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FAST AND SLOW DECAY SOLUTIONS FOR SUPERCRITICAL FRACTIONAL ELLIPTIC PROBLEMS IN EXTERIOR DOMAINS

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1 Introduction and Main Results

We construct classic solutions of the following supercritical nonlinear fractional exterior problem

$$\begin{cases} (-\Delta)^s u - u^p = 0, \ u > 0 \text{ in } \mathbb{R}^N \setminus \overline{B_1}, \\ u = 0 \ \text{ in } \ \overline{B_1}, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $p > \frac{N+2s}{N-2s}$ and B_1 is the unit ball in $\mathbb{R}^N$. As usual, the operator $(-\Delta)^s$ is the fractional Laplacian, defined at any point $x \in \mathbb{R}^N$ as

$$\begin{aligned} (-\Delta)^s u(x) : &= C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \\ &= C(N,s) \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \end{aligned}$$

here $P.V.$ is a commonly used abbreviation for "in the principal value sense" and $C(N,s)$ is a constant dependent of N and s. We refer to [6–7].

For classical Laplacian, namely, $s = 1$, which is the Lame-Emden-Fowler equation

$$\begin{cases} \Delta u + u^p = 0, \ u > 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega}, \\ u = 0 \ \text{ on } \ \partial\Omega, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.2}$$

where Ω is a bounded open set with smooth boundary in $\mathbb{R}^N$ and $p > 1$. Davila etc [4] proved (1.2) has infinitely many solutions with slow decay $O(|x|^{-\frac{2}{p-1}})$ at infinity with either $N \geq 4$ and $p > \frac{N+1}{N-3}$, or $N \geq 3$, $p > \frac{N+2}{N-2}$ and Ω is symmetric with respect to N coordinate axes. Later, this result was extended to $p > \frac{N+2}{N-2}$ and Ω is a smooth bounded domain by Davila

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etc [5]. For fractional Laplacian, we will prove that this result also holds when $s \in (0,1)$, $p > \frac{N+2s}{N-2s}$ and B_1 is the unit ball in $\mathbb{R}^N$. For problem (1.1) in general exterior domain, our method not be used to solve it, there exist some obstacles in Remark 1.

Our main results can be stated as follows:

Theorem 1.1 For any $s \in (0,1)$ and $p > \frac{N+2s}{N-2s}$, there exists a continuum of solutions u_λ, $\lambda > 0$, to problem (1.1) such that

$$u_\lambda(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2s}{p-1}} (1 + o(1)) \quad \text{as} \quad |x| \to \infty$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \overline{B_1}$.

Theorem 1.2 For any $s \in (0,1)$, there exists a number $P_s > \frac{N+2s}{N-2s}$, such that for any $p \in (\frac{N+2s}{N-2s}, P_s)$, problem (1.1) has a fast decay solution u_p, $u_p(x) = O(|x|^{2s-N})$ as $|x| \to +\infty$.

In order to prove Theorem 1.1, we will take ω as approximation of (1.1) where ω is a smooth, radially symmetric, entire solution of the following problem

$$(-\Delta)^s \omega - \omega^p = 0, \ \omega > 0 \ \text{ in } \mathbb{R}^N, \ \ \omega(0) = 1, \ \ \lim_{|x| \to \infty} \omega(x) |x|^{\frac{2s}{p-1}} = \beta^{\frac{1}{p-1}}, \tag{1.3}$$

here β is a positive constant chosen so that $\beta^{\frac{1}{p-1}} |x|^{-\frac{2s}{p-1}}$ is a singular solution to $(-\Delta)^s \omega - \omega^p = 0$ for which the existence and linear theory has been studied recently in [1] for the fractional case.

The basic idea in the proof of Theorem 1.2 is to consider as an initial approximation the function $\lambda^{\frac{N-2s}{2}} \omega_{**}(\lambda x + \xi)$, where

$$\omega_{**}(r) = \left(\frac{1}{1 + A_{N,s} r^2} \right)^{\frac{N-2s}{2}} \tag{1.4}$$

is the unique positive radial smooth solution of the problem

$$(-\Delta)^s \omega_{**} = \omega_{**}^{\frac{N+2s}{N-2s}} \ \text{ in } \mathbb{R}^N, \quad \omega_{**}(0) = 1.$$

These scalings will constitute good approximations for small λ if p is sufficiently close to $\frac{N+2s}{N-2s}$. We prove then adjusting both ξ and λ, produces a solution as desired after addition of a lower order term.

By the change of variables

$$\widetilde{u}(x) = \lambda^{-\frac{2}{p-1}} u \left(\frac{x - \xi}{\lambda} \right)$$

and the maximum principle (see the page 39 of [3]), problem (1.1) is equivalent to

$$\begin{cases} (-\Delta)^s \widetilde{u} - |\widetilde{u}|^p = 0, \ \widetilde{u} \not\equiv 0 & \text{in } \mathbb{R}^N \setminus \overline{B_{1\lambda,\xi}}, \\ \widetilde{u} = 0 \ \text{ in } \ \overline{B_{1\lambda,\xi}}, \ \lim\limits_{|x| \to \infty} \widetilde{u}(x) = 0 \end{cases} \tag{1.5}$$

where $\lambda > 0$ is a small parameter and $B_{1\lambda,\xi}$ is the shrinking domain

$$B_{1\lambda,\xi} = \{\lambda x + \xi \mid x \in B_1\}.$$

Remark 1 To prove Theorem 1.1 and Theorem 1.2, we will construct solutions of the equivalent problem (1.5) with the form $\widetilde{u} = \omega + \varphi_\lambda + \phi$ and $\widetilde{u} = \omega_{**} + \varphi_\lambda + \phi$. To obtain the decay of $\widetilde{u}$, we need to know that the decay of $\varphi_\lambda + \phi$. Using the Poisson Kernel $P(x, y)$ in $R^N \setminus B_1$, we first obtain the decay of φ_λ is no more than $O(|x - \xi|^{2s-N})$. Secondly, we can derive the decay of ϕ by the Green function $G(x, y)$ in $R^N \setminus B_1$. But for general exterior domain, there is a lack of the explicit formulas and the decay of Poisson Kernel and Green's function of fractional Laplace operator$(-\Delta)^s$.

The proof of Theorem 1.1 and Theorem 1.2 refers to [2] in detail.

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