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HYPOELLIPTIC ESTIMATE FOR SOME COMPLEX VECTOR FIELDS

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1 Introduction and Main Results

Let $\Omega \subset \mathbb{R}^n$ be a neighborhood of 0, and denote by i the square root of -1. We consider the following system of complex vector fields

$$\mathcal{P}_j = \partial_{x_j} - i\left(\partial_{x_j}\varphi(x)\right)\partial_t, \quad j = 1, \cdots, n, \quad (x,t) \in \Omega \times \mathbb{R}, \tag{1.1}$$

where $\varphi(x)$ is a real-valued function defined in Ω. This system was first studied by Treves [4], and considered therein is more general case for t varies in $\mathbb{R}^m$ rather than $\mathbb{R}$. Denote by (ξ, τ) the dual variables of (x,t). Then the principle symbol σ for the system $\{\mathcal{P}_j\}_{1\le j\le n}$ is

$$\sigma(x,t;\xi,\tau) = \left(i\xi_1 + (\partial_{x_1}\varphi)\tau, \cdots, i\xi_j + \left(\partial_{x_j}\varphi\right)\tau\right) \in \mathbb{C}^n$$

with $(x,t;\xi,\tau) \in T^*(\Omega \times \mathbb{R}_t) \setminus \{0\}$, and thus the characteristic set is

$$\left\{(x,t;\xi,\tau) \in T^*(\Omega \times \mathbb{R}_t) \setminus \{0\} \;\middle|\; \xi = 0,\ \tau \neq 0,\ \nabla\varphi(x) = 0\right\}.$$

Since outside the characteristic set the system $\{\mathcal{P}_j\}_{1\le j\le n}$ is (microlocally) elliptic, we only need to study the microlocal hypoellipticity in the two components $\{\tau > 0\}$ and $\{\tau < 0\}$ under the assumption that

$$\nabla\varphi(0) = 0. \tag{1.2}$$

Note we may assume $\varphi(0) = 0$ if replacing φ by $\varphi - \varphi(0)$. Throughout the paper we will always suppose φ satisfies the following condition of finite type

$$\sum_{1\le|\alpha|\le k-1} |\partial^\alpha\varphi(0)| = 0 \quad \text{and} \quad \sum_{|\alpha|=k} |\partial^\alpha\varphi(0)| > 0. \tag{1.3}$$

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for some positive integer k. In view of (1.2) it suffices to consider the nontrivial case of $k \geq 2$ for the maximal hypoellipticity. By maximal hypoellipticity (in the sense of Helffer-Nourrigat [2]), it means the existence of a neighborhood $\tilde{\Omega} \subset \Omega$ of 0 and a constant C such that for any $u \in C_0^\infty(\tilde{\Omega} \times \mathbb{R})$, we have

$$\|\partial_x u\|_{L^2(\mathbb{R}^{n+1})} + \|\left(\partial_x \varphi\right) \partial_t u\|_{L^2(\mathbb{R}^{n+1})} \leq C\Big(\sum_{j=1}^n \|\mathcal{P}_j u\|_{L^2(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})}\Big), \tag{1.4}$$

where and throughout paper we use the notation $\|\vec{a}\|_{L^2} = \Big(\sum\limits_{1\leq j\leq n} \|a_j\|_{L^2}^2\Big)^{1/2}$ for vector-valued functions $\vec{a} = (a_1, \cdots, a_n)$. Note that the maximal hypoellipticity along with the condition (1.3) yields the following subellptic estimate

$$\|\partial_x u\|_{L^2(\mathbb{R}^{n+1})} + \|\,|D_t|^{\frac{1}{k}}\, u\|_{L^2(\mathbb{R}^{n+1})} \leq C\Big(\sum_{j=1}^n \|\mathcal{P}_j u\|_{L^2(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})}\Big).$$

Thus the subellipticity is in some sense intermediate between the maximal hypoellipticity and the local hypoellipticity.

Observe the system $\{\,\mathcal{P}_j\,\}_{1\leq j\leq n}$ is translation invariant for t. So we may perform partial Fourier transform with respect to t, and study the maximal microhypoellipticity, in the two directions $\tau > 0$ and $\tau < 0$. Indeed we only need consider without loss of generality the maximal microhypoellipticity in positive direction $\tau > 0$, since the other direction $\tau < 0$ can be treated similarly by replacing φ by $-\varphi$. Consider the resulting system as follows after taking partial Fourier transform for $t \in \mathbb{R}$.

$$\partial_{x_j} + \tau\left(\partial_{x_j}\varphi\right), \quad j = 1, \cdots, n, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.5}$$

and we will show the maximal microhypoellipticity at 0 in the positive direction in $\tau > 0$, which means the existence of a positive number $\tau_0 > 0$, a constant $C > 0$ and a neighborhood $\tilde{\Omega} \subset \Omega$ of 0 such that

$$\forall\, \tau \geq \tau_0,\ \forall\, u \in C_0^\infty(\tilde{\Omega}),$$
$$\|\partial_x u\|_{L^2}^2 + \|\tau\left(\partial_x\varphi\right) u\|_{L^2}^2 \leq C\Big(\|\partial_x + \tau\left(\partial_x\varphi\right) u\|_{L^2}^2 + \|u\|_{L^2}^2\Big), \tag{1.6}$$

where and throughout the paper we denote $\|\cdot\|_{L^2(\mathbb{R}^n)}$ by $\|\cdot\|_{L^2}$ for short if no confusion occurs. We remark the operators defined in (1.5) is closely related to the semi-classical Witten Laplacian $\triangle_{\tau V}^{(0)} = -\triangle_x + \tau^2\left|\partial_x V\right|^2 - \tau\triangle_x V$ with τ^{-1} the semi-classical parameter, by the relationship

$$\|\partial_x + \tau\left(\partial_x\varphi\right) u\|_{L^2}^2 = \Big(\triangle_{\tau V}^{(0)} u,\ u\Big)_{L^2},$$

where $(\cdot,\cdot)_{L^2}$ stands for the inner product in $L^2(\mathbb{R}^n)$. Helffer-Nier [1] conjectured $\triangle_{\tau V}^{(0)}$ is subelliptic near 0 if φ is analytic and has no local minimum near 0, and this still remains open so far. Note (1.6) is a local estimate concerning the sharp regularity near $0 \in \mathbb{R}^n$ for $\tau > 0$, and we have also its global counterpart, which is of independent interest for analyzing the spectral property of the resolvent and the semi-classical lower bound of Witten Laplacian. We refer to Helffer-Nier's work [1] for the detailed presentation on the topic of global maximal hypoellipticity and its application to the spectral analysis on Witten Laplacian.

Theorem 1.1 (Maximal microhypoellipticity for $\tau > 0$) Let φ be a polynomial satisfying condition (1.3) with $k \geq 2$. Denote by $\lambda_j, 1 \leq j \leq n$, the eigenvalues of the Hessian matrix $(\partial_{x_i}\partial_{x_j}\varphi)_{n\times n}$. Suppose there exists a constant $C_* > 0$ such that for any $x \in \Omega$, we have the following estimates: if $k = 2$, then

$$\sum_{\lambda_j(x)>0} \lambda_j(x) \leq C_* \Big(\sum_{\lambda_j(x)<0} |\lambda_j(x)| + |\partial_x\varphi(x)|^{\epsilon_0} \Big), \tag{1.7}$$

and if $k > 2$, then

$$\sum_{\lambda_j(x)>0} \lambda_j(x) \leq C_* \Big(\sum_{\lambda_j(x)<0} |\lambda_j(x)| + |\partial_x\varphi(x)|^{\frac{k-2}{k-1}} + \sum_{2\leq|\beta|\leq k-1} \big|\partial_x^\beta\varphi(x)\big|^{\mu_\beta} \Big), \tag{1.8}$$

where $\epsilon_0 > 0$ is an arbitrarily small number and μ_β are given numbers with $\mu_\beta > (k-2)/(k-|\beta|)$ for $2 \leq |\beta| \leq k-1$. Then the system $\mathcal{P}_j$ defined in (1.1) is maximally microhypoelliptic in positive position $\tau > 0$, that is, estimate (1.6) holds.

Replacing φ by $-\varphi$ we can get the maximal microhypoellipticity for $\tau < 0$, and thus the maximal hypoellipticity for all τ.

Corollary 1.2 (Maximal hypoellipticity) Under the same assumption as Theorem 1.1 with (1.7) and (1.8) replaced by the estimate that for any $x \in \Omega$,

$$\sum_{j=1}^n |\lambda_j(x)| \leq \begin{cases} C_* \, |\partial_x\varphi(x)|^{\epsilon_0}, & \text{if } k = 2, \\ C_* \Big(|\partial_x\varphi(x)|^{\frac{k-2}{k-1}} + \sum\limits_{2\leq|\beta|\leq k-1} \big|\partial_x^\beta\varphi(x)\big|^{\mu_\beta} \Big), & \text{if } k > 2, \end{cases}$$

the system $\mathcal{P}_j$ defined in (1.1) is maximally hypoelliptic, that is, estimate (1.4) holds.

Remark 1.3 We need only verify conditions (1.7) and (1.8) for these points where $\Delta\varphi$ is positive, since it obviously holds for the points where $\Delta\varphi \leq 0$.

The details of the proof for the main result were given by [3].

References

[1] Helffer B, Nier F. Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians[M]. Lecture Notes in Mathematics Volume 1862, Berlin: Springer–Verlag, 2005.

[2] Helffer B, Nourrigat J. Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs[M]. Progress in Mathematics Volume 58. Boston, MA: Birkhäuser Boston Inc., 1985.

[3] Li Weixi, Liu Lvqiao, Zeng Juan. Hypoelliptic estimate for some complex vector fields[J]. Preprint.

[4] François Treves. Study of a model in the theory of complexes of pseudodifferential operators[J]. Ann. of Math., 1976, 104(2): 269–324.