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TILTING COMODULES OVER TRIVIAL EXTENSIONS OF COALGEBRAS

LI Ye-xuan, YAO Hai-lou
(*College of Mathematics, Faculty of Science, Beijing University of Technology, Beijing 100124, China*)

Abstract: In this paper, we study the tilting comodules over trivial extensions of coalgebras. On the basis of the tilting theory, we get the upper bound of the global dimension of trivial extensions of coalgebras, and then we obtain the equivalent condition for one comodule to be a tilting comodules over trivial extensions of coalgebras. These results generalize the conclusion of tilting modules.

Keywords: trivial extension of coalgebra; tilting comodule; Gorenstein injective comodule; global dimension

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1 Introduction

The trivial extensions of algebras play an important role in ring theory and representation theory of algebras, especially in triangular matrix rings and triangular matrix algebras. In 1975, Fossum [1] made a systematic and comprehensive summary of the trivial extensions of abelian categories and algebras. On the basis of tilting theory, in 1985, Miyachi [2] obtained the equivalent condition of tilting modules over the trivial extensions of artin algebras, and applied it to the triangular matrix algebras. Dually, the trivial extensions of coalgebras, triangular matrix coalgebras [3] and especially the category of comodules over the triangular matrix coalgebras have attracted extensive attension of scholars at home and abroad. In 2008, Zhu [4] gave the definition and properties of trivial extensions of coalgebras. In 1998, Wang [5] defined the concepts of classical tilting comodules for comodule categories. In 1999, Wang [6] introduced the concepts of tilting comodules and tilting injective comodules over coalgebras. In particular, he proved that each tilting comodule induces a torsion theory. In 2001, Simson [7] defined the concepts of cotilting comodules and he hoped to develop a (co)tilting theory for comodule categories. In 2008, Simson [8] introduced the notion of an f-cotilting comodule and a cotilting procedure for coalgebras, and constructed a pair of cotilting functors of Brenner-Butler type for coalgebras. In 2010, Kosakowska and Simson [9] gave the definition of triangular matrix coalgebras and studied its properties. In 2016,

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Biography: Li Yexuan(1994–), female, born at Changzhi, Shanxi, doctor, major in representation theory of algebras. E-mail: leeyexuan@126.com.

Fu [10] gave the equivalent condition of tilting comodules over the triangular matrix coalgebras. In 1999, Asensio [11] introduced Gorenstein injective comodules as a generalization of injective comodules. Inspired by this, we aim to generalize the triangular matrix coalgebras and obtain the bound of the global dimension of trivial extensions of coalgebras, and construct the tilting comodules and Gorenstein injective comodules over the trivial extensions of coalgebras as well.

2 Preliminaries

Let K be a fixed field, and C be a K-coalgebra. $\mathcal{M}^C$ denotes the category of right C-comodules. Suppose that M, in addition to being a left C-comodule with structure map $\rho_l : m \mapsto \Sigma m_{[-1]} \otimes m_{[0]}$, is also a right C-comodule with structure map $\rho_r : m \mapsto \Sigma m_{[0]} \otimes m_{[1]}$ and that $(id_c \otimes \rho_r)\rho_l = (\rho_l \otimes id_c)\rho_r$. Let $C \oplus M = \{(c, m) | c \in C, m \in M\}$, with componentwise addition and multiplication given, elementwise, by $(c, m)(c', m') = (cc', cm' + mc')$. $C \oplus M$ is made into a coalgebra in [4] by defining comultiplication $\Delta : C \oplus M \to (C \oplus M) \otimes (C \oplus M)$ and the counit $\varepsilon : C \oplus M \to k$ as follows:

$$\Delta(c, m) = \Sigma(c_{(1)}, 0)(c_{(2)}, 0) + \Sigma(m_{[-1]}, 0)(0, m_{[0]}) + \Sigma(0, m_{[0]})(m_{[1]}, 0) \quad \varepsilon : (c, m) \mapsto \varepsilon_c(C)$$

which is called the trivial extension of C by M, denoted by $\Gamma = C \ltimes M$.

Definition 2.1 [12] A right C-comodule X is quasi-finite, if $\dim\mathrm{Com}_c(F, M) < \infty$ for all finite dimensional comodule F.

Remark Unless otherwise specified, this thesis is all conducted under the condition of quasi-finite comodules.

Definition 2.2 [12] If M is a quasi-finite right C-comodule, we denote by $h_c(M, -)$ the left adjoint functor of $-\Box_c M$, and we have $h_c(M, N) = \varinjlim D\mathrm{Com}_c(N_\lambda, M)$, where $\{N_\lambda\}_\lambda$ is the family of finite dimensional subcomodules of N.

Definition 2.3 Let $F = -\Box_c M : \mathcal{M}^C \to \mathcal{M}^C$ be a left exact endofunctor and M be a C-bicomodule. We define the left trivial extension of $\mathcal{M}^C$ by F, denoted by $F \rtimes \mathcal{M}^C$.

(1) An object in $F \rtimes \mathcal{M}^C$ is a right C-comodule morphism $\alpha : X \to X\Box_c M$ such that the composition$X \xrightarrow{\alpha} X\Box_c M \xrightarrow{\alpha\Box_c M} X\Box_c M\Box_c M$is zero. i.e., $(\alpha\Box_c M) \circ \alpha = 0$.

(2) If $\alpha : X \to X\Box_c M$ and $\beta : Y \to Y\Box_c M$ are objects in $F \rtimes \mathcal{M}^C$, then a morphism $\gamma : \alpha \to \beta$ is a morphism $\gamma : X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & X\Box_c M \arrow[d, "\gamma\Box_c M"] \\ Y \arrow[r, "\beta"] & Y\Box_c M \end{tikzcd} \tag{2.1}$$

is commutative.

(3) Composition in $(-\Box_c M) \rtimes \mathcal{M}^C$ is just composition in $\mathcal{M}^C$.

Definition 2.4 Let $G = h_c(M, -) : \mathcal{M}^C \to \mathcal{M}^C$ be a right exact endofunctor and M be a C-bicomodule. We define the right trivial extension of $\mathcal{M}^C$ by G, denoted by $\mathcal{M}^C \ltimes G$.

(1) An object in $\mathcal{M}^C \ltimes G$ is a right C-comodule morphism $\alpha : h_c(M, X) \to X$ such that the composition $\alpha \circ h_c(M, \alpha) = 0$.

(2) If $\alpha : h_c(M, X) \to X$ and $\beta : h_c(M, Y) \to Y$ are objects in $\mathcal{M}^C \ltimes G$, then a morphism $\gamma : \alpha \to \beta$ is a morphism $\gamma : X \to Y$ such that the diagram

$$\begin{tikzcd} h_c(M,X) \arrow[r, "\alpha"] \arrow[d, "h_c(M,\gamma)"'] & X \arrow[d, "\gamma"] \\ h_c(M,Y) \arrow[r, "\beta"] & Y \end{tikzcd} \tag{2.2}$$

is commutative.

(3) Composition in $\mathcal{M}^C \ltimes G$ is just composition in $\mathcal{M}^C$.

From [1, Proposition 1.1 and Corollary 1.2], we have the following characterizations of $\mathcal{M}^C \ltimes h_c(M, -)$ and $(-\Box_c M) \rtimes \mathcal{M}^C$.

Proposition 2.5 $\mathcal{M}^C \ltimes h_c(M, -)$ and $(-\Box_c M) \rtimes \mathcal{M}^C$ are abelian categories.

Proposition 2.6 (1) A sequence of objects in $\mathcal{M}^C \ltimes h_c(M, -)$ are exact if and only if the sequence of codomains is exact;

(2) A sequence of objects in $(-\Box_c M) \rtimes \mathcal{M}^C$ are exact if and only if the sequence of domains is exact.

The next two definitions 2.7 and 2.8 come from [1].

Definition 2.7 For the endofunctor $-\Box_c M : \mathcal{M}^C \to \mathcal{M}^C$, there are pairs of adjoint functors

$$\mathcal{M}^C \underset{K}{\overset{Z}{\rightleftarrows}} (-\Box_c M) \rtimes \mathcal{M}^C \underset{H}{\overset{U}{\rightleftarrows}} \mathcal{M}^c \tag{2.3}$$

which satisfy the relations $KH = id_{\mathcal{M}^C}, UZ = id_{\mathcal{M}^C}$. They are defined on objects and morphisms as follows:

(1) The functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is defined on objects by

$$H(X) : X\Box_c M \oplus X \xrightarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} X\Box_c M\Box_c M \oplus X\Box_c M$$

and on morphisms by

$$H(\alpha) : H(X) \xrightarrow{\begin{bmatrix} \alpha\Box_c M & 0 \\ 0 & \alpha \end{bmatrix}} H(X').$$

(2) The functor $U : (-\Box_c M) \rtimes \mathcal{M}^C \to \mathcal{M}^C$ is defined on objects by $U(\alpha : X \to X\Box_c M) = \text{domain}(\alpha) = X$ and on morphisms by $U(\gamma : \alpha \to \beta) = \gamma : X \to Y$, where $\beta : Y \to Y\Box_c M$.

(3) The zero functor $Z : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is defined on objects by $Z(X) = 0 : X \to X\Box_c M$ and on morphisms by $Z(\gamma : X \to Y) = \gamma : \alpha \to \beta$, where $\beta : Y \to Y\Box_c M$.

(4) The kernel functor $K : (-\Box_c M) \rtimes \mathcal{M}^C \to \mathcal{M}^C$ is defined on objects by $K(\alpha : X \to X\Box_c M) = ker\alpha$ and on morphisms by $K(\gamma : \alpha \to \beta) = \gamma \mid_{ker\alpha}$, where $\beta : Y \to Y\Box_c M$.

Dually, we have the following notions.

Definition 2.8 For the endofunctor $h_c(M,-):\mathcal{M}^C\to\mathcal{M}^C$, there are pairs of adjoint functors

$$\mathcal{M}^C \underset{U}{\overset{T}{\rightleftarrows}} \mathcal{M}^C \ltimes h_c(M,-) \underset{Z}{\overset{C}{\rightleftarrows}} \mathcal{M}^C \tag{2.4}$$

which satisfy the relations

$$CT = id_{\mathcal{M}^C}, UZ = id_{\mathcal{M}^C}.$$

They are defined on objects and morphisms as follows.

(1) The functor $T:\mathcal{M}^C\to\mathcal{M}^C\ltimes h_c(M,-)$ is defined on objects by

$$T(X): h_c(M,X)\oplus h_c(M,h_c(M,X)) \xrightarrow{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}} X\oplus h_c(M,X)$$

and on morphisms by

$$T(\alpha): T(X) \xrightarrow{\begin{bmatrix}\alpha & 0\\ 0 & h_c(M,\alpha)\end{bmatrix}} T(X').$$

(2) The functor $U:\mathcal{M}^C\ltimes h_c(M,-)\to\mathcal{M}^C$ is defined on objects by $U(\alpha: h_c(M,X)\to X) = codom(\alpha) = X$ and on morphisms by $U(\gamma:\alpha\to\beta)=\gamma: X\to Y$, where $\beta: h_c(M,Y)\to Y$.

(3) The zero functor $Z:\mathcal{M}^C\to\mathcal{M}^C\ltimes h_c(M,-)$ is defined on objects by $Z(X)=0: h_c(M,X)\to X$ and on morphisms by $Z(\gamma:X\to Y)=\gamma:\alpha\to\beta$, where $\beta: h_c(M,Y)\to Y$.

(4) The cokernel functor $C:\mathcal{M}^C\ltimes h_c(M,-)\to\mathcal{M}^C$ is defined on objects by $C(\alpha: h_c(M,X)\to X) = coker\alpha$ and on morphisms by $K(\gamma:\alpha\to\beta)=\gamma\mid_{coker\alpha}$.

By Corollary 1.6 in [1], we have the following conclusions.

Proposition 2.9 By the Definition 2.7 and 2.8, we have the following.

(1) $(T,U),(C,Z),(U,H),(Z,K)$ are adjoint pairs;

(2) Functors K, H are left exact, T, C are right exact and Z, U are exact;

(3) If P is projective in $\mathcal{M}^C$ (*resp.* : $\mathcal{M}^C\ltimes h_c(M,-)$), then $T(P)$ (*resp.* : $C(P)$) is projective in $\mathcal{M}^C\ltimes h_c(M,-)$) (*resp.* : $\mathcal{M}^C$). Consequently, π is projective in $\mathcal{M}^C\ltimes h_c(M,-)$ if and only if $C(\pi)$ is projective in $\mathcal{M}^C$ and $\pi\cong T(C(\pi))$;

(4) If E is injective in $\mathcal{M}^C$ (*resp.* : $(-\Box_c M)\rtimes\mathcal{M}^C$), then $H(E)$ (*resp.* : $K(E)$) is injective in $(-\Box_c M)\rtimes\mathcal{M}^C$ (*resp.* : $\mathcal{M}^C$). Consequently, ϵ is injective in $(-\Box_c M)\rtimes\mathcal{M}^C$ if and only if $K(\epsilon)$ is injective in $\mathcal{M}^C$ and $\epsilon\cong H(K(\epsilon))$.

Definition 2.10 [6] A right C-comodule T_c is called a tilting comodule if

(1) there is an exact sequence $0\to T_2\to T_1\to C\to 0$ with $T_i\in AddT=\{M\mid M\oplus M_i'=T^X$ for some cardinal $X\}$ for $i=1,2$;

(2) $Ext_c^1(T^X,T)=0$ for any cardinal X;

(3) inj.dim $T_c\le 1$.

Definition 2.11 [6] Let T_c be a tilting comodule. A right C-comodule X is called tilting injective relative to T_c if $\mathrm{Com}_c(-,X)$ preserves the exactness of sequence in $CogenT=\{M\in\mathcal{M}^C\mid 0\to M\to T^X$ for some cardinal $X\}$.

Definition 2.12 [11] A right C-comodule M is called Gorenstein injective if there exists an exact sequence

$$\mathcal{E} \equiv \cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective right C-comodules with $M \cong ker(E^0 \to E^1)$ and such that the functor $\mathrm{Com}_c(E, -)$ leaves it exact for any injective right C-comodule E.

Definition 2.13 [13] A right C-comodule M is called Gorenstein coflat if there exists an exact sequence

$$\mathcal{E} \equiv \cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective right C-comodules with $M \cong ker(E^0 \to E^1)$ and such that the functor $-\Box_c Q$ leaves it exact for any projective left C-comodule Q.

Definition 2.14 [14] For any right C-comodule $M \in \mathcal{M}^C$ the injective dimension of M denoted by $id_c M$, is defined as the least number n, such that there is one injective resolution

$$0 \to M \to E_0 \to \cdots \to E_1 \to E_n \to 0$$

and there is no shorter injective resolution for M. If there exists no such n, we say that the injective dimension of M is infinite, $id_c M = \infty$.

Definition 2.15 [14] The (right)global dimension of the coalgebra C is defined as

$$\text{rgl.dim}C = \sup\{id_c M; M \in \mathcal{M}^C\}.$$

Similarly, one may have the definition of left global dimension of C.

3 Bounds for the Global Dimension of $\Gamma = C \ltimes M$

In this section, by the concept of functors in Definition 2.7, we will get the upper bound of the global dimension of Γ.

Lemma 3.1 Let C be a semiperfect coalgebra and M be a C-bicomodule. Then the trivial extension of C by M is also semiperfect.

Proof By the assumption, C is a semiperfect coalgebra that is the category $\mathcal{M}^C$ has enough projectives. By [1, Proposition 1.11 and Proposition 1.13], we have the categories $\mathcal{M}^C \ltimes h_c(M, -)$, $(-\Box_c M) \rtimes \mathcal{M}^C$ and $\mathcal{M}^{C \ltimes M}$ those are all isomorphic. So we only need to prove that the category $\mathcal{M}^C \ltimes h_c(M, -)$ has enough projectives. Suppose P is projective in $\mathcal{M}^C$ and that $\alpha \to \alpha''$ is an epimorphism in $\mathcal{M}^C \ltimes h_c(M, -)$. Let $D = \mathcal{M}^C \ltimes h_c(M, -)$, then we have

$$\begin{array}{ccc} \mathrm{Com}_D(TP, \alpha) & \longrightarrow & \mathrm{Com}_D(TP, \alpha'') \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Com}_{\mathcal{M}^C}(P, U\alpha) & \longrightarrow & \mathrm{Com}_{\mathcal{M}^C}(P, U\alpha''). \end{array}$$

But $U\alpha \to U\alpha''$ is an epimorphism. Since P is projective in $\mathcal{M}^C$, it follows that the homomorphism $\mathrm{Com}_D(TP, \alpha) \to \mathrm{Com}_D(TP, \alpha'')$ is surjective. Thus, we have that TP is a projective right $C \ltimes M$-comodule (more details see [1, Corollary 1.6 and Corollary 1.7]).

Lemma 3.2 Let C be a semiperfect coalgebra, M be a C-bicomodule and $\Gamma = C \ltimes M$ be a trivial extension of C by M. Let $X \in \mathcal{M}^C$. Then

$$id_\Gamma Z(X) \leq 1 + \max\{id_\Gamma Z(\Omega_c^{-1}(X)), id_\Gamma Z(M)\}.$$

Proof Let $\alpha : X \to E$ be an injective envelope of X with cokernel $\Omega_c^{-1}(X)$. Then we have a short exact sequence of Γ-comodules:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & E\square_c M \oplus E \arrow[r] \arrow[d] & E\square_c M \oplus \Omega_c^{-1}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X\square_c M \arrow[r] & E\square_c M\square_c M \oplus E\square_c M \arrow[r] & E\square_c M\square_c M \oplus \Omega_c^{-1}(X)\square_c M \arrow[r] & 0
\end{tikzcd}$$

where the middle term is an injective Γ-comodule. Then we get

$$id_\Gamma Z(X) \leq 1 + id_\Gamma(E\square_c M \oplus \Omega_c^{-1}(X) \to E\square_c M\square_c M \oplus \Omega_c^{-1}(X)\square_c M).$$

Next, we have the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E\square_c M \arrow[r] \arrow[d] & E\square_c M \oplus \Omega_c^{-1}(X) \arrow[r] \arrow[d] & \Omega_c^{-1}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E\square_c M\square_c M \arrow[r] & E\square_c M\square_c M \oplus \Omega_c^{-1}(X)\square_c M \arrow[r] & \Omega_c^{-1}(X)\square_c M \arrow[r] & 0
\end{tikzcd}$$

and so we obtain

$$\begin{aligned} & id_\Gamma(E\square_c M \oplus \Omega_c^{-1}(X) \to E\square_c M\square_c M \oplus \Omega_c^{-1}(X)\square_c M) \\ \leq \ & \max\{id_\Gamma Z(E\square_c M), id_\Gamma Z(\Omega_c^{-1}(X))\}. \end{aligned}$$

It follows from $id_\Gamma Z(E\square_c M) \leq id_\Gamma Z(M)$ that

$$id_\Gamma Z(X) \leq 1 + \max\{id_\Gamma Z(\Omega_c^{-1}(X)), id_\Gamma Z(M)\}.$$

Proposition 3.3 Let C be a semiperfect coalgebra, M be a C-bicomodule and $\Gamma = C \ltimes M$ be a trivial extension of C by M. Then

$$gl.\dim\Gamma \leq gl.\dim C + id_\Gamma Z(M) + 1.$$

Proof Let X be a right C-comodule. We will first prove that

$$id_\Gamma Z(X) \leq id_c X + id_\Gamma Z(M) + 1.$$

If $id_c X = \infty$, then the result follows.

Assume that $id_c X = n$. Applying Lemma 3.2 first to $Z(X)$ and then to $Z(\Omega_c^{-1}(X))$, we get

$$id_\Gamma Z(X) \leq 2 + \max\{id_\Gamma Z(\Omega_c^{-2}(X)), id_\Gamma Z(M)\}.$$

Continuing in this fashion, we obtain

$$id_\Gamma Z(X) \leq n + \max\{id_\Gamma Z(\Omega_c^{-n}(X)), id_\Gamma Z(M)\},$$

$$id_\Gamma Z(\Omega_c^{-n}(X)) \leq 1 + \max\{id_\Gamma Z(\Omega_c^{-(n+1)}(X)), id_\Gamma Z(M)\} = 1 + id_\Gamma Z(M).$$

Hence

$$id_\Gamma Z(X) \leq n + 1 + id_\Gamma Z(M) = id_c X + id_\Gamma Z(M) + 1.$$

By the definition of global dimension of a coalgebra, we have $id_\Gamma Z(X) \leq gl.\mathrm{dim}C + id_\Gamma Z(M) + 1$. Furthermore, we get

$$gl.\mathrm{dim}\Gamma \leq gl.\mathrm{dim}C + id_\Gamma Z(M) + 1.$$

Corollary 3.4 Let C be a semiperfect coalgebra, M be a C-bicomodule and $\Gamma = C \ltimes M$ be a trivial extension of C by M. If $id_c M = id_\Gamma Z(M)$, then

$$gl.\mathrm{dim}\Gamma \leq 2 \cdot gl.\mathrm{dim}C + 1.$$

Proof It follows from $id_\Gamma Z(M) = id_c M \leq gl.dimC$ and Proposition 3.3.

4 Tilting Comodules over $\Gamma = C \ltimes M$

In this section, we shall study tilting comodules over the trivial extension of a coalgebra C by a bicomodule M.

Lemma 4.1 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. If $X \in \mathcal{M}^C$ and $0 \to X \to I_0 \to I_1 \to I_2 \to \cdots$ is an injective resolution of X in $\mathcal{M}^C$, then $0 \to X\square_c\Gamma \to I_0\square_c\Gamma \to I_1\square_c\Gamma \to I_2\square_c\Gamma \to \cdots$ is an injective resolution of $X\square_c\Gamma$ in $\mathcal{M}^\Gamma$.

Proof It follows from [15, Proposition 1] that $I_i\square_c\Gamma$ is an injective Γ-comodule.

Theorem 4.2 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. If X is a tilting right C-comodule and $X\square_c M$ is cogenerated by X, then $X\square_c\Gamma$ is a tilting right Γ-comodule.

Proof Firstly, since X is a tilting right C-comodule, it follows that inj.dim$X \leq 1$. Then we get an injective resolution of X as follows.

$$0 \to X \to I_0 \to I_1 \to 0 \tag{4.1}$$

where I_0, I_1 are injective right C-comodules. Hence, we have an exact sequence

$$0 \to X\square_c\Gamma \to I_0\square_c\Gamma \to I_1\square_c\Gamma \to 0 \tag{4.2}$$

which is an injective resolution of $X\square_c\Gamma$, where $I_0\square_c\Gamma$ and $I_1\square_c\Gamma$ are injective Γ-comodules. Thus, inj.dim$(X\square_c\Gamma) \leq 1$.

Secondly, applying the functor $\mathrm{Com}_\Gamma(X\square_c\Gamma, -)$ on the exact sequence (4.2), we have the following long exact sequence

$$\begin{aligned} 0 \to \mathrm{Com}_\Gamma(X\square_c\Gamma, X\square_c\Gamma) &\to \mathrm{Com}_\Gamma(X\square_c\Gamma, I_0\square_c\Gamma) \to \mathrm{Com}_\Gamma(X\square_c\Gamma, I_1\square_c\Gamma) \to \\ \mathrm{Ext}^1_\Gamma(X\square_c\Gamma, X\square_c\Gamma) &\to \mathrm{Ext}^1_\Gamma(X\square_c\Gamma, I_0\square_c\Gamma) \to \mathrm{Ext}^1_\Gamma(X\square_c\Gamma, I_1\square_c\Gamma) \to \cdots \end{aligned}$$

Since $I_0\Box_c\Gamma$ and $I_1\Box_c\Gamma$ are both injective Γ-comodules, we obtain that

$$Ext^1_\Gamma(X\Box_c\Gamma, I_0\Box_c\Gamma) = Ext^1_\Gamma(X\Box_c\Gamma, I_1\Box_c\Gamma) = 0.$$

By the adjoint isomorphism, we have

$$\begin{aligned}
\mathrm{Com}_\Gamma(X\Box_c\Gamma, X\Box_c\Gamma) &\cong \mathrm{Com}_c(h_\Gamma(\Gamma, X\Box_c\Gamma), X) \cong \mathrm{Com}_c(X\Box_c\Gamma, X),\\
\mathrm{Com}_\Gamma(X\Box_c\Gamma, I_0\Box_c\Gamma) &\cong \mathrm{Com}_c(h_\Gamma(\Gamma, X\Box_c\Gamma), I_0) \cong \mathrm{Com}_c(X\Box_c\Gamma, I_0),\\
\mathrm{Com}_\Gamma(X\Box_c\Gamma, I_1\Box_c\Gamma) &\cong \mathrm{Com}_c(h_\Gamma(\Gamma, X\Box_c\Gamma), I_1) \cong \mathrm{Com}_c(X\Box_c\Gamma, I_1).
\end{aligned}$$

Applying the functor $\mathrm{Com}_c(X\Box_c\Gamma, -)$ on the exact sequence (4.1), we have the following long exact sequence

$$\begin{aligned}
0 \to \mathrm{Com}_c(X\Box_c\Gamma, X) \to \mathrm{Com}_c(X\Box_c\Gamma, I_0) \to \mathrm{Com}_c(X\Box_c\Gamma, I_1) \to\\
\mathrm{Ext}^1_c(X\Box_c\Gamma, X) \to \mathrm{Ext}^1_c(X\Box_c\Gamma, I_0) \to \mathrm{Ext}^1_c(X\Box_c\Gamma, I_1) \to \cdots
\end{aligned}$$

Because I_0 and I_1 are injective, we obtain that $Ext^1_c(X\Box_c\Gamma, I_0) = Ext^1_c(X\Box_c\Gamma, I_1) = 0$. Next, it suffices to prove that $Ext^1_c(X\Box_c\Gamma, X) = 0$. Since $inj.\mathrm{dim}X \leq 1$, it follows by [16, Corollary 2.12] that there exist the following isomorphisms:

$$\begin{aligned}
Ext^1_c(X\Box_c\Gamma, X) &\cong D\mathrm{Com}_c(\tau^{-1}X, X\Box_c\Gamma),\\
D\mathrm{Com}_c(\tau^{-1}X, X) &\cong Ext^1_c(X, X) = 0,\\
D\mathrm{Com}_c(\tau^{-1}X, X\Box_c M) &\cong Ext^1_c(X\Box_c M, X) = 0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
D\mathrm{Com}_c(\tau^{-1}X, X\Box_c\Gamma) &= D\mathrm{Com}_c(\tau^{-1}X, X \oplus (X\Box_c M))\\
&= D\mathrm{Com}_c(\tau^{-1}X, X)\Pi D\mathrm{Com}_c(\tau^{-1}X, X\Box_c M) = 0.
\end{aligned}$$

That is, $Ext^1_c(X\Box_c\Gamma, X) = 0$ and we know that the functor $\mathrm{Com}_c(X\Box_c\Gamma, -)$ leaves the sequence (4.1) exact. Hence, we obtain that the functor $\mathrm{Com}_\Gamma(X\Box_c\Gamma, -)$ keeps the sequence (4.2) exact. Indeed, by the above adjoint isomorphism and the functor $\mathrm{Com}_c(X\Box_c\Gamma, -)$ leaves the sequence (4.1) exact, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Com}_\Gamma(X\Box_c\Gamma, I_0\Box_c\Gamma) \arrow[r] \arrow[d, "\cong"] & \mathrm{Com}_\Gamma(X\Box_c\Gamma, I_1\Box_c\Gamma) \arrow[r] \arrow[d, "\cong"] & Ext^1_\Gamma(X\Box_c\Gamma, X\Box_c\Gamma) \arrow[d] \\
\mathrm{Com}_c(X\Box_c\Gamma, I_1) \arrow[r] & \mathrm{Com}_c(X\Box_c\Gamma, I_0) \arrow[r] & 0
\end{tikzcd}$$

It is easy to get the following exact sequence

$$0 \to \mathrm{Com}_\Gamma(X\Box_c\Gamma, X\Box_c\Gamma) \to \mathrm{Com}_\Gamma(X\Box_c\Gamma, I_0\Box_c\Gamma) \to \mathrm{Com}_\Gamma(X\Box_c\Gamma, I_1\Box_c\Gamma) \to 0.$$

So $Ext^1_\Gamma(X\Box_c\Gamma, X\Box_c\Gamma) = 0$.

Finally, since X is a right C-comodule, there exists an exact sequence

$$0 \to X_2 \to X_1 \to C \to 0,$$

where $X_1, X_2 \in AddX$. Applying the exact functor $-\Box_c\Gamma$, we get the short exact sequence

$$0 \to X_2\Box_c\Gamma \to X_1\Box_c\Gamma \to C\Box_c\Gamma \to 0,$$

where $X_1\Box_c\Gamma, X_2\Box_c\Gamma \in Add(X\Box_c\Gamma)$.

Therefore, $X\Box_c\Gamma$ is a tilting Γ-comodule.

Lemma 4.3 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. Then the functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is exact.

Proof Assume that there is an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{M}^C$. Applying the exact functor $-\Box_c\Gamma$ on the above sequence, we have that the exact sequence

$$0 \to X\Box_c\Gamma \to Y\Box_c\Gamma \to Z\Box_c\Gamma \to 0$$

is equivalent to the exact sequence

$$0 \to X\Box_c M \oplus X \to Y\Box_c M \oplus Y \to Z\Box_c M \oplus Z \to 0.$$

By Proposition 2.6, we get an exact sequence in $(-\Box_c M) \rtimes \mathcal{M}^C$:

$$\begin{tikzcd} X\Box_c M \oplus X \arrow[r, tail] \arrow[d] & Y\Box_c M \oplus Y \arrow[r, two heads] \arrow[d] & Z\Box_c M \oplus Z \arrow[d] \\ X\Box_c M\Box_c M \oplus X\Box_c M \arrow[r, tail] & Y\Box_c M\Box_c M \oplus Y\Box_c M \arrow[r, two heads] & Z\Box_c M\Box_c M \oplus Z\Box_c M. \end{tikzcd}$$

That is, we obtain a short exact sequence $0 \to H(X) \to H(Y) \to H(Z) \to 0$ in $(-\Box_c M) \rtimes \mathcal{M}^C$. Thus, the functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is exact.

Lemma 4.4 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. If the functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is fully faithfull and X is a tilting right C-comodule, then

(1) the sequence $0 \to H(X_2) \to H(X_1) \to H(C) \to 0$ is exact in $(-\Box_c M) \rtimes \mathcal{M}^C$ if and only if $0 \to X_2 \to X_1 \to C \to 0$ is exact in $\mathcal{M}^C$, where $X_i \in AddX$ for $i = 1, 2$;

(2) $Ext^1_\Gamma(H(X)^\Lambda, H(X)) = 0$ if and only if $Ext^1_c(X^\Lambda, X) = 0$;

(3) inj.dim$H(X) =$ inj.dim$X \le 1$

Proof (1) The proof follows from that H and U are exact functors.

(2) By the assumption, X is a tilting right C-comodule and inj.dim$X \le 1$. Then there exists an injective resolution

$$0 \to X \to E_0 \to E_1 \to 0. \tag{4.3}$$

of X, where E_0, E_1 are injective right C-comodules. If $Ext^1_c(X^\Lambda, X) = 0$, then applying covariant functor $\mathrm{Com}_c(X^\Lambda, -)$ on the exact sequence (4.3), we get the exact sequence

$$0 \to \mathrm{Com}_c(X^\Lambda, X) \to \mathrm{Com}_c(X^\Lambda, E_0) \to \mathrm{Com}_c(X^\Lambda, E_1) \to 0.$$

Applying the exact functor H on the sequence (4.3), there exists an injective resolution

$$0 \to H(X) \to H(E_0) \to H(E_1) \to 0 \tag{4.4}$$

of $H(X)$ in $(-\square_c M) \rtimes \mathcal{M}^C$ since the functor H preserves injective comodules. Applying the covariant functor $\mathrm{Com}_\Gamma(H(X)^\Lambda, -)$ on the sequence (4.4), we have the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Com}_\Gamma(H(X)^\Lambda, H(X)) \to \mathrm{Com}_\Gamma(H(X)^\Lambda, H(E_0)) \to \mathrm{Com}_\Gamma(H(X)^\Lambda, H(E_1)) \to \\ \mathrm{Ext}^1_\Gamma(H(X)^\Lambda, H(X)) \to \mathrm{Ext}^1_\Gamma(H(X)^\Lambda, H(E_0)) \to \mathrm{Ext}^1_\Gamma(H(X)^\Lambda, H(E_1)) \to \cdots \end{aligned}$$

Since $H(E_0)$ and $H(E_1)$ are injective, $Ext^1_\Gamma(H(X)^\Lambda, H(E_0)) = Ext^1_\Gamma(H(X)^\Lambda, H(E_1)) = 0$. We know that $Com_c(X^\Lambda, E_0) \to \mathrm{Com}_c(X^\Lambda, E_1)$ is an epimorphism and the functor H is fully faithful, then $\mathrm{Com}_\Gamma(H(X)^\Lambda, H(E_0)) \to \mathrm{Com}_\Gamma(H(X)^\Lambda, H(E_1))$ is also an epimorphism. Thus $Ext^1_\Gamma(H(X)^\Lambda, H(X)) = 0$. Similarly, if $Ext^1_\Gamma(H(X)^\Lambda, H(X)) = 0$, then $Ext^1_c(X^\Lambda, X) = 0$ since the functor U is exact.

(3) Firstly, we prove that $\mathrm{inj.dim} H(X) \leq \mathrm{inj.dim} X$. By the assumption, X is a tilting right C-comodule and $\mathrm{inj.dim} X \leq 1$. Then there exists an injective resolution $0 \to X \to E_0 \to E_1 \to 0$ of X, where E_0, E_1 are injective right C-comodules. Since the functor H is exact, it follows that $\mathrm{inj.dim} H(X) \leq \mathrm{inj.dim} X$.

Next, we prove that $\mathrm{inj.dim} X \leq \mathrm{inj.dim} H(X)$. Assume that $\mathrm{inj.dim} H(X) = n < \infty$. There is a short exact sequence

$$0 \to X \xrightarrow{\alpha_0} E_0 \to K_0 \to 0,$$

where E_0 is an injective envelope of X and $K_0 = coker\alpha_0$. Then we have an exact sequence

$$0 \to H(X) \to H(E_0) \to H(K_0) \to 0$$

since the functor H is exact. Continuing in this fashion, we could take a monomorphism $\alpha_1 : K_0 \to E_1$, where E_1 is an injective envelope of K_0 and $K_1 = coker\alpha_1$. Applying the exact functor H on the exact sequence

$$0 \to X \xrightarrow{\alpha_0} E_0 \to E_1 \to K_1 \to 0.$$

We obtain an exact sequence

$$0 \to H(X) \xrightarrow{H(\alpha_0)} H(E_0) \to H(E_1) \to H(K_1) \to 0.$$

Since $\mathrm{inj.dim} H(X) = n < \infty$, till step n we obtain an exact sequence

$$0 \to H(X) \to H(E_0) \to H(E_1) \to \cdots \to H(E_{n-1}) \to H(K_{n-1}) \to 0,$$

where $H(E_{n-1})$ is an injective right Γ-comodule. Then applying the functor U on the above sequence, we get an exact sequence

$$0 \to X \to E_0 \to E_1 \to \cdots \to E_{n-1} \to K_{n-1} \to 0.$$

Thus, we have $\text{inj.dim}X \leq n = \text{inj.dim}H(X)$. So $\text{inj.dim}X = \text{inj.dim}H(X)$.

Theorem 4.5 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. If the functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is fully faithfull, then X is a tilting right C-comodule if and only if $H(X)$ is a tilting right Γ-comodule.

Proof By Lemma 4.4, the sufficiency is easy to know.

Conversely, we assume that X is a right C-comodule and $H(X)$ is a tilting right Γ-comodule. By the assumption, there exists an exact sequence $0 \to H(X_2) \to H(X_1) \to H(C) \to 0$ in $(-\Box_c M) \rtimes \mathcal{M}^C$, where $X_i \in AddX$ for $i = 1, 2$. Since the functor U is exact, we obtain that there exists an exact sequence $0 \to X_2 \to X_1 \to C \to 0$ in $\mathcal{M}^C$.

By the assumption, $\text{inj.dim}H(X) \leq 1$ and the exact functor H preserves injectives, it follows that there is an exact sequence

$$0 \to H(X) \to H(E_0) \to H(E_1) \to 0$$

in $(-\Box_c M) \rtimes \mathcal{M}^C$, where E_0 and E_1 are injective in $\mathcal{M}^C$. Then applying the exact functor U on the above sequence, we get an exact sequence

$$0 \to X \to E_0 \to E_1 \to 0$$

in $\mathcal{M}^C$. Thus, we have $\text{inj.dim}X \leq 1$.

Since $H(X)$ is a tilting right Γ-comodule, we have $Ext^1_\Gamma(H(X)^\Lambda, H(X)) = 0$. It is easy to obtain that $Com_\Gamma(H(X)^\Lambda, H(E_0)) \to \mathrm{Com}_\Gamma(H(X)^\Lambda, H(E_1))$ is an epimorphism. Since the functor H is fully faithful, U is exact and left adjoint to H, we have that $\mathrm{Com}_c(X^\Lambda, E_0) \to \mathrm{Com}_c(X^\Lambda, E_1)$ is also an epimorphism. Thus, we have $Ext^1_c(X^\Lambda, X) = 0$.

Lemma 4.6 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. Then

(1) If $L \in CogenX$, then $L\Box_c\Gamma \in Cogen(X\Box_c\Gamma)$;

(2) If $L \in CogenX$, then $H(L) \in CogenH(X)$.

Proof (1) Since $L \in CogenX$, there exists an index set Λ such that the sequence $0 \to L \to X^\Lambda$ is exact. Then we get an exact sequence $0 \to L\Box_c\Gamma \to X^\Lambda\Box_c\Gamma$ since the functor $-\Box_c\Gamma$ is exact. Since the functor $-\Box_c\Gamma$ preserves products, it follows that the sequence $0 \to L\Box_c\Gamma \to (X\Box_c\Gamma)^\Lambda$ is also exact. Thus, $L\Box_c\Gamma \in Cogen(X\Box_c\Gamma)$.

(2) If $L \in CogenX$, then there exists an index set Λ such that the sequence $0 \to L \to X^\Lambda$ is exact. Applying the exact functor H, we get an exact sequence $0 \to H(L) \to H(X^\Lambda)$. The functor H is left exact and preserves products, since it has left adjoint. It follows that the sequence $0 \to H(L) \to H(X)^\Lambda$ is exact. Thus, $H(L) \in CogenH(X)$.

Theorem 4.7 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. The functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is fully faithfull and $X\Box_c M$ is cogenerated by X. Then

(1) X is a tilting injective right C-comodule if and only if $H(X)$ is a tilting injective right Γ-comodule;

(2) If X is a tilting injective right C-comodule, then $X\Box_c\Gamma$ is a tilting injective right Γ-comodule.

Proof (1) X is a tilting injective right C-comodule if and only if $Ext_c^1(L,X)=0$ for any $L\in CogenX$ (see [6, Proposition 3.2]). By Lemma 4.6 and Lemma 4.4(2), for any $L\in CogenX$, $Ext_c^1(L,X)=0$ if and only if $Ext_\Gamma^1(H(L),H(X))=0$ for any $H(L)\in CogenH(X)$. And by [6, Proposition 3.2] again, we know that X is a tilting injective right C-comodule if and only if $H(X)$ is a tilting injective right Γ-comodule.

(2) It follows from Theorem 4.2 that if T is a tilting right C-comodule, then $T\Box_c\Gamma$ is a tilting right Γ-comodule. Since X is a tilting injective right C-comodule, it follows that the functor $\mathrm{Com}_c(-,X)$ leaves exact in $CogenT$. It suffices to prove that the functor $\mathrm{Com}_\Gamma(-,X\Box_c\Gamma)$ leaves exact in $Cogen(T\Box_c\Gamma)$. We take an exact sequence

$$0\to D\to E\to F\to 0 \tag{4.5}$$

of right Γ-comodules in $Cogen(T\Box_c\Gamma)$. Since $Cogen(T\Box_c\Gamma)=kerExt_\Gamma^1(-,T\Box_c\Gamma)$, we obtain

$$Ext_\Gamma^1(D,T\Box_c\Gamma)\cong Ext_\Gamma^1(E,T\Box_c\Gamma)\cong Ext_\Gamma^1(F,T\Box_c\Gamma)=0.$$

Applying the functor $\mathrm{Com}_\Gamma(-,T\Box_c\Gamma)$ on the sequence (4.5), we obtain a short exact sequence

$$0\to \mathrm{Com}_\Gamma(F,T\Box_c\Gamma)\to \mathrm{Com}_\Gamma(E,T\Box_c\Gamma)\to \mathrm{Com}_\Gamma(D,T\Box_c\Gamma)\to 0.$$

It follows from adjoint isomorphism that

$$\begin{aligned}\mathrm{Com}_\Gamma(D,T\Box_c\Gamma)&\cong \mathrm{Com}_c(h_\Gamma(\Gamma,D),T)\cong \mathrm{Com}_c(D,T),\\ \mathrm{Com}_\Gamma(E,T\Box_c\Gamma)&\cong \mathrm{Com}_c(h_\Gamma(\Gamma,E),T)\cong \mathrm{Com}_c(E,T),\\ \mathrm{Com}_\Gamma(F,T\Box_c\Gamma)&\cong \mathrm{Com}_c(h_\Gamma(\Gamma,F),T)\cong \mathrm{Com}_c(F,T).\end{aligned}$$

Thus, we get a short exact sequence

$$0\to \mathrm{Com}_c(F,T)\to \mathrm{Com}_c(E,T)\to \mathrm{Com}_c(D,T)\to 0$$

where $D,E,F\in CogenT$. Applying the functor $\mathrm{Com}_\Gamma(-,X\Box_c\Gamma)$ on the sequence (4.5), we have the long exact sequence

$$\begin{aligned}0\to&\mathrm{Com}_\Gamma(F,X\Box_c\Gamma)\to \mathrm{Com}_\Gamma(E,X\Box_c\Gamma)\to \mathrm{Com}_\Gamma(D,X\Box_c\Gamma)\to\\ &Ext_\Gamma^1(F,X\Box_c\Gamma)\to Ext_\Gamma^1(E,X\Box_c\Gamma)\to Ext_\Gamma^1(D,X\Box_c\Gamma)\to\cdots\end{aligned}$$

It follows from adjoint isomorphism that

$$\begin{aligned}\mathrm{Com}_\Gamma(D,X\Box_c\Gamma)&\cong \mathrm{Com}_c(h_\Gamma(\Gamma,D),X)\cong \mathrm{Com}_c(D,X),\\ \mathrm{Com}_\Gamma(E,X\Box_c\Gamma)&\cong \mathrm{Com}_c(h_\Gamma(\Gamma,E),X)\cong \mathrm{Com}_c(E,X),\\ \mathrm{Com}_\Gamma(F,X\Box_c\Gamma)&\cong \mathrm{Com}_c(h_\Gamma(\Gamma,F),X)\cong \mathrm{Com}_c(F,X).\end{aligned}$$

Since the functor $\mathrm{Com}_c(-,X)$ leaves exact in $CogenT$, it follows that $\mathrm{Com}_\Gamma(-,X\Box_c\Gamma)$ leaves the sequence (4.5) exact. That is, $\mathrm{Com}_\Gamma(-,X\Box_c\Gamma)$ leaves the sequence exact in $Cogen(T\Box_c\Gamma)$. Thus, $X\Box_c\Gamma$ is a tilting injective right Γ-comodule relative to $T\Box_c\Gamma$.

5 Gorenstein Injective Comodules over $\Gamma = C \ltimes M$

In this section, we construct Gorenstein injective comodules over $\Gamma = C \ltimes M$ and obtain the equivalent condition for a comodule to be a Gorenstein injective Γ-comodule.

Proposition 5.1 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. If X is a Gorenstein injective right C-comodule, then $X\Box_c\Gamma$ is a Gorenstein injective right Γ-comodule.

Proof By the assumption, X is a Gorenstein injective right C-comodule, then there exists an exact sequence

$$\mathcal{E} \equiv \cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective right C-comodules with $X \cong ker(E^0 \to E^1)$ and such that the functor $\mathrm{Com}_c(E,-)$ leaves it exact for any injective right C-comodule E. Applying the exact functor $-\Box_c\Gamma$ on the sequence $\mathcal{E}$, we get the following exact sequence:

$$\mathcal{E}' \equiv \cdots \to E^{-2}\Box_c\Gamma \to E^{-1}\Box_c\Gamma \to E^0\Box_c\Gamma \to E^1\Box_c\Gamma \to \cdots$$

where each $E^i\Box_c\Gamma$ is an injective Γ-comodule. Next, we prove that the functor $\mathrm{Com}_\Gamma(W,-)$ is applied on the sequence $\mathcal{E}'$ and leaves exact for any injective Γ-comodule W. It follows from adjoint isomorphism that

$$\mathrm{Com}_\Gamma(W, E^i\Box_c\Gamma) \cong \mathrm{Com}_C(h_\Gamma(\Gamma, W), E^i) \cong \mathrm{Com}_C(W, E^i).$$

Since W is an injective Γ-comodule, W can be represented by $\omega : W \to W\Box_cM$. By the definition of an injective comodule, if we have a monomorphism $\gamma : \alpha \to \beta$, where $\alpha : N \to N\Box_cM$ and $\beta : N' \to N'\Box_cM$, then for any morphism $\sigma : \alpha \to \omega$, there exists a morphism $\tau : \beta \to \omega$, such that $\sigma = \tau\gamma$. It is easy to see that W is also an injective C-comodule. Therefore, $\mathrm{Com}_\Gamma(W,-)$ applied on the sequence $\mathcal{E}'$ leaves exact since the functor $\mathrm{Com}_c(W,-)$ leaves $\mathcal{E}$ exact. Finally, since $X \cong ker(E^0 \to E^1)$, it follows that $X\Box_c\Gamma \cong ker(E^0 \to E^1)\Box_c\Gamma \cong ker(E^0\Box_c\Gamma \to E^1\Box_c\Gamma)$. Thus $X\Box_c\Gamma$ is a Gorenstein injective Γ-comodule.

Proposition 5.2 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. Then X is a Gorenstein injective right C-comodule if and only if $H(X)$ is a Gorenstein injective right Γ-comodule.

Proof Let X be a Gorenstein injective right C-comodule, then there exists an exact sequence

$$\mathcal{E} \equiv \cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective right C-comodules with $X \cong ker(E^0 \to E^1)$ and such that the functor $\mathrm{Com}_c(E,-)$ leaves it exact for any injective right C-comodule E. By Lemma 4.3 and Proposition 2.9, we have an exact sequence

$$H(\mathcal{E}) \equiv \cdots \to H(E^{-2}) \to H(E^{-1}) \to H(E^0) \to H(E^1) \to \cdots$$

where each $H(E^i)$ is injective in $(-\Box_c M) \rtimes \mathcal{M}^C$ and $H(X) \cong H(ker(E^0 \to E^1))$, since the functor $H : \mathcal{M}^C \to (-\Box_c M) \rtimes \mathcal{M}^C$ is exact and preserves injectivity.

Next, we prove that the functor $\mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(\epsilon,-)$ preserves the sequence $H(\mathcal{E})$ exact for any injective object $\epsilon \in (-\Box_c M) \rtimes \mathcal{M}^C$. It follows from Proposition 2.9 that

$$\begin{aligned}\mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(\epsilon, H(\mathcal{E})) &\cong \mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(H(K(\epsilon)), H(\mathcal{E}))\\ &\cong \mathrm{Com}_{\mathcal{M}^C}(UH(K(\epsilon)), \mathcal{E}) \cong \mathrm{Com}_{\mathcal{M}^C}(K(\epsilon), \mathcal{E}).\end{aligned}$$

Since $\mathrm{Com}_{\mathcal{M}^C}(K(\epsilon), \mathcal{E})$ is exact, where $K(\epsilon)$ in $\mathcal{M}^C$ is injective, $\mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(\epsilon, H(\mathcal{E}))$ is exact. Thus, $H(X) \in (-\Box_c M) \rtimes \mathcal{M}^C$ is Gorenstein injective.

Conversely, if $H(X) \in (-\Box_c M) \rtimes \mathcal{M}^C$ is Gorenstein injective, there exists an exact sequence of injective comodules:

$$\mathcal{F} \equiv \cdots \to H(E^{-2}) \to H(E^{-1}) \to H(E^0) \to H(E^1) \to \cdots$$

where each $H(E^i)$ is an injective right Γ-comodule with $H(X) \cong ker(H(E^0) \to H(E^1))$ and such that the functor $\mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(\epsilon,-)$ leaves it exact for any injective object $\epsilon \in (-\Box_c M) \rtimes \mathcal{M}^C$. Applying the exact functor U on the sequence $\mathcal{F}$, we obtain an exact sequence

$$U(\mathcal{F}) \equiv \cdots \to E^{-2} \to E^{-1} \to E^0 \to E^1 \to \cdots$$

and $X \cong ker(E^0 \to E^1)$. Next, we prove that the functor $\mathrm{Com}_{\mathcal{M}^C}(K(\epsilon),-)$ preserves the sequence $U(\mathcal{F})$ exact for any injective object $\epsilon \in (-\Box_c M) \rtimes \mathcal{M}^C$ and so $K(\epsilon) \in \mathcal{M}^C$ is injective. By Proposition 2.9, we have

$$\mathrm{Com}_{\mathcal{M}^C}(K(\epsilon), U(\mathcal{F})) \cong \mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(H(K(\epsilon)), \mathcal{F}) \cong \mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(\epsilon, \mathcal{F}).$$

Since $\mathrm{Com}_{(-\Box_c M)\rtimes\mathcal{M}^C}(\epsilon, \mathcal{F})$ is exact, it follows that $\mathrm{Com}_{\mathcal{M}^C}(K(\epsilon), U(\mathcal{F}))$ is also exact. Thus, X is a Gorenstein injective right C-comodule.

Corollary 5.3 Let C be a semiperfect coalgebra, M be a coflat left C-comodule and $\Gamma = C \ltimes M$. Then $X\Box_c\Gamma$ is a Gorenstein injective right Γ-comodule if and only if $X\Box_c\Gamma$ is a Gorenstein coflat right Γ-comodule.

Proof It follows from [13, Proposition 3.4] and Lemma 3.1.

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平凡扩张余代数上的倾斜余模

李烨暄，姚海楼

(北京工业大学理学部数学学院, 北京 100124)

摘要: 本文研究了平凡扩张余代数上的倾斜余模. 在倾斜理论的基础上, 首先得到了平凡扩张余代数整体维数的上界, 然后获得了平凡扩张余代数上的倾斜余模的等价条件. 这些结果推广了倾斜模的结论.

关键词: 平凡扩张余代数; 倾斜余模; Gorenstein 内射余模; 总体维数

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