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A NONLINEAR PICONE IDENTITY FOR ANISOTROPIC LAPLACE OPERATOR AND ITS APPLICATIONS

FENG Ting-fu [1], ZHANG Ke-lei [2]

(*1.School of Mathematics, Kunming University, Kunming 650214, China*)

(*2.School of Mathematics and Computating Sciences, Guilin University of Electronic Science and Technology, Guilin 541004, China*)

Abstract: A nonlinear Picone identity for anisotropic Laplace operator is established in this paper. As applications, a Sturmian comparison principle to an anisotropic elliptic equation, a Liouville's theorem to an anisotropic elliptic system and a generalized anisotropic Hardy type inequality are obtained.

Keywords: anisotropic Laplace operator; nonlinear Picone identity; applications

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1 Introduction and Main Results

In 1910, Picone [1] considered the homogeneous linear second order differential system

$$\begin{cases} (a_1(x)u')' + b_1(x)u = 0, \\ (a_2(x)v')' + b_2(x)v = 0, \end{cases} \tag{1.1}$$

where $a_1(x)$, $a_2(x)$, u and v are differentiable functions to x, sign $'$ denotes $\frac{d}{dx}$, he established the following identity: for $v(x) \neq 0$,

$$(\frac{u}{v}(a_1u'v - a_2uv'))' = (b_2 - b_1)u^2 + (a_1 - a_2)(u')^2 + a_2(u' - v'\frac{u}{v})^2. \tag{1.2}$$

As applications, a Sturmian comparison principle and the oscilation theory of solutions of (1.1) were obtained by (1.2). Moreover, the extension of (1.2) to the multidimensional case (Laplace operator Δu) is the following identity: for differentiable functions $v > 0$ and $u \geqslant 0$,

$$(\nabla u - \frac{u}{v}\nabla v)^2 = |\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla v \cdot \nabla u = |\nabla u|^2 - \nabla(\frac{u^2}{v})\nabla v \geq 0. \tag{1.3}$$

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Biography: Feng Tingfu(1986–), male, born at Dali, Yunnan, lecturer, major in the theory of partial differential equations and its applications. E-mail: ftfml@mail.nwpu.edu.cn.

Dunninger [2] and Allegretto and Huang [3] independently extended (1.3) to p-Laplace operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$, respectively, namely, for differentiable functions $v > 0$ and $u \geqslant 0$,

$$|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v\cdot\nabla u = |\nabla u|^p - \nabla(\frac{u^p}{v^{p-1}})|\nabla v|^{p-2}\nabla v \geq 0.$$

As well known, There is a lot of literatures on the study of linear Picone identity, which show Picone identity is an important tool in the analysis of partial differential equations, we refer to [4–8] and related references. However, there is very little literatures on the study of nonlinear Picone identity. We first briefly review the research development of nonlinear Picone identity for Laplace operator and p-Laplace operator, respectively.

Recently, a nonlinear Picone identity for Laplace operator was presented by Tyagi [9] as follows.

Theorem 1.1 [9] Let $v > 0$ and $u \geqslant 0$ be two differentiable functions in the domain $\Omega \subset \mathbb{R}^n (n \geqslant 3)$. Denote

$$L(u,v) = \alpha|\nabla u|^2 - \frac{|\nabla u|^2}{f'(v)} + (\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}})^2,$$

$$R(u,v) = \alpha|\nabla u|^2 - \nabla(\frac{u^2}{f(v)})\nabla v,$$

where $f(y) > 0, 0 < y \in R$, and $f'(y) \geqslant \frac{1}{\alpha}$ for some $\alpha > 0$. Then $R(u,v) = L(u,v)$. Moreover, $L(u,v) \geqslant 0$, and $L(u,v) = 0$ a.e. in Ω if and only if $u = c_1 v + c_2$ a.e. in Ω for some constants c_1, c_2.

Bal [10] extended the nonlinear Picone identity of Tyagi [9] from Laplace operator to p-Laplace operator as follows. Moreover, Bal [11] also obtained the nonlinear Picone identity of [10] to p-sub-Laplace operator in Heisenberg group, which is an extension of the result in Euclidean space.

Theorem 1.2 [10] Let $v > 0$ and $u \geqslant 0$ be two differentiable functions in the domain $\Omega \subset \mathbb{R}^n (n \geqslant 3)$. Denote

$$L(u,v) = |\nabla u|^p - \frac{pu^{p-1}|\nabla v|^{p-2}\nabla v\cdot\nabla u}{f(v)} + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2},$$

$$R(u,v) = |\nabla u|^p - \nabla(\frac{u^p}{f(v)})|\nabla v|^{p-2}\nabla v,$$

where $f(y) > 0, 0 < y \in R$, and $f'(y) \geqslant (p-1)[f(y)^{\frac{p-2}{p-1}}], p > 1$. Then $R(u,v) = L(u,v)$. Moreover, $L(u,v) \geqslant 0$, and $L(u,v) = 0$ a.e. in Ω if and only if $u = c_1 v + c_2$ a.e. in Ω for some constants c_1, c_2.

Let us recall our previous works in Feng and Cui [12], in the following, we establish a linear Picone identity to anisotropic Laplace operator $\sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}), p_i > 1$.

Theorem 1.3 [12] Let $v > 0$ and $u \geqslant 0$ be two differentiable functions in the domain $\Omega \subset \mathbb{R}^n (n \geqslant 3)$. Denote

$$L(u,v)=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}p_i\frac{u^{p_i-1}}{v^{p_i-1}}|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}+\sum_{i=1}^{n}(p_i-1)\frac{u^{p_i}}{v^{p_i}}|\frac{\partial v}{\partial x_i}|^{p_i},$$
$$R(u,v)=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\frac{u^{p_i}}{v^{p_i-1}})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i},$$

where $p_i>1$ $(i=1,\cdots,n)$. Then $R(u,v)=L(u,v)$. Moreover, $L(u,v)\geqslant 0$, and $L(u,v)=0$ a.e. in Ω if and only if $u=c_1v+c_2$ a.e. in Ω for some constants c_1,c_2.

Based on our previous works in [12], in this paper, our aim is to obtain a nonlinear Picone identity to anisotropic Laplace operator and give its applications. Including that a Sturmian comparison principle to an anisotropic elliptic equation, a Liouville's theorem to an anisotropic elliptic system and a generalized anisotropic Hardy type inequality are obtained. Our main result is the following.

Theorem 1.4 Let $v>0$ and $u\geqslant 0$ be two differentiable functions in the domain $\Omega\subset\mathbb{R}^n(n\geqslant 3)$. Denote

$$L(u,v)=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}+\sum_{i=1}^{n}\frac{u^{p_i}f'(v)}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i},$$
$$R(u,v)=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i},$$

where $f(v)>0$ and $f'(v)\geqslant(p_i-1)[f(v)]^{\frac{p_i-2}{p_i-1}},p_i>1$. Then $R(u,v)=L(u,v)$. Moreover, $L(u,v)\geqslant 0$, and $L(u,v)=0$ a.e. in Ω if and only if $u=cv$ a.e. in Ω for a constant c.

Remark 1.5 We mention that if $p_i=2$, then Theorem 1.4 is the result in Tyagi [9] with $\alpha=1$; if $p_i=2$ and $f(v)=v$, then Theorem 1.4 is the result in Picone [1]; if $p_i>2$ and $f(v)=v^{p_i-1}$, then Theorem 1.4 is the result in Feng and Cui [12]; if $p_i=p>2$, then Theorem 1.4 is the result in Feng and Yu [13]; if $p_i=p>2$ and $f(v)=v^{p-1}$, then Theorem 1.4 is the result in Jaroš [8] with $p=r$.

This paper is organized as follows. The proof of Theorem 1.4 is given in Section 2; Section 3 is devoted to applications of Theorem 1.4.

2 Proof of Theorem 1.4

Proof of Theorem 1.4 We first prove $R(u,v)=L(u,v)$. Expanding $R(u,v)$ by a direct calculation,

$$\begin{aligned}R(u,v)&=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}\\&=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}\frac{p_iu^{p_i-1}\frac{\partial u}{\partial x_i}f(v)-u^{p_i}f'(v)\frac{\partial v}{\partial x_i}}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}\\&=\sum_{i=1}^{n}|\frac{\partial u}{\partial x_i}|^{p_i}-\sum_{i=1}^{n}p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}+\sum_{i=1}^{n}\frac{u^{p_i}f'(v)}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i}\\&=L(u,v).\end{aligned}$$

We next prove $L(u,v)\geqslant 0$. We can rewrite $L(u,v)$ as

$$\begin{aligned}L(u,v)=&\sum_{i=1}^{n}(p_i[\frac{1}{p_i}|\frac{\partial u}{\partial x_i}|^{p_i}+\frac{p_i-1}{p_i}(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}]-p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1}|\frac{\partial u}{\partial x_i}|)\\&+\sum_{i=1}^{n}(-(p_i-1)(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}+\frac{u^{p_i}f'(v)}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i})\\&+\sum_{i=1}^{n}p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-2}(|\frac{\partial v}{\partial x_i}||\frac{\partial u}{\partial x_i}|-\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i})\\:=&I+II+III,\end{aligned}\tag{2.1}$$

where

$$I=\sum_{i=1}^{n}(p_i[\frac{1}{p_i}|\frac{\partial u}{\partial x_i}|^{p_i}+\frac{p_i-1}{p_i}(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}]-p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1}|\frac{\partial u}{\partial x_i}|),$$

$$II=\sum_{i=1}^{n}(-(p_i-1)(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}+\frac{u^{p_i}f'(v)}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i}),$$

$$III=\sum_{i=1}^{n}p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-2}(|\frac{\partial v}{\partial x_i}||\frac{\partial u}{\partial x_i}|-\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}).$$

Let us recall Young's inequality

$$ab\leqslant\frac{a^{p_i}}{p_i}+\frac{b^{q_i}}{q_i}\tag{2.2}$$

for $a\geqslant 0,b\geqslant 0$, where $p_i>1,q_i>1$ and $\frac{1}{p_i}+\frac{1}{q_i}=1$, and the equality holds if and only if $a=b^{\frac{1}{p_i-1}}$. Taking $a=|\frac{\partial u}{\partial x_i}|$ and $b=\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1}$ in (2.2), it yields

$$p_i\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p-1}|\frac{\partial u}{\partial x_i}|\leqslant p_i[\frac{1}{p_i}|\frac{\partial u}{\partial x_i}|^{p_i}+\frac{p_i-1}{p_i}(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}],\tag{2.3}$$

and thus $I\geqslant 0$ by (2.3). Since $f(v)>0$ and $f'(v)\geqslant(p_i-1)[f(v)]^{\frac{p_i-2}{p_i-1}}$, we obtain

$$\begin{aligned}&-(p_i-1)(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}+\frac{u^{p_i}f'(v)}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i}\\\geqslant&-(p_i-1)(\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{p_i}{p_i-1}}+\frac{u^{p_i}(p_i-1)[f(v)]^{\frac{p_i-2}{p_i-1}}}{[f(v)]^2}|\frac{\partial v}{\partial x_i}|^{p_i}=0,\end{aligned}\tag{2.4}$$

then $II\geqslant 0$ by (2.4). Let us recall Hölder's inequality

$$a_0b_0\leqslant({a_0}^2)^{\frac{1}{2}}({b_0}^2)^{\frac{1}{2}}\tag{2.5}$$

for $a_0\geqslant 0,b_0\geqslant 0$, and the equality holds if and only if $a_0=cb_0$, c is a constant. Taking $a_0=\frac{\partial v}{\partial x_i}$ and $b_0=\frac{\partial u}{\partial x_i}$ in (2.5), it yields

$$|\frac{\partial v}{\partial x_i}||\frac{\partial u}{\partial x_i}|-\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}\geqslant 0,\tag{2.6}$$

it follows from $f(v) > 0, u \geqslant 0$ and (2.6) that $III \geqslant 0$. Hence $L(u, v) \geqslant 0$ by (2.1).

The proof process above also shows that $L(u, v) = 0$ a.e. in Ω if and only if

$$|\frac{\partial u}{\partial x_i}| = (\frac{u^{p_i-1}}{f(v)}|\frac{\partial v}{\partial x_i}|^{p_i-1})^{\frac{1}{p_i-1}}, f(v) > 0, \tag{2.7}$$

$$f'(v) = (p_i - 1)[f(v)]^{\frac{p_i-2}{p_i-1}}, f(v) > 0, \tag{2.8}$$

$$\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i} = |\frac{\partial v}{\partial x_i}||\frac{\partial u}{\partial x_i}| \tag{2.9}$$

a.e. in Ω. It follows from (2.7), (2.8) and (2.9) that $u = cv$ a.e. in Ω for a constant c. Hence $L(u, v) = 0$ a.e. in Ω if and only if $u = cv$ a.e. in Ω . The proof of Theorem 1.4 is ended.

3 Applications

Throughout this section, we always assume that $v > 0, f(v) > 0$ in Ω, and $f'(v) \geqslant (p_i - 1)[f(v)^{\frac{p_i-2}{p_i-1}}]$ for $p_i > 1, i = 1, \cdots, n$.

Let us state anisotropic Sobolev space $W_0^{1,(p_i)}(\Omega)$ needed in this paper, see Adams [14], Lu [15] and Troisi [16] et al. Let $\Omega \subset \mathbb{R}^n (n \geqslant 3)$ be a bounded domain, $p_i > 1$, $i = 1, 2, \cdots, n$. We define anisotropic Sobolev space $W_0^{1,(p_i)}(\Omega)$ by

$$W_0^{1,(p_i)}(\Omega) = \{u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, \cdots, n\}$$

with the norm $\|u\|_{W_0^{1,(p_i)}(\Omega)} = \sum\limits_{i=1}^{n} (\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i} dx)^{\frac{1}{p_i}}$. It is well known that $W_0^{1,p_i}(\Omega)$ is a separable reflexive Banach space.

We give three examples of applications of Theorem 1.4. The first is a Liouville's theorem to an anisotropic elliptic system.

Example 3.1 Let $g(u, v)$ be an integrable function in Ω. Assume that $(u, v) \in W_0^{1,(p_i)}(\Omega) \times W_0^{1,(p_i)}(\Omega)$ is a pair of solution to an anisotropic elliptic system

$$\begin{cases} \sum\limits_{i=1}^{n} |\frac{\partial u}{\partial x_i}|^{p_i} = g(u, v), & x \in \Omega, \\ \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i} = g(u, v), & x \in \Omega, \\ u > 0, v > 0, & x \in \Omega, \\ u = 0, v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Then $u = cv$ a.e. in Ω for some constants c.

Proof It follows from (3.1) that

$$\sum_{i=1}^{n}\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i} dx = \int_\Omega g(u, v)dx = \sum_{i=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}dx,$$

which implies

$$\int_\Omega L(u,v)dx=\int_\Omega R(u,v)dx=\sum_{i=1}^n\int_\Omega|\frac{\partial u}{\partial x_i}|^{p_i}dx-\sum_{i=1}^n\int_\Omega\frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}dx=0,$$

hence the conclusion is true by Theorem 1.4.

The next is a Sturmian comparison principle to an anisotropic elliptic equation.

Example 3.2 Let $f_1(x)$ and $f_2(x)$ be two continuous functions with $f_1(x)<f_2(x)$. Assume that $u\in W_0^{1,(p_i)}(\Omega)$ satisfies the following Dirichlet problem

$$\begin{cases}-\sum\limits_{i=1}^n\frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i})=\sum\limits_{i=1}^n f_1(x)u^{p_i-1}, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}\tag{3.2}$$

Then any nontrivial solution v of the equation

$$-\sum_{i=1}^n\frac{u^{p_i}}{f(v)}\frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i})=\sum_{i=1}^n f_2(x)u^{p_i},x\in\Omega\tag{3.3}$$

must change sign.

Proof Assume that the solution v in (3.3) does not change sign and without loss of generality, let $v>0$ in Ω. We have by (3.2) and Theorem 1.4 that

$$\begin{aligned}0&\leqslant\int_\Omega L(u,v)dx=\int_\Omega R(u,v)dx\\&=\sum_{i=1}^n\int_\Omega|\frac{\partial u}{\partial x_i}|^{p_i}dx+\sum_{i=1}^n\int_\Omega\frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}dx\\&=\sum_{i=1}^n\int_\Omega|\frac{\partial u}{\partial x_i}|^{p_i}dx-\sum_{i=1}^n\int_\Omega\frac{u^{p_i}}{f(v)}\frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i})dx\\&=\sum_{i=1}^n\int_\Omega(f_1(x)-f_2(x))u^{p_i}dx<0,\end{aligned}$$

which is a contradiction. This accomplishes the proof.

The end is a generalized anisotropic Hardy type inequality.

Example 3.3 Assume that there exists a constant $k>0$ and a function $h(x)$ such that

$$-\frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i})\geqslant kh(x)f(v),x\in\Omega\tag{3.4}$$

for every $i=1,\cdots,n$. Then,

$$\sum_{i=1}^n\int_\Omega|\frac{\partial u}{\partial x_i}|^{p_i}dx\geqslant\sum_{i=1}^n k\int_\Omega h(x)u^{p_i}dx\tag{3.5}$$

for any $0 \leqslant u \in C_0^1(\Omega)$.

Proof By (3.4) and Theorem 1.4, we have

$$\begin{aligned}
0 &\leqslant \int_\Omega L(u,v)dx = \int_\Omega R(u,v)dx \\
&= \sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i} dx + \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}(\frac{u^{p_i}}{f(v)})|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i}dx \\
&= \sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i} dx - \sum_{i=1}^n \int_\Omega \frac{u^{p_i}}{f(v)}\frac{\partial}{\partial x_i}(|\frac{\partial v}{\partial x_i}|^{p_i-2}\frac{\partial v}{\partial x_i})dx \\
&\leqslant \sum_{i=1}^n \int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i} dx - \sum_{i=1}^n k\int_\Omega h(x)u^{p_i}dx,
\end{aligned}$$

which implies (3.5).

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各向异性 Laplace 算子的一个非线性 Picone 恒等式及其应用

冯廷福[1], 张克磊[2]

(1.昆明学院数学学院, 云南 昆明 650214)

(2.桂林电子科技大学数学与计算科学学院, 广西 桂林 541004)

摘要: 本文针对各向异性 Laplace 算子, 建立了一个非线性 Picone 恒等式. 作为它的应用, 得到了各向异性椭圆方程的 Sturmian 比较原理、各向异性椭圆系统的 Liouville 定理和广义的各向异性 Hardy 型不等式.

关键词: 各向异性 Laplace 算子; 非线性 Picone 恒等式; 应用

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