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RESEARCH ANNOUNCEMENTS ON "MAXIMAL ERGODIC INEQUALITIES FOR SOME POSITIVE OPERATORS ON NONCOMMUTATIVE L_P-SPACE"

HONG Gui-xiang [1], RAY Samya Kumar [1], WANG Si-meng [2]

(*1. School of Mathematics and Statistics, Wuhan university, Wuhan 430072, China*)

(*1. Universié Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, Orsay 91405, France*)

1 Introduction and Main Results

In classical ergodic theory, one of the earliest pointwise ergodic convergence theorems was obtained by Birkhoff [1] in 1931. Dunford and Schwartz [2] greatly generalized the previous situation; they established the strong (p, p) maximal inequalities for all $1 < p < \infty$ for time averages of positive L_1-L_∞ contractions and obtained the pointwise convergence result as a corollary. However, the most general result in this direction was obtained by Akcoglu [3], who established a maximal ergodic inequality for general positive contractions on L_p-spaces for a fixed $1 < p < \infty$. The proof is based on an ingenious dilation theorem which reduces the problem to the case of positive isometries, and the latter was already studied by Tuleca. Akcoglu's dilation theorem has found numerous applications in various directions; let us mention (among others) Peller's work on Matsaev's conjecture for contractions on L_p-spaces, Coifman-Rochberg-Weiss' approach to Stein's Littlewood-Paley theory, g-function type estimates on compact Riemannian manifolds by Coifman-Weiss, as well as functional calculus of Ritt and sectorial operators and references therein). On the other hand, we would like to remark that the Lamperti contractions consist of a typical class of general L_p-contractions. In particular, Kan [4] established a maximal ergodic inequality for power bounded Lamperti operators whose adjoints are also Lamperti. Many more results for positive operators and Lamperti operators in the context of ergodic theory were studied further by various authors.

Motivated by quantum physics, noncommutative mathematics have advanced in a rapid speed. As foundations of noncommutative mathematics, let us recall briefly the definition of noncommutative L_p spaces. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ. We denote by $s(x)$ the support of x for a positive element $x \in \mathcal{M}$. Let $\mathcal{S}(\mathcal{M})$ be the linear span of set of all positive elements in $\mathcal{M}$ such that $\tau(s(x)) < \infty$. For $1 \le p < \infty$, we define the noncommutative L_p-space $L_p(\mathcal{M}, \tau)$

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Biography: Hong Guixiang (1983–), male, Jiangxi, professor, major in harmonic analysis and non-commutative analysis.

to be the completion of $\mathcal{S}(\mathcal{M})$ with respect to the norm

$$\|x\|_{L_p(\mathcal{M})} := (\tau(|x|^p))^{\frac{1}{p}}, \quad \text{where } |x| := (x^*x)^{\frac{1}{2}}.$$

We refer the reader to [5] for detailed presentation of noncommutative L_p spaces.

The connection between ergodic theory and von Neumann algebras is intimate and goes back to the earlier development of the theory of rings of operators. However, the study of pointwise ergodic theorems only took off with the pioneering work of Lance [6]. The topic was then stupendously studied in a series of works due to Conze, Dang-Ngoc, Kümmerer, Yeadon and others. However, the maximal inequalities and pointwise ergodic theorems in L_p-spaces remained out of reach for many years until the path-breaking work of Junge and Xu [7]. In [7], the authors established a noncommutative analogue of Dunford-Schwartz maximal ergodic theorem. This breakthrough motivated further research to develop various noncommutative ergodic theorems. We refer to [8, 9] and references therein. Notice that the general positive contractions considered by Akcoglu do not fall into the category of Junge-Xu [7]. In the noncommutative setting, there are very few results for operators beyond L_1-L_∞ contractions except some isolated cases studied in [8]. In particular, the following noncommutative analogue of Akcoglu's maximal ergodic inequalities remains open.

Question 1.1 Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful semifinite trace τ. Let $1 < p < \infty$ and $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive contraction. Does there exist a positive constant C, such that

$$\Big\| \sup_{n\geq 0}{}^{+} \frac{1}{n+1} \sum_{k=0}^{n} T^k x \Big\|_p \leq C\|x\|_p$$

for all $x \in L_p(\mathcal{M})$?

In the article [10], we answer Question 1.1 for a large class of positive contractions which do not fall into the category of aforementioned works. Indeed, this class recovers all positive contractions concerned in Question 1.1 if $\mathcal{M}$ is the classical space $L_\infty([0,1])$. To introduce our main results we set some notation and definitions.

Definition 1.2 Let $1 \leq p < \infty$. A bounded linear map $T : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}, \tau)$ is called a Lamperti (or support separating) operator, if for any two τ-finite projections $e, f \in \mathcal{M}$ with $ef = 0$, we have that

$$(Te)^*Tf = Te(Tf)^* = 0.$$

By the standard approximation argument, it is easy to observe that the above definition of Lamperti operators agrees with the known definition in the commutative setting.

The following is one of our main results. We will denote by C_p a fixed distinguished constant depending only on p, which is given by the best constant of Junge-Xu's maximal ergodic inequality [7, Theorem 0.1].

Theorem 1.3 Let $1 < p < \infty$. Assume that $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ belongs to the family

$$\overline{\text{conv}}^{\,sot}\{S : L_p(\mathcal{M}) \to L_p(\mathcal{M}) \text{ positive Lamperti contractions}\}, \tag{1.1}$$

that is, the closed convex hull of all positive Lamperti contractions on $L_p(\mathcal{M})$ with respect to the strong operator topology. Then

$$\Big\| {\sup_{n\geq 0}}^{+} \frac{1}{n+1} \sum_{k=0}^{n} T^k x \Big\|_p \leq C_p \|x\|_p$$

for all $x \in L_p(\mathcal{M})$.

It is worth noticing that the class introduced in (1.1) is quite large in the classical setting. Indeed, it is known that for $\mathcal{M} = L_\infty([0,1])$ equipped with the Lebesgue measure, we have

$$\begin{aligned} &\{S : L_p([0,1]) \to L_p([0,1]) \text{ positive contractions}\} \\ = \ &\overline{\text{conv}}^{\,sot}\{S : L_p([0,1]) \to L_p([0,1]) \text{ positive Lamperti contractions}\}, \end{aligned}$$

which does recover the classical Akcoglu's ergodic theorem on $L_p([0,1])$. Moreover, our methods also help to establish a completely bounded version of Ackoglu's ergodic theorem.

Corollary 1.4 Let $1 < p < \infty$. Let (Ω, μ) be a measure space and $T : L_p(\Omega) \to L_p(\Omega)$ be a positive contraction. Then for any semifinite von Neumann algebra $\mathcal{M}$, we have

$$\Big\| {\sup_{n\geq 0}}^{+} \frac{1}{n+1} \sum_{k=0}^{n} (T \otimes I_{L_p(\mathcal{M})})^k x \Big\|_p \leq C_p \|x\|_p, \quad \forall\, x \in L_p(L_\infty(\Omega)\overline{\otimes}\mathcal{M}).$$

As mentioned earlier, Akcoglu's arguments for ergodic theorem essentially rely on the study of dilations of positive contractions. In spite of various works on dilations on von Neumann algebras, Junge and Le Merdy showed in their remarkable paper [11] that there is no 'reasonable' analogue of Akcoglu's dilation theorem on noncommutative L_p-spaces. This becomes a serious difficulty in establishing a noncommutative analogue of Akcoglu's ergodic theorem. Our proof of the above theorem is based on the study of structural properties and dilations of convex combinations of Lamperti operators as in (1.1). This route seems to be different from that of Akcoglu's original one. Let us mention some of the key steps and new ingredients in the proof, which might be of independent interest.

(i) **Noncommutative ergodic theorem for positive isometries**: Following the classical case, the first natural step would be to establish a maximal ergodic inequality for positive isometries. Here we give an analogue of this result in the noncommutative setting.

Theorem 1.5 Let $1 < p < \infty$. Let $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive isometry. Then

$$\Big\| {\sup_{n\geq 0}}^{+} \frac{1}{n+1} \sum_{k=0}^{n} T^k x \Big\| \leq C_p \|x\|_p, \quad \forall\, x \in L_p(\mathcal{M}).$$

The key ingredient is to extend positive isometries on $L_p(\mathcal{M})$ to the vector-valued space $L_p(\mathcal{M};\ell_\infty)$. This fact seems to be highly non-obvious for the isometry not completely isometric. Then based on the methods recently developed in [8], we obtain Theorem 1.5.

(ii) **Structural theorems for Lamperti operators**: In the classical setting, Peller and Kan obtained a dilation theorem for Lamperti contractions. Their constructions are different from Akcoglu's and rely on structural description of Lamperti operators. In the noncommutative setting, we first prove a similar characterization for Lamperti operators by using techniques from [12], which constitutes the second step in proving Theorem 1.3.

Theorem 1.6 Let $1 \le p < \infty$. Let $T : L_p(\mathcal{M},\tau) \to L_p(\mathcal{M},\tau)$ be a Lamperti operator with norm C. Then, there exist, uniquely, a partial isometry $w \in \mathcal{M}$, a positive self-adjoint operator b affiliated with $\mathcal{M}$ and a normal Jordan $*$-homomorphism $J : \mathcal{M} \to \mathcal{M}$, such that

(i) $w^*w = J(1) = s(b)$; moreover we have $w = J(1) = s(b)$ if additionally T is positive;

(ii) Every spectral projection of b commutes with $J(x)$ for all $x \in \mathcal{M}$;

(iii) $T(x) = wbJ(x)$, $x \in \mathcal{S}(\mathcal{M})$;

(iv) We have $\tau(b^pJ(x)) \le C\tau(x)$ for all $x \in \mathcal{M}_+$; if additionally T is isometric, then the equality holds with $C = 1$.

(iii) **Dilation theorem for the convex hull of Lamperti contractions**: In order to establish ergodic theorems for a large class beyond Lamperti contractions, as the last step, we deploy tools from [13] to obtain an N-dilation theorem for the convex hull of Lamperti contractions for all $N \in \mathbb{N}$.

Definition 1.7 Let $1 \le p \le \infty$. Let $S \subseteq B(L_p(\mathcal{M},\tau_{\mathcal{M}}))$. We say that S has a simultaneous N-dilation if there exist a von Neumann algebra $\mathcal{N}$ with a normal faithful semifinite trace $\tau_{\mathcal{N}}$, contractive linear maps $Q : L_p(\mathcal{N},\tau_{\mathcal{N}}) \to L_p(\mathcal{M},\tau_{\mathcal{M}})$, $J : L_p(\mathcal{M},\tau_{\mathcal{M}}) \to L_p(\mathcal{N},\tau_{\mathcal{N}})$, and a set of isometries $\mathcal{U} \subseteq L_p(\mathcal{N},\tau_{\mathcal{N}})$ such that for all $n \in \{0,1,\ldots,N\}$ and $T_i \in S$, $1 \le i \le n$, there exist $U_{T_1}, U_{T_2}, \ldots, U_{T_n} \in \mathcal{U}$ such that

$$T_1T_2\ldots T_n = QU_{T_1}U_{T_2}\ldots U_{T_n}J. \tag{1.2}$$

In terms of commutative diagram, we have

$$\begin{tikzcd} L_p(\mathcal{M},\tau_{\mathcal{M}}) \arrow[r, "T_1\ldots T_n"] \arrow[d, "J"] & L_p(\mathcal{M},\tau_{\mathcal{M}}) \\ L_p(\mathcal{N},\tau_{\mathcal{N}}) \arrow[r, "U_{T_1}\ldots U_{T_n}"] & L_p(\mathcal{N},\tau_{\mathcal{N}}). \arrow[u, "Q"'] \end{tikzcd}$$

The empty product (i.e. $n = 0$) corresponds to the identity operator.

Theorem 1.8 Let $1 < p < \infty$. Suppose $S \subseteq B(L_p(\mathcal{M}))$ has a simultaneous dilation. Then, each operator $T \in \mathrm{conv}(S)$ has a N-dilation for all $N \in \mathbb{N}$.

In the process, we actually prove a simultaneous dilation theorem for tuples of Lamperti contractions, that is, for all $N \in \mathbb{N}$, which is a stronger version of Peller-Kan's dilation theorem. Our approach also establishes validity of noncommutative Matsaev's conjecture for the strong closure of the closed convex hull of Lamperti contractions for $1 < p \neq 2 < \infty$ whenever the underlying von Neumann algebra has QWEP. It is worth mentioning that prior to our work all the dilatable contractions are basically those acting on the von Neumann algebra itself, except some 'loose dilation' results. In our method, we also recover partially some results in the existed literatures. Also, our result might have some applications as in the commutative case. We leave this research direction open.

Note that Theorem 1.3 only applies to contractive operators. As in the classical case, the study for non-contractive power bounded operators requires many additional efforts. In the following we also establish a general ergodic theorem for power bounded Lamperti operators as soon as their adjoints are also Lamperti (usually called doubly Lamperti operators), which is the other main result of the paper.

Theorem 1.9 Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let $\mathcal{M}$ be a finite von Neumann algebra. Assume that $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is a positive Lamperti operator with $\sup_{n\geq 1} \|T^n\|_{L_p(\mathcal{M})\to L_p(\mathcal{M})} = K < \infty$, and that the adjoint operator $T^* : L_{p'}(\mathcal{M}) \to L_{p'}(\mathcal{M})$ is also Lamperti. Then

$$\Big\| {\sup_{n\geq 0}}^{+} \frac{1}{n+1} \sum_{k=0}^{n} T^k x \Big\|_p \leq K C_p \|x\|_p$$

for all $x \in L_p(\mathcal{M})$.

The above theorem 1.9 is the noncommutative analogue of a classical result of Kan [4]. It essentially relies on a structural theorem for positive doubly completely Lamperti operators, which reduces the problem to the setting of Theorem 1.3 and is of independent interest.

Theorem 1.10 Let $\mathcal{M}$ be a finite von Neumann algebra. Let $1 < p < \infty$ and $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive Lamperti operator with the representation $Tx = bJ(x)$ as in Theorem 1.6. Then, there exist an element $\theta \in \mathcal{M}$ and a positive Lamperti contraction $S : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ such that $T^n = \theta_n S^n$, where

(i) S is a positive Lamperti contraction which vanishes on $L_p(p_0\mathcal{M}p_0)$ and is isometric on $L_p(p_1\mathcal{M}p_1)$;

(ii) θ_n is a positive element in $\mathcal{M}$ of the form $\theta_n = \theta J(\theta) \cdots J^{n-1}(\theta)$ and $\theta_n S^n(x) = S^n(x)\theta_n$ for all $n \geq 1$ and $x \in \mathcal{M}$;

(iii) for all $n \geq 1$, $\|T^n\|_{L_p(\mathcal{M})\to L_p(\mathcal{M})} \leq \|\theta_n\|_\infty$. Moreover, the equality holds if the adjoint operator $T^* : L_{p'}(\mathcal{M}) \to L_{p'}(\mathcal{M})$ for $1/p + 1/p' = 1$ is also Lamperti.

To prove this structural result, we follow the path of Kan. However, since the structures and orthogonal relations of von Neumann subalgebras are completely different from those in classical measure theory, our proof is much more lengthy and numerous adjustments are needed in this new setting. Also, due to these technical reasons, we restrict our study to the case of finite von Neumann algebras only.

Moreover, we observe that the maximal ergodic inequality also holds for several other classes of operators outside the scope of Theorem 1.3 or Theorem 1.9.

(i) **Positive invertible operators which are not Lamperti**: Kan [4] discussed various examples of Lamperti operators. He showed that in the classical setting any positive invertible operator with positive inverse is Lamperti and that any positive invertible operator on a finite dimensional (commutative) L_p-space with $\sup_{n\in\mathbb{Z}}\|T^n\|_{L_p\to L_p}<\infty$ is Lamperti.. As a consequence, he reproved that any power bounded positive operator with positive inverse admits a maximal ergodic inequality; this generalized the ergodic theorem of de la Torre. A noncommutative analogue of this theorem, in a much general form, was achieved in [8]. However, we provide the following example of positive invertible operators on noncommutative L_p-spaces with positive inverses which are not even Lamperti, which illustrates that there is no reasonable analogue of Kan's above examples for the noncommutative setting.

Example 1.11 Let $1\le p<\infty$ and r be an invertible matrix 2×2 matrix. Define $T: S_p^2\to S_p^2,\quad T(x)=rxr^*$. Clearly, T is completely positive map, and so is the inverse map $T^{-1}(x)=r^{-1}x(r^{-1})^*$. Note that

$$e=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad f=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}$$

are two orthogonal projections with $ef=fe=0$. But if we take

$$r=\begin{pmatrix} 1 & 1\\ \alpha & \beta\end{pmatrix}$$

with $\alpha,\ \beta\in\mathbb{R}$, and $1+\alpha\beta\neq 0$, it is easy to see that $T(e)T(f)\neq 0$. So T is not Lamperti.

Moreover, consider $\alpha=0,\beta=-1$. Then $r^{-1}=r$ and $r^2=1_{M_2}$. So

$$\sup_{n\in\mathbb{Z}}\|T^n\|_{cb,S_\infty^2\to S_\infty^2}\le\sup_{n\in\mathbb{Z}}\|r^n\|_\infty^2<\infty.$$

Since the operator space of linear operators on M_2 is finite dimensional, so (T^k) is uniformly bounded with respect to any equivalent operator norm. Thus these are power bounded positive operators with positive inverses on a finite dimensional L_p-space, but not Lamperti.

This shows that Kan's method can not reprove de la Torre's ergodic theorem in the noncommutative setting. Nevertheless, these examples fall into the category of the aforementioned result of [8], and hence satisfy the maximal ergodic theorem. We would like to remark that Kan's aforementioned examples of Lamperti operators play an important role in many other papers. All these phenomena seem to be new.

(ii) **Junge-Le Merdy's non-dilatable example**: As mentioned earlier, there exist concrete examples of completely positive complete contractions which fail to admit a noncommutative analogue of Akcoglu's dilation, constructed by Junge and Le Merdy [11]. While we show that these operators still satisfy a maximal ergodic inequality. In particular we establish the following fact.

Proposition 1.12 Let $1 < p \neq 2 < \infty$. Then, for all $k \in \mathbb{N}$ large enough, there exists a completely positive complete contraction $T : S_p^k \to S_p^k$ such that

$$\Big\| {\sup_{n\geq 0}}^{+} \frac{1}{n+1} \sum_{k=0}^{n} T^k x \Big\|_p \leq (C_p + 1)\|x\|_p, \quad x \in L_p(\mathcal{M}),$$

but T does not admit Ackoglu's dilation.

The proof is very short and elementary; indeed it still relies on Akcoglu's ergodic theorem [3] in the classical setting. The above theorem illustrates again that the non-commutative situation is significantly different from the classical one.

For the proof of all the results mentioned above, we refer the reader to [10] for more information.

References

[1] Birkhoff G D. Proof of the ergodic theorem[J]. Proc. Nat. Acad. Sci., 1931, 17: 656–660.

[2] Dunford N, Schwartz J T. Convergence almost everywhere of operator averages[J]. J. Rational Mech. Anal., 1956, 5: 129–178.

[3] Akcoglu M A. A pointwise ergodic theorem in L_p-spaces[J]. Canad. J. Math, 1975, 27(5): 1075–1082.

[4] Kan C H. Ergodic properties of Lamperti operators[J]. Canad. J. Math, 1978, 30(6): 1206–1214.

[5] Pisier G, Xu Q. Non-commutative L^p-spaces[J]. Handbook of the geometry of Banach spaces, 2003, 354(6): 2131–2171.

[6] Lance E C. Ergodic theorems for convex sets and operator algebras[J]. Invent. Math, 1976, 37(3): 201–214.

[7] Junge M, Xu Q. Noncommutative maximal ergodic theorems[J]. J. Amer. Math. Soc., 2007, 20(2): 385–439.

[8] Hong G, Liao B, Wang S. Noncommutative maximal ergodic inequalities associated with doubling conditions[J]. To appear in Duke Math J.

[9] Hong G, Sun M. Noncommutative multi-parameter Wiener-Wintner type ergodic theorem[J]. J. Funct. Anal., 2018, 275(5): 1100–1137.

[10] Hong G, Ray S, Wang S. Maximal ergodic inequalities for some positive operators on noncommutative L_p-space[J]. arXiv:1907.12967v5.

[11] Junge M, Le Merdy C. Dilations and rigid factorisations on noncommutative L^p-spaces[J]. J. Funct. Anal., 2007, 249(1): 220–252.

[12] Yeadon F J. Isometries of noncommutative L_p-spaces[J]. Math. Proc. Cambridge Philos. Soc., 1981, 90(1): 41–50.

[13] Fackler S, Glück J. A toolkit for constructing dilations on Banach spaces[J]. Proc. Lond. Math. Soc., 2019, 118(2): 416–440.