Vol. 41 (2021)
No. 4

数 学 杂 志
J. of Math. (PRC)

ROGUE WAVE SOLUTIONS AND BREATHERS SOLUTION FOR THE $(4+1)$-DIMENSIONAL FOKAS EQUATION

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Abstract: In this paper, we study the high order rogue wave solutions, breathers and higher order breathers of the (4 + 1) - dimensional Fokas equation. By using the Hirota bilinear form and the simplified Hirota bilinear form, the diversity of solutions of the (4 + 1) - dimensional Fokas equation is enriched. Finally, the dynamic characteristics of some classical solutions are analyzed.

Keywords: $(4+1)$ - dimensional Fokas equation; high order rogue wave solution; breathers solution; high order breathers solution

2010 MR Subject Classification: 35C99; 35G20

Document code: A **Article ID:** 0255-7797(2021)04-0298-13

1 Introduction

In the field of the soliton theory, it is vital to find the exact solution of nonlinear evolution equations, including soliton solutions, breathers solution, rogue wave solutions, lump solutions, period line wave solutions, interaction solutions, etc. Among these exact solutions, rogue wave solutions are a class of analytic rational solutions which can reach very high amplitudes in a short time. Because of its great destructive power and unpredictability, the rouge wave has been found in the ocean for a long time. As another non-linear scientific revolution after soliton, the rouge wave has attracted wide attention of researchers, and has gradually risen in the category of social and scientific contexts, such as Bose-Einstein condensates[1–3], oceanic[4, 5], even in finance[6] and hydrodynamics[7]. In 1983, some analytical solutions of nonlinear system (NLS) equations were obtained by D. H. Peregrine[8, 9], then the rouge wave solutions were researched for NLS equation[10, 11]. In recent years, the rouge wave solution of the nonlinear Schrödinger equation [12] and the coupled Hirota systems[13] have been studied. And the abound rogue wave type solutions to the extended (3+1)-dimensional Jimbo − Miwa equation are obtained by Liu, Yang, et al. in[14]. Furthermore, breathers [15] are considered as a special type of soliton, which can periodically occur and propagate in a localized and oscillatory way. Breathers can be divided into three types:

* **Received date:** 2020-09-27 **Accepted date:** 2020-12-22

Foundation item: Supported by The Shandong Natural Science Foundation of China (ZR2017JL008).

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generalized breathers, Akhmediev breathers[16] and Kuznetsov - Ma breathers[17, 18]. In 2019, Liu obtained breathers solution and higher order breathers solution of some equations in [19]. Recently, Mandel and Scheider[20] obtained the breathers solution of nonlinear equation by variational method. And Guo and Scheider studied the higher order breathers solutions of Jimbo-Miwa equation and the breathers solutions to the cubic Klein Gordon equation in[21] and[22], respectively.

High rogue waves play an critical role in the study of phase shift, propagation direction, shape and energy distribution, which makes the significance of these solutions a very practical research topic. In addition, the dynamics feature has become a central subject because it can reveal the collision, rebound and absorption characteristics of particles in the process of interaction, as well as the underlying laws of physics.

In this paper, we would like to consider the (4+1)-dimensional nonlinear Fokas equation as follows:

$$u_{xt} - \frac{1}{4}u_{xxxy} + \frac{1}{4}u_{xyyy} + \frac{3}{2}(u^2)_{xy} - \frac{3}{2}u_{zw} = 0. \tag{1.1}$$

The nonelasticity of the equation (1.1) and the interaction between elasticity were described[23, 24]. The meaning of the Fokas equation in water waves originates in the physical application of Kadomtsev-Petviashvili (KP) equation and Davey-Stewartson (DS) equation. The latter two equations are used to describe the surface wave and internal wave in channel or straits with different depths and widths, respectively[25–28].

The (4+1)-dimensional Fokas Eq.(1.1) has been studied by different scholars. He et al.[29, 30] obtained a few new exact solutions of Eq.(1.1) by applying extended F-expansion method. Zhang et al.[31] constructed multiple-soliton of Eq.(1.1). And Two different methods are used to look for exact traveling wave solutions of a modified simple equation about Eq.(1.1) in Ref [32]. More recently, Cheng et al.[33] have derived lump-type solutions of a reduced Fokas equation utilizing the positive quadratic function method. Wazwaz[34] made use of the simplified Hirota's method[35], whereafter demonstrated a variety of multiple soliton solutions. And the M-lump solutions of the Fokas equation are ascertained by taking limit method on multi-soliton solutions by Zhang and Xia [36]. As far as we know, the high order rogue wave solution, breathers solution and high order breathers solution of the Fokas equation by the simplified Hirota's method have not been given.

The paper is arranged as follows. Section 2, the $(4+1)$-dimensional Fokas equation is transformed into a $(1+1)$ dimensional equation by variable transformation. Then based on the Hirota bilinear form of the $(1+1)$-dimensional equation, high order rogue wave solutions are constructed. Further, their typical dynamics behaviors are analyzed and explained. In Section 3, by means of the simplified Hirota's method for the $(4+1)$-dimensional equation, breathers solution and high order breathers solution are obtained, then the corresponding dynamics behaviors are illustrated. Some conclusions will be given in the final section.

2 Rogue Wave Solution

If one sets $X = \alpha x + \beta y + \theta t, Z = \gamma z + \varepsilon w$ in Eq.(1.1), Eq.(1.1) can be transformed into the following $(1+1)$-dimensional equation

$$\theta u_{XX} + \frac{1}{4}\beta(\beta^2-\alpha^2)u_{XXXX} + 3\beta(uu_X)_X - \frac{3\gamma\varepsilon}{2\alpha}u_{ZZ} = 0, \tag{2.1}$$

where $\alpha, \beta, \theta, \gamma$ and ε are five real parameters.

Under the logarithmic transformation

$$u = (\beta^2-\alpha^2)(\ln f)_{XX} + u_0, \tag{2.2}$$

where $|\alpha| \neq |\beta|$ is a constant, the bilinear form to Eq.(2.1) is generated as

$$((\theta+3\beta u_0)D_X^2 + \frac{1}{4}\beta(\beta^2-\alpha^2)D_X^4 - \frac{3\gamma\varepsilon}{2\alpha}D_Z^2)f\cdot f = 0, \tag{2.3}$$

where the bilinear operator[37] D is defined by

$$D_X^m D_Z^n f(X,Z)g(X',Z') = (\tfrac{\partial}{\partial X} - \tfrac{\partial}{\partial X'})^m(\tfrac{\partial}{\partial Z} - \tfrac{\partial}{\partial Z'})^n f(X,Z)g(X',Z')\,|_{X'=X,Z'=Z}\ , \tag{2.4}$$

m and n are nonnegative integers. Eq.(2.1) is reduced to the bilinear equation

$$2(\theta+3\beta u_0)(f_{XX}f-f_X^2) - \tfrac{3\gamma\varepsilon}{\alpha}(f_{ZZ}f-f_Z^2) + \tfrac{1}{2}\beta(\beta^2-\alpha^2)(f_{XXXX}f-4f_{XXX}f_X+f_{XX}^2) = 0. \tag{2.5}$$

An improved ansatz was proposed about multiple rogue solutions in 2018, as shown below[13]

$$f = F_{n+1}(X,Z) + 2\alpha_1 ZP_n(X,Z) + 2\beta_1 XQ_n(X,Z) + (\alpha_1^2+\beta_1^2)F_{n-1}(X,Z), \tag{2.6}$$

with

$$\begin{aligned} F_n(X,Z) &= \sum_{k=0}^{n(n+1)/2}\sum_{i=0}^{k} a_{n(n+1)-2k,2i}X^{n(n+1)-2k}Z^{2i},\\ P_n(X,Z) &= \sum_{k=0}^{n(n+1)/2}\sum_{i=0}^{k} b_{n(n+1)-2k,2i}X^{n(n+1)-2k}Z^{2i},\\ Q_n(X,Z) &= \sum_{k=0}^{n(n+1)/2}\sum_{i=0}^{k} c_{n(n+1)-2k,2i}X^{n(n+1)-2k}Z^{2i}, \end{aligned} \tag{2.7}$$

$F_0 = 1, F_{-1} = P_0 = Q_0 = 0$, where $a_{m,l}, b_{m,l}, c_{m,l}(m,l \in \{0,2,4,...,n(n+1)/2\})$ and $\alpha_1, \beta_1 \in \mathbb{R}$. The coefficients $a_{m,l}, b_{m,l}, c_{m,l}$ can be determined and the wave center can be controlled by arbitrary constants α_1, β_1. Then by substituting these values into (2.2), some rational solutions of the Eq. (1.1) are obtained. Rogue wave solutions come from these solutions. And this kind of rogue wave is localized in X and Z.

Through research and Maple direct symbolic computations, we derive the central controllable higher order rogue wave solutions of the Eq. (2.1).

2.1 The First Order Rogue Wave Solution

We will receive the first order rogue wave solution to the Eq.(2.1) if one sets $n = 0$ in (2.6)

$$f = F_1(X, Z) = X^2 + a_{0,2}Z^2 + a_{0,0}. \tag{2.8}$$

Then substituting (2.8) into (2.5) and we conduct a direct symbolic computation, a polynomial equation system is obtained. By solving the polynomial equations, we get

$$\{a_{0,0} = \frac{3\beta(\alpha^2 - \beta^2)}{4(3\beta u_0 + \theta)},\ a_{0,2} = -\frac{2(3\beta u_0 + \theta)\alpha}{3\gamma\varepsilon}\}. \tag{2.9}$$

Further

$$f = F_1'(X, Z) = (X - \alpha_1)^2 - \frac{2(3\beta u_0 + \theta)\alpha}{3\gamma\varepsilon}(Z - \beta_1)^2 + \frac{3\beta(\alpha^2 - \beta^2)}{4(3\beta u_0 + \theta)}, \tag{2.10}$$

is a solution to Eq. (2.5), where $\alpha_1, \beta_1 \in \mathbb{R}$.

When $a_{0,0} = \frac{3\beta(\alpha^2-\beta^2)}{4(3\beta u_0+\theta)} > 0$ and $a_{0,2} = \frac{2(3\beta u_0+\theta)\alpha}{3\gamma\varepsilon} < 0$, Eq.(2.5) will have a positive polynomial solution. Substituting (2.10) into (2.2), the rogue wave solution of Eq. (2.1) is obtained

$$u = u_0 + (\beta^2 - \alpha^2)(\frac{2}{F_1'(X, Z)} - \frac{4(X - \alpha_1)^2}{F_1'(X, Z)^2}). \tag{2.11}$$

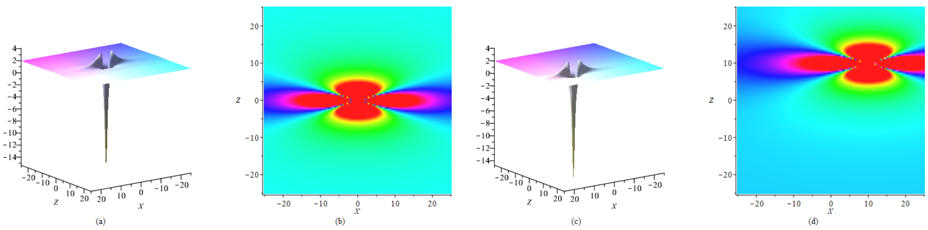

figure 1: Plots of the first order rogue wave (2.1) with $\alpha = -5, \beta = 2, \theta = 1, \varepsilon = 2, \gamma = 2, u_0 = 2$. (a) Evolution graphs of the rogue wave with $\alpha_1 = 0, \beta_1 = 0$. (b) The density plot of (a) ; (c) Evolution graphs of the rogue wave with $\alpha_1 = 10, \beta_1 = 10$. (d) The density plot of (c).

Figure. 1 shows the first order rogue wave. From the figure, we can clearly see that the rogue wave has three peaks. Two of the peaks are higher and the other is lower than the water level. Two parameters (α_1, β_1) are used to control the center of the rogue wave. By computation we can find that when $a_{0,0} = \frac{3\beta(\alpha^2-\beta^2)}{4(3\beta u_0+\theta)} > 0$ and $a_{0,2} = \frac{2(3\beta u_0+\theta)\alpha}{3\gamma\varepsilon} < 0$. The rogue wave reaches the peaks at the point (α_1, β_1) and $(\frac{\pm 3\sqrt{\beta(3\beta u_0+\theta)(\alpha^2-\beta^2)}+(6\beta u_0+2\theta)\alpha_1}{6\beta u_0+2\theta}, \beta_1)$ in the (X, Z)-plane, respectively. The first order rogue wave have extreme values $\frac{6\beta u_0+\theta}{3\beta}$ and $\frac{-21\beta u_0-8\theta}{3\beta}$. In figures. 1 (a) and 1 (c), it is shown that the rogue waves are concentrated near $(0, 0)$ and $(10, 10)$, respectively.

2.2 The Second Order Rogue Wave Solution

By (2.6) we can derive

$$f = F_2'(X, Z) = F_2(X, Z) + 2\alpha_1 Z P_1(X, Z) + 2\beta_1 X Q_1(X, Z) + (\alpha_1^2 + \beta_1^2)F_0(X, Z), \tag{2.12}$$

where

$$
\begin{aligned}
&F_2(X,Z) = X^6 + a_{4,0}X^4 + a_{4,2}Z^2X^4 + (a_{2,0} + a_{2,2}Z^2 + a_{2,4}Z^4)X^2 + a_{0,0} + a_{0,4}Z^4 + a_{0,6}Z^6,\\
&P_1(X,Z) = b_{0,0} + b_{0,2}X^2 + b_{2,0}Z^2, Q_1(X,Z) = c_{0,0} + c_{0,2}Z^2 + c_{2,0}X^2,\\
&F_0(X,Z) = 1.
\end{aligned}
\tag{2.13}
$$

Substituting (2.12) into (2.5) and setting all the coefficients of the different powers of Z^pX^q to zero. The set of solutions read as

$$
\left\{
\begin{array}{l}
\quad a_{0,0} = \frac{1}{192\alpha(3\beta u_0+\theta)^3}(5625\alpha^3\beta^3(\alpha^4 - 3\alpha^2\beta^2 + 3\beta^4) - 5625\beta^9\alpha - 192\alpha(\beta_1^2 + \alpha_1^2\\
\quad -\beta_1^2 c_{2,0}^2)(3\beta u_0 + \theta)^3 - 288\varepsilon\gamma\alpha_1^2 b_{2,0}^2(\beta u_0 + \frac{\theta}{3})^2), a_{0,2} = -\frac{475\alpha(\alpha^2-\beta^2)^2\beta^2}{24\varepsilon\gamma(3\beta u_0+\theta)},\\
\quad a_{0,4} = \frac{17(3\beta u_0+\theta)\alpha^2(\alpha^2-\beta^2)\beta}{9\varepsilon^2\gamma^2}, a_{2,0} = \frac{-125(\alpha^2-\beta^2)^2\beta^2}{16(3\beta u_0+\theta)^2}, a_{0,6} = \frac{8\alpha^3(3\beta u_0+\theta)^3}{-27\gamma^3\varepsilon^3},\\
a_{2,2} = \frac{15\beta\alpha(\beta^2-\alpha^2)}{\varepsilon\gamma}, a_{2,4} = \frac{4\alpha^2(3\beta u_0+\theta)^2}{3\varepsilon^2\gamma^2}, a_{4,0} = \frac{25\beta(\alpha^2-\beta^2)}{4(3\beta u_0+\theta)}, b_{2,0} = b_{2,0},\\
b_{0,0} = \frac{5\beta(\alpha^2-\beta^2)b_{2,0}}{12(3\beta u_0+\theta)}, b_{0,2} = \frac{2b_{2,0}(3\beta u_0+\theta)\alpha}{9\gamma\varepsilon}, c_{0,0} = \frac{\beta(\beta^2-\alpha^2)c_{2,0}}{4(3\beta u_0+\theta)}, c_{2,0} = c_{2,0},\\
c_{0,2} = \frac{2\alpha(3\beta u_0+\theta)c_{2,0}}{\gamma\varepsilon}, a_{4,2} = \frac{-2\alpha(3\beta u_0+\theta)}{\gamma\varepsilon},
\end{array}
\right\}
\tag{2.14}
$$

where $b_{2,0}, c_{2,0} \in C$. The second order rogue wave solution of Eq. (2.1) is obtained by Substituting (2.14) and (2.12) into (2.2).

$$
u = u_0 + (\beta^2 - \alpha^2)(\ln F_2'(X,Z))_{XX} \tag{2.15}
$$

in which $X = \alpha x + \beta y + \theta z, Z = \gamma z + \varepsilon w, \alpha, \beta, \theta, \gamma, \varepsilon, u_0 \in C$ that guarantee the analyticity of u. When $\alpha_1 = \beta_1 = 0$, the generated rogue wave is shown in figure 2. The second order rogue wave solution wave is concentrated around $(0,0)$. When the parameter values α_1 and β_1 reach certain level, the second order waves begin to separate, forming three first order rogue waves. The second-order rogue waves are drawn in figure 3. From (b) and (c) of figure 3, it has a clear display of three first-order rogue waves with three centers forming a triangle. So the second order rogue wave is also called triplet lump wave.

2.3 The Third Order Rogue Wave Solution

To construct the third order rogue wave solutions to the equation (2.1), we set the function f as

$$
f = F_3'(X,Z) = F_3(X,Z) + 2\alpha_1 Z P_2(X,Z) + 2\beta_1 X Q_2(X,Z) + (\alpha_1^2 + \beta_1^2)F_1(X,Z), \tag{2.16}
$$

with

$$
\begin{aligned}
F_3(X,Z) \;=\; & X^{12} + (a_{10,0} + a_{10,2}Z^2)X^{10} + (a_{8,0} + a_{8,2}Z^2 + a_{8,4}Z^4)X^8 + (a_{6,0}\\
& + a_{6,2}Z^2 + a_{6,4}Z^4 + a_{6,6}Z^6)X^6 + (a_{4,0} + a_{4,2}Z^2 + a_{4,4}Z^4 + a_{4,6}Z^6 + a_{4,8}Z^8)X^4\\
& + (a_{2,0} + a_{2,2}Z^2 + a_{2,4}Z^4 + a_{2,6}Z^6 + a_{2,8}Z^8 + a_{2,10}Z^{10})X^2 + a_{0,0} + a_{0,2}Z^2\\
& + a_{0,4}Z^4 + a_{0,6}Z^6 + a_{0,8}Z^8 + a_{0,10}Z^{10} + a_{0,12}Z^{12},\\
P_2(X,Z) \;=\; & Z^6 + (b_{4,0} + b_{4,2}X^2)Z^4 + (b_{2,0} + b_{2,2}X^2 + b_{2,4}X^4)Z^2\\
& + (b_{0,0} + b_{0,2}X^2 + b_{0,4}X^4 + b_{0,6}X^6),
\end{aligned}
$$

$$
\begin{aligned}
Q_2(X,Z) &= X^6+(c_{4,0}+c_{4,2}Z^2)X^4+(c_{2,0}+c_{2,2}Z^2+c_{2,4}Z^4)X^2 \\
&\quad +(c_{0,0}+c_{0,2}Z^2+c_{0,4}Z^4+c_{0,6}Z^6), \\
F_1(X,Z) &= X^2-\frac{2(3\beta u_0+\theta)\alpha}{3\gamma\varepsilon}Z^2+\frac{3\beta(\alpha^2-\beta^2)}{4(3\beta u_0+\theta)}. \qquad (2.17)
\end{aligned}
$$

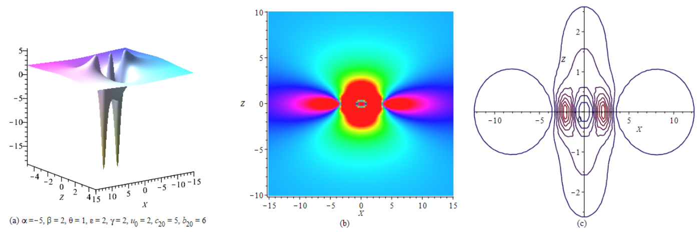

figure 2: Plots of the rogue wave (2.15) with
$\alpha=-5,\beta=2,\theta=1,\gamma=2,\varepsilon=2,u_0=2,c_{2,0}=5,b_{2,0}=6,\alpha_1=0,\beta_1=0$. (a) Evolution graphs of the wave $u(X,Z)$. (b) The density plot. (c) The contour plot.

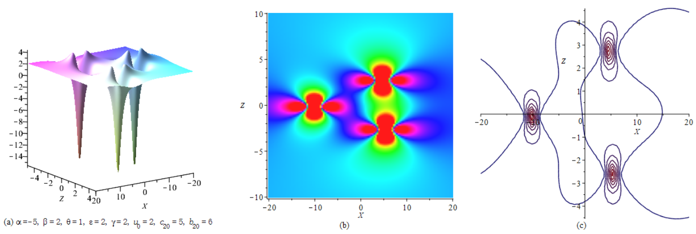

figure 3: Plots of the rogue wave (2.15) with
$\alpha=-5,\beta=2,\theta=1,\gamma=2,\varepsilon=2,u_0=2,c_{2,0}=5,b_{2,0}=6,\alpha_1=222,\beta_1=222$. (a) Evolution graphs of the wave $u(X,Z)$. (b) The density plot. (c) The contour plot.

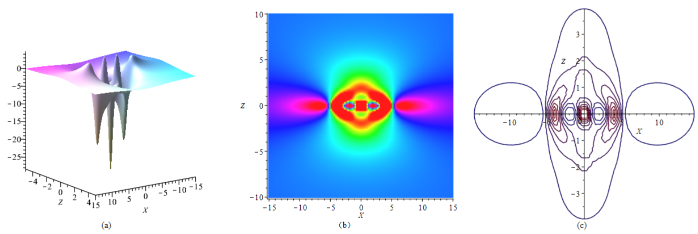

figure 4: Plots of the rogue wave (2.19) with
$\alpha=-5,\beta=2,\theta=1,\gamma=2,\varepsilon=2,u_0=2,\alpha_1=0,\beta_1=0$. (a) Evolution graphs of the wave $u(X,Z)$. (b) The density plot. (c) The contour plot.

Similarly, substituting (2.16) into (2.5), we can obtain the following set of constraining equations for the parameters

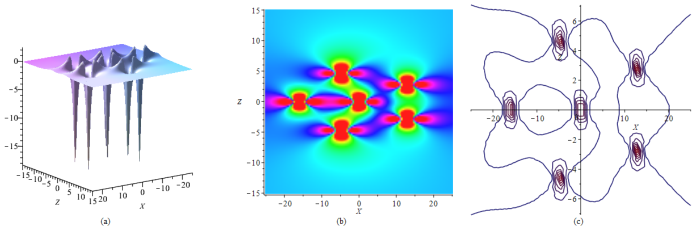

figure 5: Plots of the rogue wave (2.19) with $\alpha=-5,\beta=2,\theta=1,\gamma=2,\varepsilon=2,u_0=2,\alpha_1=1200000,\beta_1=1200000$. (a) Evolution graphs of the wave $u(X,Z)$. (b) The density plot. (c) The contour plot.

$$\left\{\begin{array}{l}
a_{0,0}=\frac{6561(\alpha^2-\beta^2)}{2\alpha^{43}\beta^{19}}\Big[\frac{70212289292481875A}{120932352}-\frac{878826025\beta^{15}}{13436928}+(\beta_1^2+\frac{\alpha_1^2}{2})(u_0\beta+\frac{\theta}{3})^5\alpha^7\\
-\frac{\gamma^7\alpha_1^2\varepsilon^7}{256}\Big],a_{0,2}=\frac{B-\frac{145}{77}A}{2304\gamma\varepsilon(3u_0\beta+\theta)^6\alpha^6},a_{0,4}=\frac{16391725\alpha^{11}\beta^9D^4}{1728\gamma^2\varepsilon^2C},a_{0,6}=\frac{199745D^3\alpha^3\beta^3}{162\gamma^3\varepsilon^3},\\
a_{0,8}=\frac{1445D^2C\alpha^5\beta^3}{27\gamma^4\varepsilon^4},a_{0,10}=\frac{464C^2D^2}{243\gamma^5\varepsilon^5\alpha^{13}\beta^9},a_{0,12}=\frac{46656\alpha^{33}\beta^{15}}{C^3\gamma^6\varepsilon^6},a_{2,8}=\frac{760DC}{-9\alpha^5\beta^4\gamma^4\varepsilon^4},\\
a_{2,0}=\frac{A-B}{1536\alpha^7(3u_0\beta+\theta)^7},a_{2,2}=\frac{94325D^4\alpha\beta^4}{-64\gamma\varepsilon(3u_0\beta+\theta)^3},a_{2,4}=\frac{1225D^3\alpha^6\beta^5}{-12\gamma^2\varepsilon^2}\sqrt{\frac{\alpha\beta}{C}},\\
a_{2,6}=\frac{17710D^2}{-27\alpha\gamma^3\varepsilon^3}\sqrt{\frac{C}{\alpha\beta}},a_{2,10}=\frac{192(u_0\beta+\frac{\theta}{3})^5\alpha^5}{-\gamma^5\varepsilon^5},a_{4,0}=\frac{5187875D^4\beta^{12}\alpha^{16}}{-768C^2}.\\
a_{4,2}=\frac{18375D^3\alpha^{10}\beta^8}{-8\gamma\varepsilon C},a_{4,4}=\frac{18725D^2\alpha^2\beta^2}{18\gamma^2\varepsilon^2},a_{4,6}=\frac{2920DC}{-27\gamma^3\varepsilon^3\alpha^6\beta^4},a_{4,8}=\frac{80C^2}{27\alpha^{14}\beta^{10}\gamma^4\varepsilon^4},\\
a_{6,0}=\frac{18865D^3\beta^3}{48(3u_0\beta+\theta)^3},a_{6,2}=\frac{4655D^2\alpha\beta^2}{-6(3u_0\beta+\theta)\gamma\varepsilon},a_{6,4}=\frac{1540D}{9\alpha^2\beta\gamma^2\varepsilon^2}\sqrt{\frac{C}{\alpha\beta}},a_{8,0}=\frac{735D^2\beta^7\alpha^9}{16C},\\
a_{6,6}=\frac{160\alpha^3(u_0\beta+\frac{\theta}{3})^3}{-\gamma^3\varepsilon^3},a_{8,2}=\frac{115D\alpha\beta}{-\gamma\varepsilon},a_{8,4}=\frac{20C}{3\alpha\gamma^2\varepsilon^2\beta^5},a_{10,0}=\frac{49\beta D\alpha^4\beta^2}{2(3u_0\beta+\theta)}\sqrt{\frac{\alpha\beta}{C}},\\
a_{10,2}=\frac{-4(3u_0\beta+\theta)\alpha}{\gamma\varepsilon},b_{0,0}=\frac{169785D^3\alpha^{24}\beta^{15}\gamma^3\varepsilon^3}{512C^3},b_{0,2}=\frac{17955D^2\beta^2\gamma^3\varepsilon^3}{128(3u_0\beta+\theta)^5\alpha^3},\\
b_{0,4}=\frac{2835D\alpha^{15}\beta^{10}\gamma^3\varepsilon^3}{32C},b_{0,6}=\frac{135\gamma^3\varepsilon^3}{8(3u_0\beta+\theta)^3\alpha^3},b_{2,0}=\frac{2205D^2\beta^8\alpha^{16}\gamma^2\varepsilon^2}{64C^2},\\
b_{2,2}=\frac{855D\beta\gamma^2\varepsilon^2}{8(3u_0\beta+\theta)^3\alpha^2},b_{2,4}=\frac{-45\gamma^2\varepsilon^2}{4\beta^5\alpha^{11}},b_{4,0}=\frac{21D\gamma\varepsilon\alpha^8\beta^6}{8C},c_{2,0}=\frac{245D^2\beta^7\alpha^9}{-16C},\\
b_{4,2}=\frac{27\alpha^8\beta^5\gamma\varepsilon}{2C},c_{0,0}=\frac{12005D^3\beta^3}{192(3u_0\beta+\theta)^3},c_{0,2}=\frac{535D^2\alpha\beta^2}{-24(3u_0\beta+\theta)\gamma\varepsilon},c_{0,4}=\frac{5D}{\alpha^2\beta\gamma^2\varepsilon^2}\sqrt{\frac{C}{\alpha\beta}},\\
c_{0,6}=\frac{40(u_0\beta+\frac{\theta}{3})^3\alpha^3}{-\gamma^3\varepsilon^3},c_{2,2}=\frac{115D\alpha\beta}{3\gamma\varepsilon},c_{2,4}=\frac{20C}{9\alpha^5\beta^7\gamma^2\varepsilon^2},c_{4,0}=\frac{13D\beta}{4(3u_0\beta+\theta)},\\
c_{4,2}=\frac{6}{\alpha^3\beta^2\gamma\varepsilon}\sqrt{\frac{C}{\alpha\beta}},A=79893275C(\alpha^8-5\alpha^6\beta^2+10\alpha^4\beta^4-10\alpha^2\beta^6+5\beta^8),\\
B=373248\frac{C}{\beta^5\alpha^9}(\frac{79893275\beta^{15}}{373248}+\frac{C\alpha_1^2}{9\beta^5\alpha^9})\alpha^7+26244\gamma^7\alpha_1^2\varepsilon^7,\\
C=\alpha^9\beta^5(3u_0+\theta)^2,D=\beta^2-\alpha^2,
\end{array}\right\} \tag{2.18}$$

where $\alpha,\beta,\theta,\gamma,\varepsilon,u_0,\alpha_1,\beta_1$ are arbitrary constants. Similarly, substituting (2.16) and (2.18) into (2.2), we have the rogue wave solution of Eq.(2.1) as

$$u=u_0+2\ln(F_3'(X,Z))_{XX}. \tag{2.19}$$

When $\alpha=-5,\beta=2,\theta=1,\varepsilon=2,\gamma=2,u_0=2,\alpha_1=0$ and $\beta_1=0$, from the figure 4 we find that the third order rogue wave has four peaks of wave and three troughs of wave. When we take $\alpha_1=\beta_1=1200000$ in figure 5, the rogue wave (figure 4) will gradually split into six first order rogue waves. In this case, from the figure, it is found that the structure has six peaks, which form a pentagram.

3 Breathers Solution

In this section, We apply the simplified Hirota's bilinear method to investigate the breathers solution of the Eq.(1.1). Substituting

$$u = e^{\theta_i}, \theta_i = k_i x + r_i y + s_i z + q_i w - c_i t, i = 1, 2, ..., N, \tag{3.1}$$

into the linear terms of (1.1) and solve it. We find that there exists the soliton solution when the following relation is satisfied

$$c_i = -\frac{k_i^3 r_i - k_i r_i^3 + 6 s_i q_i}{4k_i}, i = 1, 2, ..., N. \tag{3.2}$$

Under the transformations

$$u(x, y, z, w, t) = R(\ln f(x, y, z, w, t))_{xx}, \tag{3.3}$$

where

$$f = 1 + e^{k_1 x + r_1 y + s_1 z + q_1 w + \frac{k_1^3 r_1 - k_1 r_1^3 + 6 s_1 q_1}{4k_1} t}. \tag{3.4}$$

Substituting (3.4) into Eq.(1.1), we find that

$$m{k_1}^2 = {r_1}^2, m = R + 1, m \neq -1, \tag{3.5}$$

further

$$\theta_i = k_i x + \sqrt{m} k_i y + s_i z + q_i w + \frac{\sqrt{m}(1-m)k_i^4 + 6 s_i q_i}{4k_i} t, \tag{3.6}$$

under the constraint (3.3). Through the above calculation, we obtain the following two-soliton solution

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}. \tag{3.7}$$

Therefore, the solution (3.3) can also be written as

$$u(x, y, z, w, t) = (m-1)(\ln f(x, y, z, w, t))_{xx}. \tag{3.8}$$

Based on (3.7),(3.8), the a_{12} is expressed as

$$a_{12} = \frac{-k_1^2 k_2^2 (k_1 - k_2)^2 m^{\frac{3}{2}} + k_1^2 k_2^2 (k_1 - k_2)^2 m^{\frac{1}{2}} - 2(k_1 s_2 - k_2 s_1)(k_1 q_2 - k_2 q_1)}{-k_1^2 k_2^2 (k_1 + k_2)^2 m^{\frac{3}{2}} + k_1^2 k_2^2 (k_1 + k_2)^2 m^{\frac{1}{2}} - 2(k_1 s_2 - k_2 s_1)(k_1 q_2 - k_2 q_1)}, \tag{3.9}$$

and the phase shift can be generalized in the following ways

$$a_{ij} = \frac{-k_i^2 k_j^2 (k_i - k_j)^2 m^{\frac{3}{2}} + k_i^2 k_j^2 (k_i - k_j)^2 m^{\frac{1}{2}} - 2(k_i s_j - k_j s_i)(k_i q_j - k_j q_i)}{-k_i^2 k_j^2 (k_i + k_j)^2 m^{\frac{3}{2}} + k_i^2 k_j^2 (k_i + k_j)^2 m^{\frac{1}{2}} - 2(k_i s_j - k_j s_i)(k_i q_j - k_j q_i)}. \tag{3.10}$$

The breathers solution to the Fokas equation can be presented by choosing

$$k_1 = k_2^* = a_1 + b_1 I, s_1 = s_2^* = a_2 + b_2 I, q_1 = q_2^* = a_3 + b_3 I, \tag{3.11}$$

where $*$ denotes the conjugate operator and $I^2 = -1$.

Setting $k_1 = I, s_1 = 2 + I, q_1 = 2, m = 1$, the f can be expressed as

$$\begin{aligned} f &= 1 + 2(\cosh(2z + 2w + 3t) + \sinh(2z + 2w + 3t))\cos(\frac{t + 4y}{4}\sqrt{2} - 6t + x + z) \\ &\quad +(1 + \frac{\sqrt{2}}{8})(\cosh(4z + 4w + 6t) + \sinh(4z + 4w + 6t). \end{aligned} \tag{3.12}$$

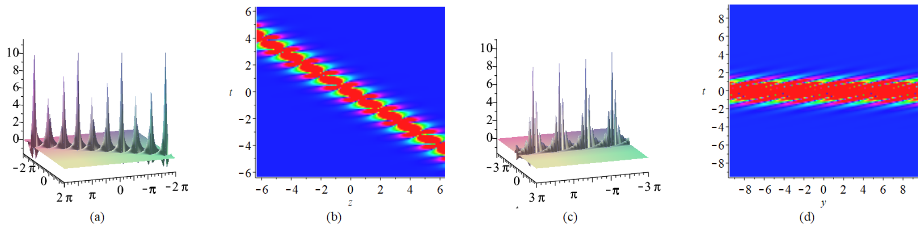

figure 6: (a) The General breather solution of Eq.(1.1) for the values $x = y = w = 0$. (b) Overhead view of the wave (a). (c) The Akhmediev breather solution of Eq.(1.1) for the values $x = z = w = 0$. (d) Overhead view of the wave (c). When $y = z = w = 0$, the plot is similar to (c) also (not shown here).

By analyzing function f (Eq. (3.12)), we can see that the spatial variables x and y are similar, but the spatial variable z is essentially different from the spatial variables x and y when $w = 0$. Eq. (3.12) constitutes the general solution of the Fokas Eq.(1.1). Figure 6(a) shows that the behavior of the general breathers is periodic in both space and time. The solution shown in Figure 6 is Akhmediev breather solution, which is periodic in x and localized in t.

To acquire the high order breathers solution of the Eq.(1.1), we analyse the following four-soliton solution

$$\begin{aligned} f &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{14}e^{\theta_1+\theta_4} + a_{23}e^{\theta_2+\theta_3} \\ &\quad + a_{24}e^{\theta_2+\theta_4} + a_{34}e^{\theta_3+\theta_4} + a_{123}e^{\theta_1+\theta_2+\theta_3} + a_{124}e^{\theta_1+\theta_2+\theta_4} + a_{134}e^{\theta_1+\theta_3+\theta_4} \\ &\quad + a_{234}e^{\theta_2+\theta_3+\theta_4} + a_{1234}e^{\theta_1+\theta_2+\theta_3+\theta_4}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &a_{123} = a_{12}a_{13}a_{23}, a_{124} = a_{12}a_{14}a_{24}, a_{134} = a_{13}a_{14}a_{34}, \\ &a_{234} = a_{23}a_{24}a_{34}, a_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}. \end{aligned} \tag{3.14}$$

Meanwhile $\theta_i, 1 \le i \le 4$, and $a_{ij}, 1 \le i < j \le 4$ are given in (3.1),(3.10), respectively.

In a similar way, by choosing the following parameters

$$\begin{aligned} &k_1 = k_2^* = a_1 + b_1 I, k_3 = k_4^* = a_2 + b_2 I, s_1 = s_2^* = a_3 + b_3 I, s_3 = s_4^* = a_4 + b_4 I, \\ &q_1 = q_2^* = a_5 + b_5 I, q_3 = q_4^* = a_6 + b_6 I. \end{aligned} \tag{3.15}$$

And by the Eq. (3.13), the high order breathers solution are obtained. As shown

$$
\begin{aligned}
f &= \frac{1}{17904\sqrt{2}+43832}\Bigg(2(8408+7920\sqrt{2})\cos(\frac{(9t+12y)\sqrt{2}}{4}-\frac{15t}{2}+3x+2z) \\
&\quad \times e^{3z+4w+\frac{9t}{2}}-2(408+1888\sqrt{2})\sin(\frac{(9t+12y)\sqrt{2}}{4}-\frac{15t}{2}+3x+2z)e^{3z+4w+\frac{9t}{2}} \\
&\quad +2(18612+13719\sqrt{2})\cos((2t+2y)\sqrt{2}-\frac{3t}{2}+2x+z)e^{5z+6w+\frac{15t}{2}} \\
&\quad -2(4512+3168\sqrt{2})\sin((2t+2y)\sqrt{2}-\frac{3t}{2}+2x+z)e^{5z+6w+\frac{15t}{2}} \\
&\quad +2(110520+77520\sqrt{2})\cos(\frac{(t+4y)\sqrt{2}}{4}-6t+x+z)e^{4z+6w+6t} \\
&\quad -2(11712+5184\sqrt{2})\sin(\frac{(t+4y)\sqrt{2}}{4}-6t+x+z)e^{4z+6w+6t} \\
&\quad +2(59544+39408\sqrt{2})\cos(\frac{(7t+4y)\sqrt{2}}{4}+x+\frac{9}{2}t)e^{3z+4w+\frac{9}{2}t} \\
&\quad -2(2592+4128\sqrt{2})\sin(\frac{(7t+4y)\sqrt{2}}{4}+x+\frac{9}{2}t)e^{3z+4w+\frac{9}{2}t} \\
&\quad +2(43832+17904\sqrt{2})\cos(\frac{(t+4y)\sqrt{2}}{4}+x+z-6t)e^{2z+2w+3t} \\
&\quad +(193232\sqrt{2}+187064)e^{2z+4w+3t}+17904\sqrt{2}+43832+(23383\sqrt{2}+48308)e^{6z+8w+9t}\Bigg),
\end{aligned}
\tag{3.16}
$$

by taking

$$k_1=I, k_3=2I, s_1=2+I, s_3=1+I, q_1=2, q_3=2. \tag{3.17}$$

The corresponding dynamics characteristics of solution u are shown in figure 7.

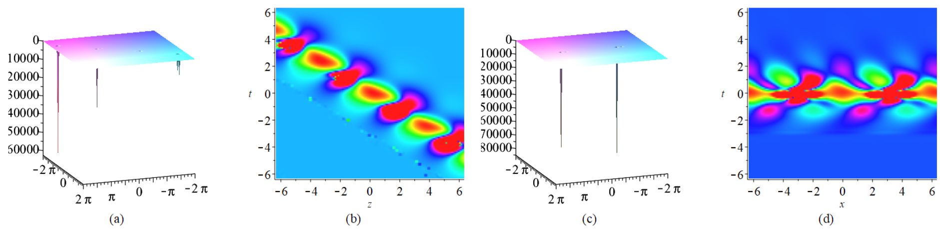

figure 7: The two breathers solution of Eq.(1.1): (a) The General breather solution of Eq.(1.1) for the values $x=y=w=0$. (b) Overhead view of the wave (a). (c) The Akhmediev breather solution of Eq.(1.1) for the values $y=z=w=0$. (d) Overhead view of the wave (c). When $x=z=w=0$, the plot is similar to (c) also (not shown here).

The high order general breathers solution of the Fokas Eq. (1.1) are shown in figure 7(a) that the behavior of the general breathers is periodic in both space and time. The solution shown in figure 7(c) is Akhmediev breather solution, which is periodic in x and localized in t.

Remark In this paper, based on the bilinear form of the $(4+1)$-dimensional Fokas equation, the rogue wave solution and breathers solution of Eq. (2.1) were constructed. And high order rogue wave solutions and high order breathers solution were presented also. The correctness of the results in this work has been verified. The results of this paper enrich the types of solutions of the $(4+1)$-dimensional Fokas equation, which is helpful to promote our understanding of various kinds wave phenomena in fluid dynamics, natural phenomenon and science problems.

The mechanism of more new kinds of hybrid solutions for different nonlinear partial differential equations will be studied in the future.

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(4+1)维Fokas方程的怪波解和呼吸子解

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摘要: 本文研究了（4+1）维Fokas方程的多阶怪波解, 呼吸子解和高阶呼吸子解. 利用Hirota双线性形式和简化的Hirota双线性形式, 丰富了（4+1）维Fokas方程的解的多样性. 最后分析了精确解的动力学行为.

关键词: (4+1)-维Fokas方程; 高阶怪波解; 呼吸子解; 高阶呼吸子解

MR(2010)**主题分类号**: 35C99; 35G20 **中图分类号**: O29; O175.29