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ANNOUNCEMENT ON "MAXIMUM PRINCIPLE FOR NON-UNIFORMLY PARABOLIC EQUATIONS AND APPLICATIONS"

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Abstract: In this note we announce the global boundedness for the solutions to a class of possibly degenerate parabolic equations by De-Giorgi's iteration. In particular, the existence of weak solutions for possibly degenerate stochastic differential equations with singular diffusion coefficients is obtained.

Keywords: maximum principle; De-Giorgi's iteration; stochastic differential equation; Krylov's estimate

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Consider the following elliptic equation of divergence form in $\mathbb{R}^d$ $(d \geqslant 2)$:

$$\operatorname{div}(a \cdot \nabla u) = 0, \tag{1}$$

where $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a Borel measurable function and $\nabla := (\partial_{x_1}, \cdots, \partial_{x_d})$. When a is uniformly elliptic, the celebrated works of De-Giorgi [1] and Nash [2] said that any weak solutions of elliptic equation (1) are bounded and Hölder continuous. Moreover, Moser [3] showed that any weak solutions of (1) satisfy the Harnack inequality. In [4], Trudinger considered the non-uniformly elliptic equation (1) under the following integrability assumptions:

$$\lambda_0^{-1} \in L^{p_0},\ \mu_0 \in L^{p_1} \text{ with } p_0, p_1 \in (1, \infty] \text{ satisfying } \tfrac{1}{p_0} + \tfrac{1}{p_1} < \tfrac{2}{d},$$

where

$$\lambda_0(x) := \inf_{|\xi|=1} \xi \cdot a(x)\xi, \quad \mu_0(x) := \sup_{|\xi|=1} \frac{|a(x)\xi|^2}{\xi \cdot a(x)\xi}. \tag{2}$$

He showed that any generalized solutions of (1) are locally bounded and weak Harnack inequality holds. Recently, Bella and Schäffner [5] showed the same results under the following *sharp* condition on p_0, p_1,

$$\tfrac{1}{p_0} + \tfrac{1}{p_1} < \tfrac{2}{d-1}, \quad p_0, p_1 \in [1, \infty], \tag{3}$$

Here we extend the main result of [5] to parabolic case. More precisely, we consider the following parabolic equation of divergence form in $\mathbb{R}^{d+1}$:

$$\partial_t u = \operatorname{div}(a \cdot \nabla u) + b \cdot \nabla u + f, \tag{4}$$

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where

$$a:\mathbb{R}^{d+1}\to\mathbb{R}^{d\times d},\ b:\mathbb{R}^{d+1}\to\mathbb{R}^{d},\ f:\mathbb{R}^{d+1}\to\mathbb{R}$$

are Borel measurable functions. As in (2), we introduce

$$\lambda(x):=\inf_{t\geqslant 0,|\xi|=1}\xi\cdot a(t,x)\xi,\quad \mu(x):=\sup_{t\geqslant 0,|\xi|=1}\frac{|a(t,x)\xi|^2}{\xi\cdot a(t,x)\xi},\tag{5}$$

and suppose that λ and μ are nonnegative Borel measurable functions.

Definition 0.1 A continuous function $u:\mathbb{R}^{d+1}\to\mathbb{R}$ is called a Lipschitz weak (super/sub)-solution of PDE (4) if ∇u is locally bounded and for any nonnegative Lipschitz function φ on $\mathbb{R}^{d+1}$ with compact support,

$$-\langle\!\langle u,\partial_t\varphi\rangle\!\rangle=(\geqslant/\leqslant)-\langle\!\langle a\cdot\nabla u,\nabla\varphi\rangle\!\rangle+\langle\!\langle b\cdot\nabla u,\varphi\rangle\!\rangle+\langle\!\langle f,\varphi\rangle\!\rangle,\tag{6}$$

where $\langle\!\langle f,g\rangle\!\rangle:=\int_{\mathbb{R}}\int_{\mathbb{R}^d}f(t,x)g(t,x)\mathrm{d}x\mathrm{d}t$.

For $p,q\in[1,\infty]$, let $\mathbb{L}^{q,p}_{t,x}:=L^q(\mathbb{R};L^p(\mathbb{R}^d))$ and $\mathbb{L}^{p,q}_{x,t}:=L^p(\mathbb{R}^d;L^q(\mathbb{R}))$ be the space of space-time functions with norms, respectively,

$$\|f\|_{\mathbb{L}^{q,p}_{t,x}}:=\left(\int_{\mathbb{R}}\|f(t,\cdot)\|_p^q\mathrm{d}t\right)^{1/q},\ \|f\|_{\mathbb{L}^{p,q}_{x,t}}:=\left(\int_{\mathbb{R}^d}\|f(\cdot,x)\|_q^p\mathrm{d}x\right)^{1/p},$$

where $\|\cdot\|_p$ stands for the usual L^p-norm. For $r>0$ and $(s,z)\in\mathbb{R}^{d+1}$, we define

$$Q_r:=[-r^2,r^2]\times B_r\subset\mathbb{R}^{d+1},\ Q_r^{s,z}:=Q_r+(s,z),\quad B_r^z:=B_r+z,$$

and for $p\in[1,\infty]$, introduce the following localized L^p-space:

$$\widetilde{L}^p:=\left\{f\in L^1_{loc}(\mathbb{R}^d):|\!|\!|f|\!|\!|_p:=\sup_z\|\mathbf{1}_{B_1^z}f\|_p<\infty\right\},\tag{7}$$

and for $p,q\in[1,\infty]$,

$$\widetilde{\mathbb{L}}^{q,p}_{t,x}:=\left\{f\in L^1_{loc}(\mathbb{R}^{d+1}):|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}}:=\sup_{s,z}\|\mathbf{1}_{Q_1^{s,z}}f\|_{\mathbb{L}^{q,p}_{t,x}}<\infty\right\},\tag{8}$$

and similarly for $\widetilde{\mathbb{L}}^{p,q}_{x,t}$.

Below we fix $p_0\in(\frac{d}{2},\infty]$ and $p_1\in[1,\infty]$ with

$$\tfrac{1}{p_0}+\tfrac{1}{p_1}<\tfrac{2}{d-1},\tag{9}$$

and introduce the index set

$$\mathbb{I}^d_{p_0}:=\left\{(p,q)\in[1,\infty]^2:\tfrac{1}{p}<(1-\tfrac{1}{q})(\tfrac{2}{d}-\tfrac{1}{p_0})\right\}.$$

We make the following assumptions about a and b:

($\mathbf{H}^a$) $|\!|\!|\lambda^{-1}|\!|\!|_{p_0}+|\!|\!|\mu|\!|\!|_{p_1}<\infty$, where λ,μ are defined by (5).

($\mathbf{H}^b$) $b=b_1+b_2$, where if $p_0\in(\frac{d}{2},d]$, $b_1\equiv 0$, and if $p_0>d$, $b_1\in\widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}$ for some $(p_2,q_2)\in[1,\infty]^2$ with

$$\tfrac{1}{2p_0}+\tfrac{1}{p_2}<(\tfrac{1}{2}-\tfrac{1}{q_2})(\tfrac{2}{d}-\tfrac{1}{p_0}),\tag{10}$$

and $b_2\in\widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}$ and $\mathrm{div}b_2\equiv 0$.

For simplicity of notations, we introduce the following parameter set

$$\Theta := \Big(d, p_i, q_i, |\!|\!|\lambda^{-1}|\!|\!|_{p_0}, |\!|\!|\mu|\!|\!|_{p_1}, |\!|\!|b_1|\!|\!|_{\widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}}, |\!|\!|b_2|\!|\!|_{\widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}}\Big). \tag{11}$$

We have the following apriori estimate.

Theorem 0.2 Under $(\mathbf{H}^a)$ and $(\mathbf{H}^b)$, for any $f \in \widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}$ with $(p_4, q_4) \in \mathbb{I}^d_{p_0}$ and for any $T > 0$, there exists a constant $C = C(T, \Theta, p_4, q_4) > 0$ such that for any Lipschitz weak solution u of PDE (4) in $\mathbb{R}^{d+1}$ with $u(t)|_{t\leqslant 0} \equiv 0$,

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant C|\!|\!|f\mathbf{1}_{[0,T]}|\!|\!|_{\widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}}. \tag{12}$$

Consider the following heat equation with divergence free drift b:

$$\partial_t u = \Delta u + b\cdot\nabla u + f,\ \ u(t)|_{t\leqslant 0} = 0. \tag{13}$$

The following apriori global boundedness estimate is a direct consequence of Theorem 0.2.

Corollary 0.3 Let $b \in \widetilde{\mathbb{L}}^{p,\infty}_{x,t}$ with $\mathrm{div} b = 0$, where $p \in [1,\infty] \cap (\frac{d-1}{2}, \infty]$. For any $T > 0$ and $f \in \widetilde{\mathbb{L}}^{q',p'}_{t,x}$, where $p', q' \in [1,\infty]$ satisfy $\frac{d}{p'} + \frac{2}{q'} < 2$, there exists a constant $C > 0$ only depending on T, d, p, p', q' and $\|b\|_{\widetilde{\mathbb{L}}^{p,\infty}_{x,t}}$ such that for any Lipschitz weak solution u of (13),

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant C|\!|\!|f\mathbf{1}_{[0,T]}|\!|\!|_{\widetilde{\mathbb{L}}^{q',p'}_{t,x}}. \tag{14}$$

Remark 0.4 Note that when $\frac{d}{p} + \frac{2}{q} < 2$ and $b \in \widetilde{\mathbb{L}}^{q,p}_{t,x}$ with $\mathrm{div} b = 0$, it is well known that (14) holds (cf. [6], [7]). When b does not depend on t, the current condition $p > \frac{d-1}{2}$ in Corollary 0.3 is clearly better than $p > \frac{d}{2}$.

As an application of the global boundedness estimate (12), we consider the following SDE:

$$\mathrm{d}X_t = \sqrt{2}\sigma(t, X_t)\mathrm{d}W_t + b(t, X_t)\mathrm{d}t, \quad X_0 = x, \tag{15}$$

where W is a d-dimensional standard Brownian motion. We recall the following notion of weak solutions to SDE (15).

Definition 0.5 Let $\mathfrak{F} := (\Omega, \mathscr{F}, \mathbf{P}; (\mathscr{F}_t)_{t\geqslant 0})$ be a stochastic basis and (X, W) a pair of $\mathscr{F}_t$-adapted processes defined thereon. We call triple $(\mathfrak{F}, X, W)$ a weak solution of SDE (15) with starting point $x \in \mathbb{R}^d$ if

(i) $\mathbf{P}(X_0 = x) = 1$ and W is an $\mathscr{F}_t$-Brownian motion;

(ii) for all $t \geqslant 0$, it holds that $\mathbf{P}$-a.s.

$$\int_0^t \Big(|\sigma(s, X_s)|^2 + |b(s, X_s)|\Big)\mathrm{d}s < \infty, \ \ a.s., \text{and } X_t = x + \sqrt{2}\int_0^t \sigma(s, X_s)\mathrm{d}W_s + \int_0^t b(s, X_s)\mathrm{d}s.$$

We make the following assumptions about σ and b.

$(\widetilde{\mathbf{H}}^\sigma)$ Suppose that there are a sequence of $d \times d$-matrix functions $\sigma_n \in L^\infty(\mathbb{R}_+; C^\infty_b)$, $(p_2, q_2) \in \mathbb{I}^d_{p_0}$ and $\kappa_0 > 0$ such that for all $n \in \mathbb{N}$,

$$|\!|\!|\lambda_n^{-1}|\!|\!|_{p_0} + |\!|\!|\mu_n|\!|\!|_{p_1} + |\!|\!|\partial_i a^{ij}_n|\!|\!|_{\widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}} + |\!|\!|(\partial_i\partial_j a^{ij}_n)^+|\!|\!|_{\widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}} \leqslant \kappa_0, \tag{16}$$

where $a_n := \sigma_n\sigma_n^*$, λ_n and μ_n are defined as in (5) by a_n. Moreover, for some $p_3, q_3 \in [2,\infty]$ with $(\frac{p_3}{2}, \frac{q_3}{2}) \in \mathbb{I}^d_{p_0}$ and for any $T, R > 0$,

$$\sup_n |\!|\!|\sigma_n|\!|\!|_{\widetilde{\mathbb{L}}^{q_3,p_3}_{t,x}} =: \kappa_1 < \infty, \quad \lim_{n\to\infty} \|(\sigma_n - \sigma)\mathbf{1}_{[0,T]\times B_R}\|_{\mathbb{L}^{q_3,p_3}_{t,x}} = 0. \tag{17}$$

$(\widetilde{\mathbf{H}}^b)$ Let $b = b_1 + b_2$ satisfy $(\mathbf{H}^b)$ and $b \in \widetilde{\mathbb{L}}_{t,x}^{q_4,p_4}$ for some $(p_4, q_4) \in \mathbb{I}_{p_0}^d$.

We have the following existence result.

Theorem 0.6 Under $(\widetilde{\mathbf{H}}^\sigma)$ and $(\widetilde{\mathbf{H}}^b)$, for any $x \in \mathbb{R}^d$, there is at least one weak solution $(\mathfrak{F}, X, W)$ for SDE (15). Moreover, for any $(p,q) \in \mathbb{I}_{p_0}^d$ and $T > 0$, there are $\theta \in (0,1)$ and constant $C = C(T, \Theta, p, q) > 0$ such that for any stopping time $\tau \leqslant T$, $\delta \in (0,1)$ and $f \in \widetilde{\mathbb{L}}_{t,x}^{q,p}$,

$$\mathbf{E}\left(\int_\tau^{\tau+\delta} f(s, X_s)\mathrm{d}s\middle|\mathscr{F}_\tau\right) \leqslant C\delta^\theta |\!|\!| f |\!|\!|_{\widetilde{\mathbb{L}}_{t,x}^{q,p}}. \tag{18}$$

The following two examples can be derived from the above existence result.

Example 0.7 Let $d \geqslant 3$ and $\alpha \in (0, (\frac{d}{2} - 1) \wedge (\frac{1}{2} + \frac{1}{d-1}))$, $\beta \in (0, 2\alpha)$. For any $\lambda \geqslant 0$ and $x \in \mathbb{R}^d$, the following SDE admits a unique strong solution:

$$\mathrm{d}X_t = |X_t|^{-\alpha}\mathrm{d}W_t + \lambda X_t |X_t|^{-\beta-1}\mathrm{d}t, \quad X_0 = x.$$

Note that the starting point can be zero and the uniqueness follows from [8].

Example 0.8 The following two dimensional degenerate SDE admits a solution:

$$\begin{cases} \mathrm{d}X_t^1 = |X_t^2|^\alpha \mathrm{d}W_t^1 + b^1(X_t)\mathrm{d}t, \\ \mathrm{d}X_t^2 = |X_t^1|^\alpha \mathrm{d}W_t^2 + b^2(X_t)\mathrm{d}t, \end{cases}$$

where $\alpha \in (0, \frac{1}{2})$ and $b = (b^1, b^2) \in \widetilde{L}^p(\mathbb{R}^2)$ for some $p > \frac{1}{1-2\alpha}$.

More details can be found in [9].

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