lambda_1(x)}\Big||f^i(x) - g^i(x)| + \sum_{i=2}^{n}\max_{x\in[0,T]}\Big|\frac{\lambda_i(x)}{\lambda_1(x)} - \frac{\mu_i(x)}{\lambda_1(x)}\Big||g^i(x)|\\
&+ \sum_{i=2}^{n}\max_{x\in[0,T]}\Big|\frac{\mu_i(x)}{\lambda_1(x)} - \frac{\mu_i(x)}{\mu_1(x)}\Big||g^i(x)|\\
\le {}& \frac{1}{k_1}\|F - G\| + \frac{L'}{k_1^2}\|\lambda_1 - \mu_1\| + \frac{1}{k_1}\sum_{i=2}^{n}\sum_{j=0}^{i-1}L_iM^j\|f - g\|\\
&+ \frac{L}{k_1}\sum_{i=2}^{n}\|\lambda_i - \mu_i\| + \frac{L}{k_1^2}\sum_{i=2}^{n}L_i\|\lambda_1 - \mu_1\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big(1 - \frac{1}{k_1}\sum_{i=2}^{n}\sum_{j=0}^{i-1}L_iM^j\Big)\|f - g\|\\
\le {}& \frac{1}{k_1}\|F - G\| + \frac{1}{k_1^2}\Big(L' + L\sum_{i=2}^{n}L_i\Big)\|\lambda_1 - \mu_1\| + \frac{L}{k_1}\sum_{i=2}^{n}\|\lambda_i - \mu_i\|, \qquad (3.2)
\end{aligned}$$

where, by (3.1), $1-\frac{1}{k_1}\sum\limits_{i=2}^{n}\sum\limits_{j=0}^{i-1}L_iM^j>0$. Hence $\|f-g\|$ tends to 0 when $G(x)$ tends to $F(x)$ in norm and $\mu_i(x)$ tends to $\lambda_i(x)$.

4 Examples

In this section, some examples are provided to illustrate that the assumptions of Theorem 2.4 and 3.1 do not self-contradict.

Example 4.1 Consider the equation

$$(4+\cos^2(x))f(x)+f(f(x))=\sin(x),\quad \forall x\in\mathbb{R}. \tag{4.1}$$

It corresponds to the case of $\lambda_1(x)=4+\cos^2(x),\lambda_2(x)=1,F(x)=\sin(x)$. Taking $k_1=4,\ L_1=5,\ L=M=L'=M'=M_1=L_2=M_2=1$. A simple calculation yields

$$L'=1\le 3=(k_1-L_2)L,$$

and

$$M'=1\le\frac{3}{2}=k_1M-\frac{L'}{k_1}M_1-\Big(L_2M^2+LM_2+\frac{LL_2}{k_1}M_1\Big)$$

then (2.5) is satisfied. Theorem 2.4 gives a continuous periodic solution for Eq. (4.1) in $\mathcal{P}_{2\pi}(1,1)$. Noting

$$\alpha=\frac{1}{k_1}L_2(1+M)=\frac{1}{2}<1,$$

(3.1) is satisfied, hence by Theorem 3.1, we know the continuous periodic solution is the unique one in $\mathcal{P}_{2\pi}(1,1)$.

Example 4.2 Consider the equation

$$(4+\cos^2(x))f(x)+\delta f(f(x))=\sin(x),\quad \forall x\in\mathbb{R}. \tag{4.2}$$

As Example 4.1, $\lambda_1(x)=4+\cos^2(x),\lambda_2(x)=\delta,F(x)=\sin(x)$, where δ is a parameter. So taking $k_1=4,\ L_1=5,\ L_2=M_2=\delta,\ L=M=L'=M'=M_1=1$. In order to apply (2.5) in Theorem 2.4, we need

$$L'=1\le 4-\delta=(k_1-L_2)L, \tag{4.3}$$

and

$$M'=1\le\frac{15}{4}-\frac{9}{4}\delta=k_1M-\frac{L'}{k_1}M_1-\Big(L_2M^2+LM_2+\frac{LL_2}{k_1}M_1\Big). \tag{4.4}$$

Then we see $\delta\le\frac{11}{9}$ by (4.3) and (4.4).

From (3.1), we have

$$\alpha=\frac{1}{k_1}L_2(1+M)=\frac{\delta}{2}<1. \tag{4.5}$$

From (4.3)–(4.5), by Theorem 3.1, we know (4.2) has an unique continuous periodic solution in $\mathcal{P}_{2\pi}(1,1)$ with $\delta\le\frac{11}{9}$. This improves estimates of Example 4.1.

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变系数多项式型迭代函数方程的连续周期解

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摘要: 本文利用Schauder's 不动点定理和Banach 压缩映像原理讨论了一类变系数多项式型迭代函数方程$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \ldots + \lambda_n(x)f^n(x) = F(x)$, 得到了此类方程的连续周期解的存在性、唯一性和稳定性的结论，并通过几个例子验证了所得定理的正确性. 所得结果丰富和推广了多项式型迭代函数方程的相关理论.

关键词: 迭代函数方程; 周期解; 不动点定理

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