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INTEGRABILITY AND BOUNDEDNESS OF MINIMIZERS FOR INTEGRAL FUNCTIONAL OF HÖRMANDER'S VECTOR FIELDS

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Abstract: The integral functional of Hörmander's vector fields is considered, by virtue of the Sobolev inequality related to Hörmander's vector fields and the iteration formula of Stampacchia, it is proved that the minimizers of integral functional have higher integrability with the boundary data allowing the higher integrability. Moreover, the $L^1(\Omega)$ and $L^\infty(\Omega)$ boundedness of minimizers are also given, which extends the results of Leonetti and Siepe[12] and Leonetti and Petricca[13] from Euclidean spaces to Hörmander's vector fields.

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1 Introduction

We consider the integral functional of Hörmander's vector fields

$$I(u)=\int_\Omega f(x,Xu(x))dx, x\in\Omega, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n(n\geq 3)$ is a bounded open set, $X=\{X_1,\cdots,X_m\}$ $(m\geq n)$ are C^∞ vector fields in Ω satisfying the Hörmander's finite rank condition[11], rank Lie$[X_1,\cdots,X_m]=n$, where $X_j=\sum\limits_{i=1}^{n}b_{ij}(x)\frac{\partial}{\partial x_i}, b_{ij}(x)\in C^\infty(\Omega), j=1,\ \cdots,m$. Note that, when $f(x,z)$ in (1.1) is a Carathéodory function and satisfies the standard growth condition $|z|^p\leq f(x,z)\leq c(1+|z|^p), 1<p<Q$, where c is a positive constant, Q is homogeneous dimension of Ω relative to $\{X_1,\cdots,X_m\}$ and $Q\geq n$, Xu[16] proved the existence of minimizers of (1.1)

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by direct method and obtained Hölder continuity by Moser's method. Furthermore, Xu[17] obtained C^∞ continuity by similar method. Afterwards, Giannetti[7] obtained higher integrability of the minimizers of (1.1) under the growth condition

$$|f(\xi)-f(\eta)| \leq c\,|\xi-\eta|\left(|\xi|^{p-1}+|\xi-\eta|^{p-1}\right), 1<p<Q.$$

Namely, he proved $u \in W^{1,p}_{loc}(\Omega)$ if $u \in W^{1,r}_{loc}(\Omega), r_1 \leq r < p, \max\{1, p-1\} < r_1 < p$. When $m = n$, $Xu = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$, (1.1) is an integral function in Euclidean spaces, the study on the regularity and boundedness of minimizers of such integral function can also be seen in [4, 8, 10].

In this paper we assume that $f(x,z)$ in (1.1) is a Carathéodory function and satisfies the standard growth condition

$$c_1|z|^p - g_1(x) \leq f(x,z) \leq c_2|z|^p + g_2(x), 1<p<Q, \tag{1.2}$$

where c_1 and c_2 are positive constants, functions $g_1(x), g_2(x) \in L^{\frac{1}{1-b}}(\Omega), b = \frac{t-p}{t}, 1 < p < t \leq q \leq Q$. By virtue of the Sobolev inequality (see Lemma 2.5) related to Hörmander's vector fields and the iteration formula of Stampacchia (see Lemma 2.6), it is proved that the minimizers of integral functional have higher integrability with the boundary data allowing the higher integrability. Moreover, the $L^1(\Omega)$ and $L^\infty(\Omega)$ boundedness of minimizers are also given, which extends the results of Leonetti and Siepe[12] and Leonetti and Petricca[13] from Euclidean spaces to Hörmander's vector fields. Throughout this paper c denotes the different positive constants in different places.

2 Main Results and Preliminary Knowledge

Definition 2.1[3, 6] For any $1 < p < Q$, Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega), X_j u \in L^p(\Omega), j = 1, \cdots, m\},$$

whose norm is $\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|u|^p + |Xu|^p)\,dx\right)^{\frac{1}{p}}$, where $|Xu| = \left(\sum\limits_{j=1}^{m}(X_j u)^2\right)^{\frac{1}{2}}$. We also denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ by $W_0^{1,p}(\Omega)$, whose norm is $\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |Xu|^p dx\right)^{\frac{1}{p}}$.

Definition 2.2[1, 2, 9] The weak $L^p(\Omega)$ space which is also known as the Marcinkiewicz space, denoted by $L^p_{weak}(\Omega)$, is the set of all measurable functions $f(x)$ satisfying

$$meas\,\{x \in \Omega : |f(x)| > t_0\} \leq \frac{c}{{t_0}^p} \tag{2.1}$$

for any $t_0 > 0$ and some positive constants $c = c(f)$, where $meas\,E$ denotes the n-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$. If $f \in L^p_{weak}(\Omega)$, then $f \in L^{q_0}(\Omega)$ for any $1 \leq q_0 < p$. Moreover, when $p = \infty$, it follows that $L^\infty_{weak}(\Omega) = L^\infty(\Omega)$.

Definition 2.3 A function $u \in u_* + W_0^{1,p}(\Omega)$ is called a minimizer of (1.1) with the condition (1.2) if

$$\int_\Omega f(x, Xu)dx \le \int_\Omega f(x, Xw)dx \tag{2.2}$$

for any $w \in u_* + W_0^{1,p}(\Omega)$, where u_* denotes the boundary data.

In this paper, our mian results are sated as follows.

Theorem 2.4 Let $u \in u_* + W_0^{1,p}(\Omega)$ be a minimizer of (1.1) with the condition (1.2), where $u_* \in W^{1,q}(\Omega)$, $1 < p < q \le Q$. Suppose that $g_1(x), g_2(x) \in L^{\frac{1}{1-b}}(\Omega), b = \frac{t-p}{t}, p < t \le q$. Hence,

(i) (integrability) if $b < \frac{Q-p}{Q}$, then $u - u_* \in L_{weak}^{\frac{Qp}{Q-p-bQ}}(\Omega)$, where $\frac{Qp}{Q-p-bQ} > p^* = \frac{Qp}{Q-p} > p$;

(ii) ($L^1(\Omega)$ boundedness) if $b = \frac{Q-p}{Q}$, then there exists a positive constant θ such that $e^{\theta|u-u_*|} \in L^1(\Omega)$;

(iii) ($L^\infty(\Omega)$ boundedness) if $b > \frac{Q-p}{Q}$, then $u - u_* \in L^\infty(\Omega)$.

Inspired by Leonetti and Siepe[12], for a minimizer u of (1.1) with the condition (1.2), we can rewrite u as $u = u_* + (u - u_*)$, our aim is to prove when the boundary datum u_* has the higher integrability, $u - u_*$ also has the higher integrability. The following two lemmas are needed for the proof of Theorem 2.4.

Lemma 2.5[3, 6] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then for any $u \in W_0^{1,p}(\Omega), 1 < p < Q$, there holds $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, where $p^* = \frac{Qp}{Q-p}$, namely there exists a positive constant c such that

$$\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{1}{p^*}} \le c\left(\int_\Omega |Xu|^p dx\right)^{\frac{1}{p}}$$

for any $u \in W_0^{1,p}(\Omega), 1 < p < Q$.

Lemma 2.6[5, 14, 15] Let $\varphi(t)$ be a nonnegative and nonincreasing function on $[k_0, +\infty)$ satisfying $\varphi(h) \le \frac{c}{(h-k)^\alpha}[\varphi(k)]^\beta, h > k \ge k_0$, where c, α and β are positive constants. If $\beta < 1,\ k_0 > 0$, then

$$\varphi(h) \le \left[c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}}\varphi(k_0)\right] 2^{\frac{\alpha}{(1-\beta)^2}} \left(\frac{1}{h}\right)^{\frac{\alpha}{1-\beta}}; \tag{2.3}$$

if β=1, then

$$\varphi(h) \le e^{1-\tau(h-k_0)}\varphi(k_0), \tag{2.4}$$

where $\tau = (ec)^{-\frac{1}{\alpha}} > 0$; if $\beta > 1$, then

$$\varphi(k_0 + d) = 0, \tag{2.5}$$

where $d = c(\varphi(k_0))^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}$.

3 Proof of Theorem 2.4

Proof of Theorem 2.4 For any $k \in (0, +\infty)$, suppose that $T_k : \mathbb{R} \to \mathbb{R}$ is a function such that

$$T_k(u - u_*) = \begin{cases} u - u_*, & |u - u_*| \le k, \\ k\frac{u-u_*}{|u-u_*|}, & |u - u_*| > k, \end{cases} \tag{3.1}$$

setting $\psi = u - u_* - T_k(u - u_*)$, it follows from (3.1) that

$$\psi = \begin{cases} u - u_* + k, & u - u_* < -k, \\ 0, & -k \le u - u_* \le k, \\ u - u_* - k, & u - u_* > k. \end{cases} \tag{3.2}$$

Using $u \in u_* + W_0^{1,p}(\Omega)$ and (3.2) we have $\psi \in W_0^{1,p}(\Omega)$ and

$$X\psi = (Xu - Xu_*)1_{\{|u-u_*|>k\}}, \tag{3.3}$$

$$|\psi| = (|u - u_*| - k)\,1_{\{|u-u_*|>k\}}, \tag{3.4}$$

where $1_A(x) = 1$ if $x \in A$, $1_A(x) = 0$ if $x \notin A$. Let us consider

$$w = u - \psi, \tag{3.5}$$

which implies $w \in u_* + W_0^{1,p}(\Omega)$. By (3.3) and (3.5) we obtain

$$\begin{aligned} Xw &= Xu - X\psi = Xu - (Xu - Xu_*)1_{\{|u-u_*|>k\}} = Xu - (Xu)1_{\{|u-u_*|>k\}} + (Xu_*)1_{\{|u-u_*|>k\}} \\ &= (Xu)1_{\{|u-u_*|\le k\}} + (Xu)1_{\{|u-u_*|>k\}} - (Xu)1_{\{|u-u_*|>k\}} + (Xu_*)1_{\{|u-u_*|>k\}} \\ &= (Xu)1_{\{|u-u_*|\le k\}} + (Xu_*)1_{\{|u-u_*|>k\}}. \end{aligned} \tag{3.6}$$

Combining (2.2), (3.6) and $\Omega = \{|u - u_*| \le k\} \cup \{|u - u_*| > k\}$, it concludes

$$\begin{aligned} & \int_{\{|u-u_*|\le k\}} f(x, Xu)dx + \int_{\{|u-u_*|>k\}} f(x, Xu)dx \\ \le & \int_{\{|u-u_*|\le k\}} f(x, Xw)dx + \int_{\{|u-u_*|>k\}} f(x, Xw)dx \\ = & \int_{\{|u-u_*|\le k\}} f(x, Xu)dx + \int_{\{|u-u_*|>k\}} f(x, Xu_*)dx, \end{aligned} \tag{3.7}$$

and then by (3.7),

$$\int_{\{|u-u_*|>k\}} f(x, Xu)dx \le \int_{\{|u-u_*|>k\}} f(x, Xu_*)dx. \tag{3.8}$$

It follows from (1.2), (3.3), (3.8) and Lemma 2.5 that

$$\begin{aligned} & (\int_\Omega |\psi|^{p^*} dx)^{\frac{p}{p^*}} \le c\int_\Omega |X\psi|^p dx = c\int_{\{|u-u_*|>k\}} |Xu - Xu_*|^p dx \\ \le\ & 2^p c\int_{\{|u-u_*|>k\}} |Xu|^p dx + 2^p c\int_{\{|u-u_*|>k\}} |Xu_*|^p dx \\ \le\ & 2^p c[\int_{\{|u-u_*|>k\}} (f(x, Xu)dx + g_1)dx] + 2^p c[\int_{\{|u-u_*|>k\}} (f(x, Xu_*)dx + g_1)dx] \\ \le\ & 2^p c[\int_{\{|u-u_*|>k\}} (f(x, Xu_*)dx + g_1)dx] \ + 2^p c[\int_{\{|u-u_*|>k\}} (f(x, Xu_*)dx + g_1)dx] \\ \le\ & c[\int_{\{|u-u_*|>k\}} (|Xu_*|^p + g_1 + g_2)dx]. \end{aligned} \tag{3.9}$$

Since $p < q$, there exists a positive number t such that $p < t \leq q$, thus from (3.9), Hölder's inequality and Minkowski's inequality, we can get

$$\begin{aligned} & (\int_\Omega |\psi|^{p^*} dx)^{\frac{p}{p^*}} \qquad (3.10)\\ \leq \ & c(\int_{\{|u-u_*|>k\}} (|Xu_*|^p + g_1 + g_2)^{\frac{t}{p}} dx)^{\frac{p}{t}} [\text{meas}\{|u-u_*|>k\}]^{\frac{t-p}{t}} \\ \leq \ & c((\int_\Omega |Xu_*|^t dx)^{\frac{p}{t}} + (\int_\Omega (g_1)^{\frac{t}{p}} dx)^{\frac{p}{t}} + (\int_\Omega (g_2)^{\frac{t}{p}} dx)^{\frac{p}{t}}) \cdot [\text{meas}\{|u-u_*|>k\}]^{\frac{t-p}{t}}. \end{aligned}$$

Moreover, we can define a positive number $b = \frac{t-p}{t}$ in (3.10), namely we have $0 < b = 1 - \frac{p}{t} \leq 1 - \frac{p}{q} < 1$; and thus from $u_* \in W^{1,q}(\Omega)$ and $g_1, g_2 \in L^{\frac{1}{1-b}}(\Omega)$, we can set

$$M = \left(\int_\Omega |Xu_*|^{\frac{p}{1-b}} dx\right)^{1-b} + \left(\int_\Omega (g_1)^{\frac{1}{1-b}} dx\right)^{1-b} + \left(\int_\Omega (g_2)^{\frac{1}{1-b}} dx\right)^{1-b} < +\infty. \quad (3.11)$$

Finally we insert (3.11) into (3.10), we easily obtain

$$\int_\Omega |\psi|^{p^*} dx \leq cM^{\frac{p^*}{p}} [meas\,\{|u-u_*|>k\}]^{\frac{bp^*}{p}}. \quad (3.12)$$

For any $h > k \geq k_0$, it follows from (3.4) that

$$\begin{aligned} (h-k)^{p^*} [meas\,\{|u-u_*|>h\}] &= \int_{\{|u-u_*|>h\}} (h-k)^{p^*} dx \leq \int_{\{|u-u_*|>h\}} (|u-u_*|-k)^{p^*} dx \\ &\leq \int_{\{|u-u_*|>k\}} (|u-u_*|-k)^{p^*} dx = \int_\Omega |\psi|^{p^*} dx. \quad (3.13) \end{aligned}$$

Combining (3.12) and (3.13), it yields

$$meas\,\{|u-u_*|>h\} \leq \frac{cM^{\frac{p^*}{p}}}{(h-k)^{p^*}} [meas\,\{|u-u_*|>k\}]^{\frac{bp^*}{p}}. \quad (3.14)$$

In (3.14), setting

$$\varphi(h) = meas\,\{|u-u_*|>h\}\,, \varphi(k) = meas\,\{|u-u_*|>k\}\,, \alpha = p^*, \beta = \frac{bp^*}{p}. \quad (3.15)$$

We now apply Lemma 2.6 to (3.15). We can prove, respectively.

(i) If $b < \frac{p}{p^*} = \frac{p}{\frac{Qp}{Q-p}} = \frac{Q-p}{Q}$, then $\beta < 1$. For any $k_0 > 0$, it follows from (2.3) that

$$\begin{aligned} meas\,\{|u-u_*|>h\} &\leq \left[c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} meas\,\{|u-u_*|>k_0\}\right] 2^{\frac{\alpha}{(1-\beta)^2}} \left(\frac{1}{h}\right)^{\frac{\alpha}{1-\beta}} \\ &\leq \left[c^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}} meas\,\Omega\right] 2^{\frac{\alpha}{(1-\beta)^2}} \left(\frac{1}{h}\right)^{\frac{\alpha}{1-\beta}}, \quad (3.16) \end{aligned}$$

by (3.16) and (2.1), $u - u_* \in L^{\frac{\alpha}{1-\beta}}_{weak}(\Omega)$, where $\frac{\alpha}{1-\beta} = \frac{p^*}{1-\frac{bp^*}{p}} = \frac{Qp}{Q-p-bQ} > p^* = \frac{Qp}{Q-p}$.

(ii) If $b=\frac{Q-p}{Q}$, then $\beta=1$. It follows from (2.4) that

$$meas\left\{|u-u_*|>h\right\}\leq\ e^{1-\tau(h-k_0)}meas\left\{|u-u_*|>k_0\right\}, \tag{3.17}$$

where $\tau=(ec)^{-\frac{1}{p^*}}>0$. When $k_0\leq 0$, we have

$$e^{1-\tau(h-k_0)}=ee^{-\tau(h-k_0)}\leq ee^{-\tau h} \tag{3.18}$$

and

$$meas\left\{|u-u_*|>k_0\right\}=meas\,\Omega, \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17),

$$meas\left\{|u-u_*|>h\right\}\leq ee^{-\tau h}meas\ \Omega. \tag{3.20}$$

It is easy to see that there exists a positive constant $\theta<\tau$ satisfying

$$e^{\theta|u-u_*|}-1=\int_0^{|u-u_*|}\theta e^{\theta h}dh=\int_0^{\infty}\theta e^{\theta h}1_{\{|u-u_*|>h\}}dh. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\begin{aligned}\int_{\Omega}\left(e^{\theta|u-u_*|}-1\right)dx&=\int_{\Omega}\int_0^{\infty}\theta e^{\theta h}1_{\{|u-u_*|>h\}}dhdx=\int_{\Omega}1_{\{|u-u_*|>h\}}dx\int_0^{\infty}\theta e^{\theta h}dh\\&=\int_{\{|u-u_*|>h\}}1dx\int_0^{\infty}\theta e^{\theta h}dh=meas\left\{|u-u_*|>h\right\}\int_0^{\infty}\theta e^{\theta h}dh\\&=\int_0^{\infty}\theta e^{\theta h}meas\left\{|u-u_*|>h\right\}dh\leq\int_0^{\infty}\theta e^{\theta h}ee^{-\tau h}meas\,\Omega\,dh\\&=\frac{\theta e}{\tau-\theta}meas\,\Omega,\end{aligned}$$

which implies

$$e^{\theta|u-u_*|}\in L^1(\Omega).$$

(iii) If $b>\frac{Q-p}{Q}$, then $\beta>1$. It follows from (2.5) that

$$\varphi\left(k_0+d\right)=meas\left\{|u-u_*|>k_0+d\right\}=0, \tag{3.22}$$

where $d=c(meas\left\{|u-u_*|>k_0\right\})^{\frac{bQ-Q+b}{Qp}}2^{\frac{bQ}{bQ-Q+p}}$, hence from (3.22) and (2.1), we obtain $u-u_*\in L^{\infty}_{weak}(\Omega)$. Also since $L^{\infty}_{weak}(\Omega)=L^{\infty}(\Omega)$, then $u-u_*\in L^{\infty}(\Omega)$.

References

[1] Adams R A. Sobolev spaces [M]. New York: Academic Press, 1975.

[2] Campanato S. Sistemi ellittici in forma divergenza: regolarità all'interno[M]. Pisa: Scuola Normale Superiore, 1980.

[3] Capogna L, Danielli D, Garofalo N. An embedding theorem and the Harnack inequality for nonlinear subelliptic equations[J]. Comm. Partial Differential Equations, 1993, 18(9–10): 1765–1794.

[4] Evans L C. Quasiconvexity and partial regularity in the calculus of variations[J]. Arch. Ration. Mech. Anal., 1986, 95(3): 227–252.

[5] Gao H, Leonetti F, Wang L. Remarks on Stampacchia lemma[J]. J. Math. Anal. Appl., 2018, 458(1): 112–122.

[6] Garofalo N, Nhieu D M. Isoperimetric and Sobolev inequalities for Carnot-Carathédory spaces and the existence of minimal surfaces[J]. Comm. Pure Appl. Math., 1996, 40(10): 1081–1144.

[7] Giannetti F. Weak minima of integral functionals in Carnot-Carathéodory spaces[J]. Ricerche Mat., 2005, 54(1): 255–270.

[8] Giaquinta M. Multiple integrals in the calculus of variations and nonlinear elliptic systems[M]. Princeton: Princeton University Press, 1983.

[9] Giusti E. Metodi diretti nel calcolo delle variazioni[M]. Bologna: Unione Matematica Italiana, 1994.

[10] Hestenes M R. Calculus of variations and optimal control theory[M]. New York: Wiley, 1966.

[11] Hörmander L. Hypoelliptic second order differential equations[J]. Acta Math., 1967, 119(1): 147–171.

[12] Leonetti F, Siepe F. Global integrability for minimizers of anisotropic functionals[J]. Manuscripta Math., 2014, 144(1–2): 91–98.

[13] Leonetti F, Petricca P V. Regularity for minimizers of integrals with nonstandard growth[J]. Nonlinear Anal., 2015, 129: 258–264.

[14] Stampacchia G. Équations elliptiques du second ordre à coefficients discontinues[J]. Séminaire Jean Leray, 1963, 16(3): 1–77.

[15] Stampacchia G. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinues[J]. Ann. Inst. Fourier, 1965, 15(1): 189–257.

[16] Xu C. Subelliptic variational problems[J]. Bull. Soc. Math. France, 1990, 118(2): 147–169.

[17] Xu C. On the regularity of the minima of quasiconvex integrals[J]. Acta Math. Sinica (N.S.), 1992, 8(4): 362–374.

Hörmander 向量场型积分泛函的极小元的可积性和有界性

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摘要: 本文考虑 Hörmander 向量场型积分泛函, 当边界值具有更高可积性时, 借助 Hörmander 向量场上的 Sobolev 不等式和 Stampacchia 的迭代公式证明此积分泛函的极小元也会有更高可积性. 此外还得到极小元的 $L^1(\Omega)$ 和 $L^\infty(\Omega)$ 有界性, 从而把 Leonetti 和 Siepe[12] 以及 Leonetti 和 Petricca[13] 的结果从欧式空间延拓到 Hörmander 向量场.

关键词: Hörmander 向量场; 积分泛函; 极小元; 可积性; 有界性

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