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ON THE CHARACTERIZATION OF COMPACTNESS IN THE SPACE OF HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS

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Abstract: In this paper, we are concerned with a classical question in the space of Henstock-Kurzweil (shortly HK) integrable functions. A negative answer to this question is given by using the theory of the distributional Henstock-Kurzweil (shortly D_{HK}) integral. Furthermore, we use convergence to prove a sufficient and necessary condition for a function to be HK integral and then give a characterization of compactness in the space of the HK integrable functions. The results enrich and extend the theory of HK integrable functions space.

Keywords: Henstock-Kurzweil integral; distributional derivative; distributional Henstock-Kurzweil integral; convergence theorem; compactness

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1 Introduction

It is well known that the HK integral of real-function comprises Riemann integral, Lebesgue integral, improper integral and it is equivalent to Perron integral and restricted Denjoy integral. A distinguishing feature of HK integral is that it can integrate highly oscillatory functions which occur in nonlinear analysis and quantum theory. It is also easy to understand because its definition requires no measure theory. Such integral has very wide applications in many fields, for instance, differential and integral equations ([1–7]), Fourier analysis ([8–11]), economics ([12–15]), quantum theory ([16, 17]) and so on([18–21]). Likewise, the theory of the HK integral were widely studied by many mathematicians and physicians, for example, Gill and Zachary ([22]), Jan ([23]), Kurtz and Swartz ([24]), Lee ([25, 26]), Lee ([9]), Monteiro ([27]), etc.. However, the space of HK integrable functions is not a Banach space. The more extensive applications of HK integrals are limited by the

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incompleteness of the space of the HK integrable functions. So, many people tried to solve this problem, for instance, Kurzweil [28, 29]. Meanwhile, Lee gave some open problems on the theory of the HK integral in the last few years. One of the problems is about the two-norm convergence in the space of the HK integrable functions which will be stated in Section 5. Inspired by this problem, a related question, which is Question 5.1 in Section 5, appeared naturally.

We give a negative answer to Question 5.1 by using the completeness of the space of D_{HK} integral. The D_{HK} is a very wide integral and it includes Lebesgue and HK integrals. Denote the space of HK integrable functions by $\mathcal{HK}$ and called the Denjoy space [25], and the space of integrable distributions by $\mathcal{D}_{\mathcal{HK}}$. $\mathcal{D}_{\mathcal{HK}}$ is a Banach space and it is isometrically isomorphic to the space of continuous functions on an closed interval with uniform norm. The Denjoy space is dense in $\mathcal{D}_{\mathcal{HK}}$.

In Section 2, we present some basic definitions and preliminaries of HK integral of real functions. Section 3 is devoted to the D_{HK} integral and its properties. In Section 4, we prove two sufficient and necessary conditions for a distribution to be integrable. In Section 5 we firstly state Question 5.1 and then give a negative answer. In Section 6, we prove a sufficient and necessary condition for a function to be HK integrable and then give a characterization of compactness in the Denjoy space.

2 Basic Definitions and Preliminaries

Let $I_0 = [a, b]$ be a compact interval in $\mathbb{R}$ and $E \subset \mathbb{R}$ a measurable subset of I_0. Let $\mu(E)$ denote the Lebesgue measure. We first extend the notion of a partition of an interval.

We say that the intervals I and J are non-overlapping if int $(I) \cap \text{int}(J) = \emptyset$, where $\text{int}(J)$ denotes the interior of J.

A *partial K-partition* D in I_0 is a finite collection of interval-point pairs (I, ξ) with non-overlapping intervals $\xi \in I \subset I_0$. We write $D = \{(I, \xi)\}$. Moreover, if the union of all the intervals I equals I_0, then $D = \{(I, \xi)\}$ is a *K-partition* of I_0.

Let δ be a positive function defined on I_0 which is called a *gauge*. The symbol $\Delta(I_0)$ stands for the set of *gauge* on I_0. A K-partition $D = \{(I, \xi)\}$ is said to be *δ-fine* if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$ where $B(\xi, \delta(\xi)) = (\xi - \delta(\xi), \xi + \delta(\xi))$. Let $\mathcal{P}(\delta, I_0)$ be the set of all δ-fine K-partition of I_0.

Given $P \in \mathcal{P}(\delta, I_0)$, we write

$$S(f, P) = \sum_D f(\xi)\mu(I)$$

for integral sums over D, where $f : I_0 \to \mathbb{R}$.

Definition 2.1 [24] A function f is called HK integrable on I_0 with the HK integral $J = (HK)\int_{I_0} f(x)dx$, if there exists a $J \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta \in \Delta(I_0)$ such that

$$|S(f, P) - J| < \varepsilon$$

for every $P \in \mathcal{P}(\delta, I_0)$. The family of all HK integrable functions on I_0 is denoted by $\mathcal{HK}_{I_0}$. f is HK integrable on a set $E \subset I_0$ if the function $f \cdot \chi_E \in \mathcal{HK}_{I_0}$. We write

$$(HK)\int_E f = (HK)\int_{I_0} f\chi_E = F(E)$$

for the HK integral of f on E where F is called the primitive of f.

Definition 2.2 [26, 30] Let $E \subset I_0$ and $F : I_0 \to \mathbb{R}$. $F : I_0 \to \mathbb{R}$ is called *absolutely continuous* or AC^* on E if for every $\varepsilon > 0$ there exists a $\eta > 0$ such that

$$\sum_i |F(u_i) - F(v_i)| < \varepsilon, \tag{2.1}$$

whenever $\{[v_i, u_i]\}$ is a finite sequence of non-overlapping intervals which have an endpoint v_i or u_i in E and satisfy $\sum_i (u_i - v_i) < \eta$. A family of function $\{F_n\}$ is said to be uniformly AC^* if F_n is AC^* but uniformly in n, i.e., $\eta > 0$ independent of n with F replaced by F_n in (2.1).

$F : I_0 \to \mathbb{R}$ is ACG^* (or F_n is uniformly ACG^*) on I_0 if I_0 can be expressed as a countable union of its subsets E_n, $n \in N$ such that F is AC^* (or F_n is uniformly AC^*) on each E_n.

Lemma 2.3 [25, 26] Let f: $[a, b] \to \mathbb{R}$. If $f \in \mathcal{HK}_{[a,b]}$, then the primitive F of f is ACG^* on $[a, b]$.

Definition 2.4 [25, 26] f is called restricted Denjoy integrable (shortly D^*) on I_0 if there exists an $F \in ACG^*$ such that $F'(x) = f(x)$ a.e. on I_0.

Lemma 2.5 [25, 26] The HK-integral and the D^*-integral are equivalent.

The theory of distributions, or generalized functions, was founded by Schwartz L in the 1940's and it extended the notion of function so that all distributions have derivatives of all orders. Distributions are defined as continuous linear functionals on the space of the functions $C_c^\infty = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty \text{ and } \phi \text{ has compact support in } \mathbb{R}\}$, where the *support* of a function ϕ is the closure of the set on which ϕ does not vanish. Denote it by

$$\operatorname{supp}(\phi) = \overline{\{x \in \mathbb{R} : \phi(x) \neq 0\}}.$$

A sequence $\{\phi_n\} \subset C_c^\infty$ converges to $\phi \in C_c^\infty$ if there exists a compact set K such that all ϕ_n have support in K and for every $m \in \mathbb{N}$ the sequence of derivatives $\phi_n^{(m)}$ converges to $\phi^{(m)}$ uniformly on K. Denote C_c^∞ endows with this convergence property by $\mathcal{D}$. Here ϕ is called a test function if $\phi \in \mathcal{D}$. The distributions are defined as continuous linear functionals on $\mathcal{D}$. The space of distributions is denoted by $\mathcal{D}'$, which is the dual space of $\mathcal{D}$. That is, if $f \in \mathcal{D}'$ then $f : \mathcal{D} \to \mathbb{R}$, and we write $\langle f, \phi \rangle \in \mathbb{R}$ for $\phi \in \mathcal{D}$.

For all $f \in \mathcal{D}'$, we define the distributional derivative f' of f to be a distribution satisfying $\langle f', \phi \rangle = -\langle f, \phi' \rangle$, where ϕ is a test function. Further, we write distributional derivative as f'. If a function f is differentiable, then its ordinary pointwise derivative denotes as $f'(x)$ where $x \in \mathbb{R}$. From now on, all derivative in this paper will be distributional derivatives unless stated otherwise.

3 The Distributional Henstock-Kurzweil Integral

Let I be an open interval in $\mathbb{R}$, we define

$$\mathcal{D}(I) = \{\phi : I \to \mathbb{R} \mid \phi \in C_c^\infty \text{ and } \phi \text{ has compact support in } I\}.$$

Then the distributions on I are the continuous linear functionals on $\mathcal{D}(I)$. The space of distributions on I is denoted by $\mathcal{D}'(I)$, which is the dual space of $\mathcal{D}(I)$. Since $\mathcal{D}(I) \subset \mathcal{D}$, so $\mathcal{D}' \subset \mathcal{D}'(I)$, i.e., if $f \in \mathcal{D}'$ then $f \in \mathcal{D}'(I)$.

Denote the space of continuous functions on $[a,b]$ by $C([a,b])$. Let

$$C_0 = \{F \in C([a,b]) : F(a) = 0\}. \tag{3.1}$$

Then C_0 is a Banach space under the norm

$$\|F\|_\infty = \sup_{x\in[a,b]} |F(x)|.$$

Definition 3.1 A distribution f in $\mathcal{D}'((a,b))$ is said to be distributionally Henstock-Kurzweil integrable (shortly D_{HK}) on an interval $[a,b]$ if there exists a continuous function $F \in C_0$ such that $F' = f$, i.e., f is the distributional derivative of F. The distributional Henstock-Kurzweil integral of f on $[a,b]$ is denoted by $\int_a^b f(x)dx = F(b) - F(a)$, for short, $\int_a^b f = F(b) - F(a)$.

For every $f \in D_{HK}$, $\phi \in \mathcal{D}((a,b))$, we write $\langle f, \phi\rangle = -\langle F, \phi'\rangle = -\int_a^b F(x)\phi'(x)dx$.

In symbols,

$$D_{HK} = \{f \in \mathcal{D}'((a,b)) : f = F', F \in C_0\}. \tag{3.2}$$

Of course, D_{HK} is a subset of $\mathcal{D}'((a,b))$.

Notice that if $f \in D_{HK}$ then f has many primitives in $C([a,b])$, all differing by a constant, but f has exactly one primitive in C_0. For simplicity of notation, in what follows we use the letters $F, G, ...$ for the primitives of $f, g, ...$ in D_{HK}, respectively. Unless otherwise stated, "$\int$" denotes the D_{HK}-integral throughout this paper.

Remark 3.2 If taking $a = -\infty$ and $b = +\infty$, we obtain distributional Henstock-Kurzweil integral on $\overline{\mathbb{R}} = [-\infty, +\infty]$ as in [31]. For D_{HK} integral on $\overline{\mathbb{R}}$, we can similarly discuss all properties as on $[a,b]$.

For $f \in D_{HK}$, define the Alexiewicz norm in D_{HK} as $\|f\| = \|F\|_\infty$. With the Alexiewicz norm, D_{HK} is a Banach space (see [31]).

Lemma 3.3 [31] (a) The space of all Lebesgue integrable functions and the spaces of restricted Denjoy and wide Denjoy integrable functions are dense in D_{HK}.

(b) D_{HK} is a separable space.

Since the primitive F of a HK integrable function f is continuous and $F'(x) = f(x)$ is almost everywhere. It is easy to see that $HK \subset D_{HK}$. By Lemma 2.5 and Lemma 3.3, the following corollary holds.

Corollary 3.4 The space HK is dense in D_{HK}.

Let $g:[a,b]\to\mathbb{R}$, its variation is $V(g)=\sup\sum_n|g(y_n)-g(x_n)|$ where the supremum is taken over every sequence $\{(x_n,y_n)\}$ of disjoint intervals in $[a,b]$. A function g is of bounded variation on $[a,b]$ if $V(g)$ is finite. Denote the space of functions of bounded variation by $\mathcal{BV}$. The space $\mathcal{BV}$ is a Banach space with norm $\|g\|_{\mathcal{BV}}=|g(a)|+V(g)$.

Recall that $C([a,b])^*=\mathcal{BV}$ by the Riesz Representation Theorem. Since C_0 is the space of continuous functions on $[a,b]$ vanishing at a and D_{HK} is isometrically isomorphic to C_0 due to the definition of the integral, an obvious fact is that the dual space of D_{HK} is $\mathcal{BV}$ (see details in [31]).

Furthermore, integration by parts and Hölder inequality hold.

Lemma 3.5 [31] **(Integration by parts)** Let $f\in D_{HK}$ and $g\in\mathcal{BV}$. Then $fg\in D_{HK}$ and

$$\int_a^b fg=F(b)g(b)-\int_a^b Fdg. \tag{3.3}$$

Lemma 3.6 [31] **(Hölder inequality)** Let $f\in D_{HK}$. If $g\in\mathcal{BV}$, then

$$\left|\int_a^b fg\right|\le 2\|f\|\|g\|_{\mathcal{BV}}. \tag{3.4}$$

Note that D_{HK} is a Banach space under the Alexiewicz norm $\|.\|$, there are several equivalent norms as follows (see details in [31, Theorem 29]).

For $f\in D_{HK}$, define

$$\begin{aligned}\|f\|_1&=\sup_I\left\{\left|\int_I f\right|:I\subset[a,b]\right\},\\ \|f\|_2&=\sup_g\left\{\int_I fg:g\in\mathcal{BV},|g|\le 1,\ \ V(g)\le 1,\ I\subset[a,b]\right\}.\end{aligned}$$

Recall that HK is not a Banach space under the norm $\|f\|_{HK}=\sup\{|\int_a^x f|:x\in[a,b]\}$. So we have

Proposition 3.7 HK is not complete under the norms $\|.\|_1$, $\|.\|_2$.

Remark 3.8 $HK\subset D_{HK}$ and D_{HK} is complete. So, extending the HK to D_{HK}, it overcomes the defect of the space HK not being complete.

4 Sufficient and Necessary Condition in D_{HK}

In this section we discuss the convergence problems of the sequence $\{f_ng\}$ for $f_n\in D_{HK}$ and $g\in\mathcal{BV}$ and then we give two sufficient and necessary conditions for a distribution to be integrable, which plays a significant role in answering Question 5.1 in the next section.

Let $O(\mathcal{BV})$ be the unit ball in $\mathcal{BV}$, i.e.,

$$O(\mathcal{BV})=\{g\in\mathcal{BV}:\|g\|_{\mathcal{BV}}\le 1\}. \tag{4.1}$$

Now we use Lemma 3.3 and Corollary 3.4 to prove a weak convergence theorem.

Theorem 4.1 Let $f \in D_{HK}$. Then there exists a sequence $\{f_n\}$ of HK integrable functions such that

(1) for every $g \in \mathcal{BV}$, $\lim\limits_{n\to\infty} \int_a^b f_n g = \int_a^b fg$;

(2) $\lim\limits_{n\to\infty} \int_a^b f_n g = \int_a^b fg$ uniformly on $O(\mathcal{BV})$, that is, for arbitrary $\varepsilon > 0$, there exists $N > 0$ such that whenever $n > N$, $\left|\int_a^b f_n g - \int_a^b fg\right| < \varepsilon$ for each $g \in O(\mathcal{BV})$.

Proof (1) By Lemma 3.3, the space D_{HK} is a separable Banach space. By Corollary 3.4, the space HK is dense in D_{HK}. So, for $f \in D_{HK}$, there exists a sequence $\{f_n\}$ of Hk integrable functions satisfying

$$\|f_n - f\| = \|F_n - F\|_\infty \to 0 \ (n \to \infty), \tag{4.2}$$

where F_n and F are the primitives of f_n and f, respectively.

Since $f_n \in HK$, $f_n \in D_{HK}$. For $f \in D_{HK}$ and every $g \in \mathcal{BV}$, by Lemma 3.5, we have $fg \in D_{HK}$. By Lemma 3.6,

$$\left|\int_a^b f_n g - \int_a^b fg\right| = \left|\int_a^b (f_n - f)g\right| \le 2\|f_n - f\|\|g\|_{\mathcal{BV}} \to 0 \ (n \to \infty).$$

Thus, $\lim\limits_{n\to\infty} \int_a^b f_n g = \int_a^b fg$ for every $g \in \mathcal{BV}$.

(2) For each $g \in O(\mathcal{BV})$, in view of (4.2) and the Hölder inequality,

$$\left|\int_a^b f_n g - \int_a^b fg\right| = \left|\int_a^b (f_n - f)g\right| \le 2\|f_n - f\| \to 0 \ (n \to \infty). \tag{4.3}$$

Hence, $\lim\limits_{n\to\infty} \int_a^b f_n g = \int_a^b fg$ uniformly on $O(\mathcal{BV})$ and the proof is complete.

The converse of Theorem 4.1 is also true.

Theorem 4.2 Assume that $\{f_n\}$ is a sequence of HK integrable functions satisfying

(1) for every $g \in \mathcal{BV}$, $\int_a^b f_n g$ converges;

(2) $\int_a^b f_n g$ uniformly converges on $O(\mathcal{BV})$.

Then there exists $f \in D_{HK}$ satisfying

$$\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg, \quad \forall g \in \mathcal{BV},$$

and

$$\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg \text{ uniformly on } O(\mathcal{BV}).$$

Proof For each $n \in \mathbb{N}$, the fact that $f_n \in HK$ and $HK \subset D_{HK}$ implies $f_n \in D_{HK}$. Since $\int_a^b f_n g$ converges for every $g \in \mathcal{BV}$ then $\{\int_a^b f_n g\}$ is a Cauchy sequence. Denote

$$\langle f_n, g\rangle = \int_a^b f_n g, \quad \forall g \in \mathcal{BV}.$$

Thus, $\{\langle f_n, g\rangle\}$ is a Cauchy sequence.

Since for each $n, m \in \mathbb{N}$,

$$\|f_n - f_m\| = \sup_{g \in O(\mathcal{BV})} |\langle f_n - f_m, g\rangle| = \sup_{g \in O(\mathcal{BV})} \left| \int_a^b (f_n - f_m) g \right|.$$

It follows from uniform convergence of $\{\int_a^b f_n g\}$ on $O(\mathcal{BV})$ that $\{f_n\}$ is a Cauchy sequence in D_{HK}.

By Lemma 3.3, the space D_{HK} is a Banach space, so there exists $f \in D_{HK}$ such that $\lim\limits_{n\to\infty} f_n = f$ in D_{HK}. Hence, we have

$$\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg, \quad \forall g \in \mathcal{BV},$$

and

$$\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg \quad \text{uniformly on } O(\mathcal{BV}).$$

The proof is complete.

Combining Theorem 4.1 with Theorem 4.2, we get a sufficient and necessary condition for a distribution to be integrable.

Theorem 4.3 A distribution $f \in D_{HK}$ iff there exists a sequence $\{f_n\}$ of HK integrable functions satisfying (1) and (2) in Theorem 4.2.

Based on Theorem 4.3, we can prove another sufficient and necessary condition for a distribution to be integrable.

Theorem 4.4 A distribution $f \in D_{HK}$ with the primitive F if and only if there exists a sequence $\{f_n\}$ of HK integrable functions with the primitives F_n satisfying

(1) $\{F_n\}$ is bounded in $C([a, b])$;

(2) $\lim\limits_{n\to\infty} F_n(x) = F(x)$for each $x \in [a, b]$ and $F \in C([a, b])$.

Moreover,

$$\int_a^b fg = \lim_{n\to\infty} \int_a^b f_n g, \quad \forall g \in \mathcal{BV}.$$

Proof (Necessity) Assume that $f \in D_{HK}$. Then $F \in C_0$. By Theorem 4.3, there exists a sequence $\{f_n\}$ of Henstock-Kurzweil integrable functions satisfying (1) and (2) in Theorem 4.2. This is, for every $g \in \mathcal{BV}$, $\int_a^b f_n g$ converges to $\int_a^b fg$ and

$$\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg \text{ uniformly on } O(\mathcal{BV}).$$

It follows from Banach-Steinhaus theorem that $\{f_n\}$ is bounded in D_{HK}. Since $\|f_n\| = \|F_n\|_\infty$, then $\{F_n\}$ is bounded in $C([a, b])$ and (1) holds.

In addition, since $\lim_{n\to\infty} \int_a^b f_n g = \int_a^b fg$ uniformly on $O(\mathcal{BV})$, taking $g = \chi_{[a,x]} \in \mathcal{BV}$, we have $\lim\limits_{n\to\infty} \int_a^x f_n = \int_a^x f$, i.e.,

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \forall x \in [a, b].$$

The fact that $f \in D_{HK}$ implies $F \in C_0 \subset C([a,b])$. This shows that (2) holds.

(Sufficiency) Since $f_n \in HK$, $F_n(a) = 0$ $(n \in \mathbb{N})$. By hypothesis (2), from $F(a) = \lim_{n\to\infty} F_n(a) = 0$, it follows that $F \in C_0$. Define $f = F'$. Then $f \in D_{HK}$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Moreover, since $\{F_n\}$ is bounded in $C([a,b])$, by dominated convergence theorem of Riemann-Stieltjes integral,

$$\lim_{n\to\infty} \int_a^b F_n dg = \lim_{n\to\infty} \int_a^b F dg, \quad \forall g \in \mathcal{BV}.$$

Hence, by Lemma 3.5, we have

$$\int_a^b fg = Fg|_a^b - \int_a^b Fdg = \lim_{n\to\infty} \left(F_n g|_a^b - \int_a^b F_n dg \right) = \lim_{n\to\infty} \int_a^b f_n g, \quad \forall g \in \mathcal{BV}.$$

The sufficiency is complete.

5 Question and Answer

P. Y. Lee asked an open problem on the two-norm convergence in the Denjoy space. The two-norm convergence of a sequence of functions often means that the sequence is bounded in the strong topology and convergent in the weak topology. We call it Lee's problem which is stated as follows:

Lee's problem Two-norm convergence. The controlled convergence does not generate a topology in $\mathcal{HK}$, and the condition of almost everywhere convergence seems to be too strong. Can we define a two-norm convergence in $\mathcal{HK}$ so that the two-norm convergence will generate a topology in $\mathcal{HK}$?

Inspired by this problem, a related question appeared naturally and can be stated as follows.

Question 5.1 Let $\{f_n\} \subset \mathcal{HK}_{[a,b]}$. If the sequence $\{f_n\}$ is bounded in $\mathcal{HK}_{[a,b]}$ and $\{(HK)\int_a^b f_n g\}$ converges for every $g \in \mathcal{BV}$, whether there exists $f \in \mathcal{HK}_{[a,b]}$ such that

$$\lim_{n\to\infty} (HK) \int_a^b f_n g = (HK) \int_a^b fg \tag{5.1}$$

for every $g \in \mathcal{BV}$?

Recall that $(HK)\int_a^b f$ denotes the HK integral of f on $[a,b]$.

Let us see an example first.

Example 5.2 Suppose that $x \in [0,1]$. Let

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x\pi}{n^2}, \quad \text{and} \quad F_n(x) = \sum_{k=1}^{n} \frac{\sin k^2 x\pi}{k^2}.$$

Then

$$F(x) = \lim_{n\to\infty} F_n(x)$$

for all $x \in [0,1]$. Since for each $n \in \mathbb{N}$, the function $F_n(x)$ is continuous on $[0,1]$ and $F_n(x)$ uniformly converges to $F(x)$, so F is continuous on $[0,1]$ and $F(0) = 0$, which imply $F \in C_0$. But F, apart from certain exceptional points, is not differentiable on $[0,1]$.

Let

$$f = F'. \tag{5.2}$$

Then $f \in D_{HK}$ and

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f = F(1) = \sum_{n=1}^{\infty} \frac{\sin n^2\pi}{n^2}.$$

However, $f \notin \mathcal{HK}_{[0,1]}$.

In fact, if f in (5.2) is HK integrable on $[0,1]$, then its primitive F is differentiable for almost all $x \in [0,1]$. It is a contradiction. So $f \notin \mathcal{HK}_{[0,1]}$.

On the other hand, let

$$f_n(x) = F_n'(x) = \pi \sum_{k=1}^{n} \cos k^2 x\pi, \quad x \in [0,1].$$

Then f_n are continuous on $[0,1]$ and therefore f_n are HK integrable on $[0,1]$, and

$$\int_0^x f_n = F_n(x), \quad \forall x \in [0,1].$$

Since

$$\|F_n(x)\| = \left\| \sum_{k=1}^{n} \frac{\sin k^2 x\pi}{k^2} \right\| \le \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty, \quad \forall x \in [0,1]. \tag{5.3}$$

It follows that $F_n(x)$ are uniformly bounded on $[0,1]$. Hence, for every $g \in \mathcal{BV}$, by dominated convergence theorem of Riemann-Stieltjes integrals, we obtain that

$$\lim_{n\to\infty} \int_0^1 F_n dg = \int_0^1 F dg. \tag{5.4}$$

Combining (5.4) with Lemma 3.5, we have

$$\lim_{n\to\infty} \int_0^1 f_n g = \lim_{n\to\infty} \left(F_n g|_0^1 - \int_0^1 F_n dg \right) = Fg|_0^1 - \int_0^1 F dg = \int_0^1 fg, \quad \forall g \in \mathcal{BV}.$$

So, the sequence $\{f_n\}$ is bounded in $\mathcal{HK}_{[a,b]}$ and weakly converges to f, but f is not HK integrable.

Therefore, according to Example 5.2, we can give an answer to Question 5.1.

Theorem 5.3 Let $\{f_n\} \subset \mathcal{HK}_{[a,b]}$. Asuume that $\{f_n\}$ is bounded in $\mathcal{HK}_{[a,b]}$ and $\{(HK)\int_a^b f_n g\}$ converges for every $g \in \mathcal{BV}$. Then it is not necessary that there exists a function $f \in HK$ such that for every $g \in \mathcal{BV}$,

$$\lim_{n\to\infty} (HK) \int_a^b f_n g = (HK) \int_a^b fg.$$

In order to further discuss the related problems with Question 5.1, we give a necessary and sufficient condition for a function to be HK integrable in the next section.

6 Compactness

Firstly, we prove a necessary Lemma.

Lemma 6.1 Assume that $\{f_n\}$ is a sequence of the HK integrable functions on $[a, b]$ satisfying

(1) for every $g \in \mathcal{BV}$, $\{(HK)\int_a^b f_n g\}$ converges;

(2) the sequence $\{F_n\}$ of the primitives of f_n is uniformly ACG^*.

Then there exists $f \in \mathcal{HK}_{[a,b]}$ satisfying

$$(HK)\int_a^b f = \lim_{n\to\infty}(HK)\int_a^b f_n$$

and

$$\lim_{n\to\infty}(HK)\int_a^b f_n g = (HK)\int_a^b fg, \quad \forall g \in \mathcal{BV}.$$

Proof Taking $g(x) = \chi_{[a,x]} \in \mathcal{BV}$. Since $f_n \in HK$ and $\{(HK)\int_a^b f_n g\}$ converges for $g \in \mathcal{BV}$, one has

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\int_a^x f_n = \lim_{n\to\infty}(HK)\int_a^b f_n g$$

exists for all $x \in [a, b]$.

Let

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\int_a^x f_n, \quad x \in [a, b].$$

The facts that $\{F_n\}$ is uniformly ACG^* and $F(x) = \lim_{n\to\infty} F_n(x)$ for all $x \in [a, b]$ yields $F \in ACG^*$.

Let $f(x) = F'(x)$ for a.e. $x \in [a, b]$. Then $f \in \mathcal{HK}_{[a,b]}$ and

$$\lim_{n\to\infty}(HK)\int_a^b f_n g = (HK)\int_a^b fg, \quad \forall g \in \mathcal{BV}.$$

The proof is complete.

Lemma 6.2 Assume that $f \in \mathcal{HK}_{[a,b]}$. Then there exists a sequence $\{f_n\}$ of HK integrable functions on $[a, b]$ satisfying

(1) for every $g \in \mathcal{BV}$, $\{(HK)\int_a^b f_n g\}$ converges to $(HK)\int_a^b fg$;

(2) the sequence $\{F_n\}$ of the primitives of f_n is uniformly ACG^*.

Proof Let $f \in HK$ with primitive $F \in ACG^*$. According to [26, p198, Exercise 5.7], there is a sequence $\{\varphi_n\}$ of step functions such that $\varphi_n \to f$ is almost everywhere and the primitives Φ_n of φ_n are uniformly ACG^*. Due to the definition of uniformly ACG^*, Φ_n is uniformly bounded and equicontinuous on $[a, b]$. In view of [26, Lemma 5.5.1], there is a subsequence Φ_{k_n} of Φ_n such that Φ_{k_n} converges uniformly to F on $[a, b]$. Denote Φ_{k_n} by F_n.

Of course, F_n are uniformly ACG^*. By virtue of the controlled convergence theorem ([26, Theorem 5.5.2]), one has

$$\int_a^b F_n' = \int_a^b f_n \to \int_a^b f \quad (n \to \infty). \tag{6.1}$$

Moreover, it follows from Lemma 3.5 that

$$\int_a^b f_n g = g(b) \int_a^b f_n - \int_a^b F_n dg, \quad \forall g \in \mathcal{BV}. \tag{6.2}$$

The fact that F_n converges uniformly to F on $[a, b]$ and the dominated convergence theorem for Riemann-Stieltjes integrals yield that, for each $g \in \mathcal{BV}$,

$$\int_a^b F_n dg \to \int_a^b F dg \quad (n \to \infty), \tag{6.3}$$

which together with (6.1) implies that, for each $g \in \mathcal{BV}$,

$$\lim_{n\to\infty} \int_a^b f_n g = \lim_{n\to\infty} \left(g(b) \int_a^b f_n - \int_a^b F_n dg \right) = g(b) \int_a^b f - \int_a^b F dg = \int_a^b fg.$$

Therefore, the proof is complete.

By Lemma 6.1 and Lemma 6.2, we obtain a necessary and sufficient condition for a function to be HK integrable.

Theorem 6.3 $f \in \mathcal{HK}_{[a,b]}$ iff there exists a sequence $\{f_n\}$ of HK integrable functions on $[a, b]$ satisfying (1) and (2) in Lemma 6.2.

Now we are coming to give a characterization of the compact subsets in HK.

Theorem 6.4 Assume that $\mathcal{A} \subset HK$ and $\mathcal{B} = \{F : F(t) = \int_a^t f, f \in \mathcal{A}\}$. Then $\mathcal{A}$ is relatively compact in HK if and only if for every sequence $\{f_n\} \subset \mathcal{A}$, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $\{f_{n_j}\}$ satisfies (1) and (2) in Lemma 6.2.

Proof The proof is directly deduced from Theorem 6.3.

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Henstock-Kurzweil 可积函数空间的紧性特征

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摘要: 本文研究 Henstock-Kurzweil 可积 (HK 可积)函数空间中的一个经典问题. 文章通过研究分布Henstock-Kurzweil 积分 (DHK 积分)的性质, 给出了该问题的否定答案. 进一步, 利用收敛性获得了函数HK 可积的一个充分必要条件. 最后, 在上述结论的基础上刻画了 HK 可积函数空间的紧性. 所得结果丰富和推广了HK可积函数空间理论.

关键词: Henstock-Kurzweil 积分; 分布导数; 分布 Henstock-Kurzweil 积分; 收敛定理; 紧性

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