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RESEARCH ANNOUNCEMENTS ON "MINIMAL TIME CONTROL OF EXACT SYNCHRONIZATION FOR PARABOLIC SYSTEMS"

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1 Introduction and Main Results

Let $\Omega \subseteq \mathbb{R}^d$ (with $d \geq 1$) be a bounded domain with a C^2 boundary $\partial\Omega$. Let $\omega \subseteq \Omega$ be an open and nonempty subset with its characteristic function χ_ω. Let $A \triangleq (a_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ and $B \triangleq (b_{ij})_{1\leq i\leq n, 1\leq j\leq m} \in \mathbb{R}^{n\times m}$ be two constant matrices, where $n \geq 2$ and $m \geq 1$. Let $\boldsymbol{y}_0 \in L^2(\Omega)^n$. Consider the controlled linear parabolic system

$$\begin{cases} \boldsymbol{y}_t - \Delta \boldsymbol{y} + A\boldsymbol{y} = \chi_\omega B\boldsymbol{u} & \text{in} \quad \Omega \times (0, +\infty), \\ \boldsymbol{y} = \boldsymbol{0} & \text{on} \quad \partial\Omega \times (0, +\infty), \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} \in L^2(0, +\infty; L^2(\Omega)^m)$ is a control. Write $\boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{u})$ for the solution of system (1.1). It is well known that for each $T > 0$, $\boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{u}) \in W^{1,2}(0, T; H^{-1}(\Omega)^n) \cap L^2(0, T; H_0^1(\Omega)^n) \subseteq C([0, T]; L^2(\Omega)^n)$. We will treat this solution as a function from $[0, +\infty)$ to $L^2(\Omega)^n$.

We next define control constraint set $\mathcal{U}_M$ (with $M > 0$) and the target set S as follows

$$\begin{aligned} &\mathcal{U}_M \triangleq \left\{\boldsymbol{u} \in L^2(0, +\infty; L^2(\Omega)^m) \ : \ \|\boldsymbol{u}\|_{L^2(0,+\infty;L^2(\Omega)^m)} \leq M\right\}; \\ &S \triangleq \{(y_1, y_2, \ldots, y_n)^\top \in L^2(\Omega)^n : y_1 = y_2 = \cdots = y_n\}. \end{aligned}$$

Given $M > 0$, $\boldsymbol{y}_0 \in L^2(\Omega)^n$, we define the minimal time control problem $(TP)_M^{\boldsymbol{y}_0}$:

$$T(M, \boldsymbol{y}_0) \triangleq \inf_{\boldsymbol{u}\in\mathcal{U}_M} \{T \geq 0 \ : \ \boldsymbol{u}(\cdot) = \boldsymbol{0} \ \text{ and } \ \boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{u}) \in S \text{ over } [T, +\infty)\}.$$

About problem $(TP)_M^{\boldsymbol{y}_0}$, several notes are given in order

(a_1) We call $T(M, \boldsymbol{y}_0)$ the optimal time; we call $\boldsymbol{u} \in \mathcal{U}_M$ an admissible control if there is $T \geq 0$ so that $\boldsymbol{u}(\cdot) = \boldsymbol{0}$ and $\boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{u}) \in S$ over $[T, +\infty)$; we call $\boldsymbol{u}^* \in \mathcal{U}_M$ an optimal

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control if $T(M, \boldsymbol{y}_0) < +\infty, \boldsymbol{u}^*(\cdot) = \mathbf{0}$ and $\boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{u}^*) \in S$ over $[T(M, \boldsymbol{y}_0), +\infty)$; we agree that $T(M, \boldsymbol{y}_0) = +\infty$ if problem $(TP)_M^{\boldsymbol{y}_0}$ has no admissible control.

(a_2) One can easily check that if $\boldsymbol{y} \in L^2(\Omega)^n$, then $\boldsymbol{y} \in S$ if and only if $D\boldsymbol{y} = \mathbf{0}$, where

$$D \triangleq \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}_{(n-1)\times n}.$$

(a_3) Differing from a general minimal time control problem, our problem here is to ask for a control (from the constraint set) not only driving the corresponding solution to the target S at the shortest time, but also remaining the solution in S after the shortest time with the null control. This arises from the characteristic of the exact synchronization. When the target is an equilibrium solution of the system with the null control, this can be done by taking the null control after the shortest time. However, the elements in S may not be equilibrium solutions. Thus, we need some reasonable assumptions to fit it.

Hypotheses Our main theorems are based on one of the following two hypotheses.

(H_1) The pair (A, B) satisfies that

$$\sum_{\ell=1}^{n} a_{i\ell} = \sum_{\ell=1}^{n} a_{j\ell} \ \text{ for all } \ i, j \in \{1, 2, \dots, n\}; \tag{1.2}$$

and that

$$\operatorname{rank}(DB, DAB, \dots, DA^{n-2}B) = n - 1.$$

(H_2) The pair (A, B) satisfies that

$$\sum_{\ell=1}^{n} a_{i_0\ell} \neq \sum_{\ell=1}^{n} a_{j_0\ell} \ \text{ for some } \ i_0, j_0 \in \{1, 2, \dots, n\};$$

and that

$$\operatorname{rank}(B, AB, \dots, A^{n-1}B) = n.$$

Several remarks on these hypotheses are given in order.

(b_1) One can easily see that (H_1) differs from (H_2).

(b_2) (1.2) is equivalent to that (see [2]) there exists a unique matrix $\widetilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ so that

$$DA = \widetilde{A}D.$$

(b_3) There is a pair (A, B) satisfying (H_1). For example

$$A = \begin{pmatrix} 1 & 0 \\ 0.5 & 0.5 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b_4) There is a pair (A, B) satisfying (H_2). For example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b_5) It is proved that system (1.1) is exactly synchronizable at time T if and only if (A, B) satisfies either (H_1) or (H_2) (see [4]).

Our main results will be given by two theorems. To state them, we need to introduce one kind of minimal norm control problem and two kinds of functionals under either (H_1) or (H_2).

Minimal Norm Control Problem Given $T > 0$ and $\boldsymbol{y}_0 \in L^2(\Omega)^n$, define the minimal norm control problem $(NP)_T^{\boldsymbol{y}_0}$ in the following

$$N(T, \boldsymbol{y}_0) \triangleq \inf\{\|\boldsymbol{v}\|_{L^2(0,+\infty;L^2(\Omega)^m)} : \boldsymbol{v}(\cdot) = \boldsymbol{0} \text{ and } \boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{v}) \in S \text{ over } [T, +\infty)\}.$$

Several notes on the problem $(NP)_T^{\boldsymbol{y}_0}$ are given in order.

(c_1) We call $N(T, \boldsymbol{y}_0)$ the minimal norm; we call $\boldsymbol{v} \in L^2(0, +\infty; L^2(\Omega)^m)$ an admissible control if $\boldsymbol{v}(\cdot) = \boldsymbol{0}$ and $\boldsymbol{y}(\cdot; \boldsymbol{y}_0, \boldsymbol{v}) \in S$ over $[T, +\infty)$; we call a function $\boldsymbol{v}^*$ an optimal control if it is admissible and satisfies that $\|\boldsymbol{v}^*\|_{L^2(0,+\infty;L^2(\Omega)^m)} = N(T, \boldsymbol{y}_0)$.

(c_2) Given $\boldsymbol{y}_0 \in L^2(\Omega)^n$, we can treat $N(\cdot, \boldsymbol{y}_0)$ as a function of the time variable. We proved that if either (H_1) or (H_2) holds, then for each $\boldsymbol{y}_0 \in L^2(\Omega)^n$, $\lim\limits_{T\to+\infty} N(T, \boldsymbol{y}_0)$ exists (see [4]). Thus, under either (H_1) or (H_2), we can let

$$M(\boldsymbol{y}_0) \triangleq \lim_{T\to+\infty} N(T, \boldsymbol{y}_0) \text{ for each } \boldsymbol{y}_0 \in L^2(\Omega)^n.$$

(c_3) If either (H_1) or (H_2) holds, then for any $T > 0$ and $\boldsymbol{y}_0 \in L^2(\Omega)^n$, problem $(NP)_T^{\boldsymbol{y}_0}$ has a unique optimal control (see [4]).

Two Auxiliary Functionals The first functional is built up (under assumption (H_1)) in the following manner.

Recall the note (b_2) for the matrix $\widetilde{A}$. Let $T > 0$ and let $\boldsymbol{y}_0 \in L^2(\Omega)^n$. Write $\boldsymbol{\psi}(\cdot; T, \boldsymbol{\psi}_T)$, with $\boldsymbol{\psi}_T \in L^2(\Omega)^{n-1}$, for the solution to the system

$$\begin{cases} \boldsymbol{\psi}_t + \Delta\boldsymbol{\psi} - \widetilde{A}^\top\boldsymbol{\psi} = \boldsymbol{0} & \text{in } \quad \Omega \times (0, T), \\ \boldsymbol{\psi} = \boldsymbol{0} & \text{on } \quad \partial\Omega \times (0, T) \end{cases} \tag{1.3}$$

with the initial condition $\boldsymbol{\psi}(T) = \boldsymbol{\psi}_T$. Here and throughout this paper, we denote the transposition of a matrix J by $J^\top$. Construct two subspaces

$$X_{T,1} \triangleq \{\chi_\omega B^\top D^\top \boldsymbol{\psi}(\cdot; T, \boldsymbol{\psi}_T) : \boldsymbol{\psi}_T \in L^2(\Omega)^{n-1}\}$$

and

$$Y_{T,1} \triangleq \overline{X_{T,1}}^{\|\cdot\|_{L^2(0,T;L^2(\omega)^m)}}.$$

We can characterize elements in the space $Y_{T,1}$ (see [4]). In fact, each element in $Y_{T,1}$ can be expressed as $\chi_\omega B^\top D^\top \boldsymbol{\psi}$, where $\boldsymbol{\psi} \in C([0, T); L^2(\Omega)^{n-1})$ solves (1.3) and satisfies that

$\chi_\omega B^\top D^\top \boldsymbol{\psi}(\cdot) = \lim\limits_{i\to+\infty} \chi_\omega B^\top D^\top \boldsymbol{\psi}(\cdot; T, \boldsymbol{z}_i)$ for some sequence $\{\boldsymbol{z}_i\}_{i\geq 1} \subseteq L^2(\Omega)^{n-1}$, where the limit is taken in $L^2(0,T; L^2(\omega)^m)$. Define the first functional $J_{T,1}^{\boldsymbol{y}_0} : Y_{T,1} \to \mathbb{R}$ by

$$J_{T,1}^{\boldsymbol{y}_0}(\chi_\omega B^\top D^\top \boldsymbol{\psi}) \triangleq \frac{1}{2}\int_0^T \|\chi_\omega B^\top D^\top \boldsymbol{\psi}\|^2_{L^2(\omega)^m} \mathrm{d}t + \langle \boldsymbol{\psi}(0), D\boldsymbol{y}_0\rangle_{L^2(\Omega)^{n-1}}$$

for each $\chi_\omega B^\top D^\top \boldsymbol{\psi} \in Y_{T,1}$.

The second functional is defined (under assumption (H_2)) in the following.

Let $T > 0$ and let $\boldsymbol{y}_0 \in L^2(\Omega)^n$. Write $\boldsymbol{\varphi}(\cdot; T, \boldsymbol{\varphi}_T)$, with $\boldsymbol{\varphi}_T \in L^2(\Omega)^n$, for the solution to the system

$$\begin{cases} \boldsymbol{\varphi}_t + \Delta\boldsymbol{\varphi} - A^\top \boldsymbol{\varphi} = \mathbf{0} & \text{in} \quad \Omega\times(0,T), \\ \boldsymbol{\varphi} = \mathbf{0} & \text{on} \quad \partial\Omega\times(0,T) \end{cases} \tag{1.4}$$

with the initial condition $\boldsymbol{\varphi}(T) = \boldsymbol{\varphi}_T$. Build up two subspaces

$$X_{T,2} \triangleq \{\chi_\omega B^\top \boldsymbol{\varphi}(\cdot; T, \boldsymbol{\varphi}_T) : \boldsymbol{\varphi}_T \in L^2(\Omega)^n\} \ \text{ and } \ Y_{T,2} \triangleq \overline{X_{T,2}}^{\|\cdot\|_{L^2(0,T;L^2(\omega)^m)}}.$$

We can also characterize elements in the space $Y_{T,2}$ (see [4]). Indeed, each element in $Y_{T,2}$ can be expressed as $\chi_\omega B^\top \boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \in C([0,T); L^2(\Omega)^n)$ solves (1.4) and satisfies that $\chi_\omega B^\top \boldsymbol{\varphi}(\cdot) = \lim\limits_{i\to+\infty} \chi_\omega B^\top \boldsymbol{\varphi}(\cdot; T, \boldsymbol{z}_i)$ for some sequence $\{\boldsymbol{z}_i\}_{i\geq 1} \subseteq L^2(\Omega)^n$, where the limit is taken in $L^2(0,T; L^2(\omega)^m)$. Define the second functional $J_{T,2}^{\boldsymbol{y}_0} : Y_{T,2} \to \mathbb{R}$ by

$$J_{T,2}^{\boldsymbol{y}_0}(\chi_\omega B^\top \boldsymbol{\varphi}) \triangleq \frac{1}{2}\int_0^T \|\chi_\omega B^\top \boldsymbol{\varphi}\|^2_{L^2(\omega)^m} \mathrm{d}t + \langle \boldsymbol{\varphi}(0), \boldsymbol{y}_0\rangle_{L^2(\Omega)^n}$$

for each $\chi_\omega B^\top \boldsymbol{\varphi} \in Y_{T,2}$.

Two notes on these two functionals are given in order.

(d_1) The functional $J_{T,1}^{\boldsymbol{y}_0}$ has the following properties: (1) it is well defined on $Y_{T,1}$; (2) it has a unique nontrivial minimizer in $Y_{T,1}$ when $\boldsymbol{y}_0 \notin S$ (see [4]).

(d_2) The functional $J_{T,2}^{\boldsymbol{y}_0}$ has the following properties: (1) it is well defined on $Y_{T,2}$; (2) it has a unique nontrivial minimizer in $Y_{T,2}$ when $\boldsymbol{y}_0 \neq \mathbf{0}$ (see [4]).

The main theorems of this paper are as follows.

Theorem 1.1 Suppose that (H_1) holds. Let $\boldsymbol{y}_0 \in L^2(\Omega)^n$ and let $M > 0$. The following conclusions are true

(i) If $\boldsymbol{y}_0 \in S$, then $(TP)_M^{\boldsymbol{y}_0}$ has the unique optimal control $\mathbf{0}$ (while 0 is the optimal time); If $\boldsymbol{y}_0 \notin S$ and $M \leq M(\boldsymbol{y}_0)$, then $(TP)_M^{\boldsymbol{y}_0}$ has no optimal control; If $\boldsymbol{y}_0 \notin S$ and $M > M(\boldsymbol{y}_0)$, then $(TP)_M^{\boldsymbol{y}_0}$ has a unique nontrivial optimal control.

(ii) If $\boldsymbol{y}_0 \notin S$ and $M > M(\boldsymbol{y}_0)$, then T^* and $\boldsymbol{u}^*$ are the optimal time and the optimal control to $(TP)_M^{\boldsymbol{y}_0}$ if and only if

$$M = \Big(\int_0^{T^*} \|\chi_\omega B^\top D^\top \boldsymbol{\psi}^*(t)\|^2_{L^2(\omega)^m} \mathrm{d}t\Big)^{\frac{1}{2}}$$

and

$$\boldsymbol{u}^*(t) \triangleq \begin{cases} \chi_\omega B^\top D^\top \boldsymbol{\psi}^*(t), & t \in (0, T^*), \\ \mathbf{0}, & t \geq T^*, \end{cases}$$

where $\chi_\omega B^\top D^\top \boldsymbol{\psi}^*$, with $\boldsymbol{\psi}^* \in C([0,T^*];L^2(\Omega)^{n-1})$ solving (1.3), is the unique minimizer of $J^{\boldsymbol{y}_0}_{T^*,1}$ over $Y_{T^*,1}$.

Theorem 1.2 Suppose that (H_2) holds. Let $\boldsymbol{y}_0 \in L^2(\Omega)^n$ and let $M > 0$. The following conclusions are true

(i) If $\boldsymbol{y}_0 = \mathbf{0}$, then $(TP)^{\boldsymbol{y}_0}_M$ has the unique optimal control $\mathbf{0}$ (while 0 is the optimal time); If $\boldsymbol{y}_0 \neq \mathbf{0}$ and $M \leq M(\boldsymbol{y}_0)$, then $(TP)^{\boldsymbol{y}_0}_M$ has no optimal control; If $\boldsymbol{y}_0 \neq \mathbf{0}$ and $M > M(\boldsymbol{y}_0)$, then $(TP)^{\boldsymbol{y}_0}_M$ has a unique nontrivial optimal control.

(ii) If $\boldsymbol{y}_0 \neq \mathbf{0}$ and $M > M(\boldsymbol{y}_0)$, then T^* and $\boldsymbol{u}^*$ are the optimal time and the optimal control to $(TP)^{\boldsymbol{y}_0}_M$ if and only if

$$M = \Big(\int_0^{T^*} \|\chi_\omega B^\top \boldsymbol{\varphi}^*(t)\|^2_{L^2(\omega)^m} \mathrm{d}t \Big)^{\frac{1}{2}}$$

and

$$\boldsymbol{u}^*(t) \triangleq \begin{cases} \chi_\omega B^\top \boldsymbol{\varphi}^*(t), & t \in (0,T^*), \\ \mathbf{0}, & t \geq T^*, \end{cases}$$

where $\chi_\omega B^\top \boldsymbol{\varphi}^*$, with $\boldsymbol{\varphi}^* \in C([0,T^*];L^2(\Omega)^n)$ solving (1.4), is the unique minimizer of $J^{\boldsymbol{y}_0}_{T^*,2}$ over $Y_{T^*,2}$.

Remark Several notes on Theorem 1.1 and Theorem 1.2 are given in order.

(a) Conclusion (i) in Theorem 1.1 (or Theorem 1.2) shows how the existence of optimal controls to $(TP)^{\boldsymbol{y}_0}_M$ depends on $(M, \boldsymbol{y}_0) \in (0,+\infty) \times L^2(\Omega)^n$.

(b) Conclusion (ii) in Theorem 1.1 (or Theorem 1.2) gives characteristics of the optimal time and the optimal control via the minimizer of a given functional, under assumption (H_1) (or (H_2)).

(c) By (ii) in Theorem 1.1 (or Theorem 1.2), we can use the similar way to that used in [3] to get an algorithm for the optimal time and the optimal control.

(d) Theorems 1.1 and 1.2 can be extended to the boundary control case. For example, we consider the controlled linear parabolic system

$$\begin{cases} \boldsymbol{y}_t - \boldsymbol{y}_{xx} + A\boldsymbol{y} = \mathbf{0} & \text{in} \quad (0,\pi) \times (0,+\infty), \\ \boldsymbol{y}(0,\cdot) = B\boldsymbol{v}, \quad \boldsymbol{y}(\pi,\cdot) = \mathbf{0} & \text{on} \quad (0,+\infty), \\ \boldsymbol{y}(\cdot,0) = \boldsymbol{y}_0 & \text{in} \quad (0,\pi), \end{cases} \tag{1.5}$$

where $\boldsymbol{y}_0 \in H^{-1}(0,\pi)^n$ and $\boldsymbol{v} \in L^2(0,+\infty)^m$ is a control. For each $T > 0$, system (1.5) has a unique solution (defined by transposition, see [1]) $\boldsymbol{y}(\cdot;\boldsymbol{y}_0,\boldsymbol{v}) \in C([0,T];H^{-1}(0,\pi)^n)$. By Theorem 1.1 and Proposition 2.4 in [1], we can employ the similar method to that used in [4] to obtain similar results as Theorems 1.1 and 1.2.

The details of proofs for Theorems 1.1 and Theorem 1.2 were given by [4].

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