Vol. 40 (2020)
No. 3

数 学 杂 志
J. of Math. (PRC)

RESEARCH ANNOUNCEMENTS ON "DEFORMATION LIMIT AND BIMEROMORPHIC EMBEDDING OF MOISHEZON MANIFOLDS"

RAO Sheng[1,2], TSAI I-Hsun [3]

(1. *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*)
(2. *Université de Grenoble-Alpes, Institut Fourier (Mathématiques) UMR 5582 du C.N.R.S., 100 rue des Maths, 38610 Gières, France*)
(3. *Department of Mathematics, National Taiwan University, Taipei 10617, China*)

This note is a research announcement on [18] for the deformation limit of Moishezon manifolds and most all of the further details can be found there.

1 Main Results

The deformation limit problem is central in deformation theory, with which the following longstanding conjecture is concerned. Throughout this note, one considers the holomorphic family $\pi : \mathcal{X} \to \Delta$ of compact complex manifolds of dimension n over an open disk Δ in $\mathbb{C}$ with the fiber $X_t := \pi^{-1}(t)$ for each $t \in \Delta$. By a holomorphic family $\pi : \mathcal{Y} \to B$ of compact complex manifolds, we mean that π is a proper holomorphic surjective submersion between complex manifolds as in [9, Definition 2.8].

Conjecture 1.1 Assume that the fiber X_t is projective for each $t \in \Delta^* := \Delta \setminus \{0\}$. Then the reference fiber $X_0 := \pi^{-1}(0)$ is Moishezon.

By definition, a compact connected complex manifold X is called a Moishezon manifold if it possesses $\dim_{\mathbb{C}} X$ algebraically independent meromorphic functions. Equivalently, X is Moishezon if and only if there exist a projective algebraic manifold Y and a holomorphic modification $Y \to X$. Any connected projective manifold is Moishezon.

The following is a stronger variant of the above.

Conjecture 1.2 If the fiber X_t is Moishezon for each $t \in \Delta^*$, then the reference fiber $X_0 := \pi^{-1}(0)$ is Moishezon.

The above two conjectures are actually equivalent to

Conjecture 1.3 Let $\pi : \mathcal{X} \to Y$ be a holomorphic family of compact complex manifolds over a complex variety Y, $V \subset Y$ a proper subvariety and write $Y' = Y \setminus V$. Suppose that

* **Received date:** 2020-03-30 **Accepted date:** 2020-04-15

Foundation item: The first author is partially supported by National Natural Science Foundation of China (11671305; 11771339; 11922115).

Biography: Rao Sheng (1982–), male, born at Fengcheng, Jiangxi, professor, major in complex geometry, several complex variables, algebraic geometry. E-mail: likeanyone@whu.edu.cn.

X_t are Moishezon (or projective) for all $t \in Y'$. Then X_t are Moishezon for all $t \in V$.

In fact, fix a point t_0 of V, take D as a one-dimensional disc in Y with t_0 being the center of D and set $V' := D \cap V$. Then V' is a subvariety of D. Suppose that D is not contained in V. By the identity theorem, V' is a discrete subset of D. Hence by shrinking D, we may assume that V' is just the point t_0.

Popovici proposed proofs of Conjectures 1.1, 1.2 in [12, 13], respectively, and Barlet presented several related results to Conjecture 1.2 in [2]. Our results, some of which were involved in [12–14] for which we propose a new proof, can be summed up as follows. Recall that for a complex n-dimensional manifold X, a smooth positive-definite $(1,1)$-form α on X is said to be a strongly Gauduchon metric if the $(n, n-1)$-form $\partial\alpha^{n-1}$ is $\bar{\partial}$-exact on X. If X carries such a metric, X will be said to be a strongly Gauduchon manifold. This notion was introduced by Popovici in [14].

As the main theorem of [18], we prove the following result.

Theorem 1.4 If the fiber X_t is Moishezon for each nonzero t in an uncountable subset B of Δ, with 0 not necessarily being a limit point of B, and the reference fiber X_0 satisfies the local deformation invariance for Hodge number of type $(0,1)$ or admits a strongly Gauduchon metric, then over $\Delta_\epsilon := \{z \in \mathbb{C} : |z| < \epsilon\}$ with some small constant $\epsilon > 0$,

(i) X_t is still Moishezon for any $t \in \Delta_\epsilon$.

(ii) For some $N \in \mathbb{N}$, there exist a bimeromorphic map $\Phi : \mathcal{X}_{\Delta_\epsilon} \dashrightarrow \mathcal{Y}$ from $\mathcal{X}_{\Delta_\epsilon} := \pi^{-1}(\Delta_\epsilon)$ to a subvariety $\mathcal{Y}$ of $\mathbb{P}^N \times \Delta_\epsilon$ with every fiber $Y_t \subset \mathbb{P}^N \times \{t\}$ being a projective variety of dimension n, and also a proper analytic set $\Sigma \subset \Delta_\epsilon$, such that Φ induces a bimeromorphic map $\Phi|_{X_t} : X_t \dashrightarrow Y_t$ for every $t \in \Delta_\epsilon \setminus \Sigma$.

In the terminology of [5, Definition 3.5], we may say that our family $\pi : \mathcal{X} \to \Delta$ is Moishezon, meaning that it is bimeromorphically equivalent over Δ to a proper holomorphic map $p : \mathcal{Y} \to \Delta$ from a complex variety, which is p-ample, in particular, every fiber $Y_t \subset \mathcal{Y}$ is a projective variety. It is remarked in [5, p.334] that not every holomorphic map $f : X \to Y$ between complex spaces such that every fiber $f^{-1}(y)$ is Moishezon, is Moishezon. Here the meromorphic/bimeromorphic maps are understood and defined in the sense of Remmert (see [21], [23] and [26]).

In contrast to Popovici's approach which is analytic in nature, our approach is partly built on algebraic methods in the sense of Grauert, cf. [1]. In the strongly Gauduchon case, we resort to Monge–Ampère equations of degenerate type with solutions obtained by S.-T. Yau [28] and to Popovici's criterion on big line bundles using mass control [11], via Fujita's approximate Zariski decomposition [8]. We use an uncountable subset B (with $0 \notin \bar{B}$ allowed) for the assumed Moishezon conditions rather than the whole Δ^* as Popovici does, while the case $0 \in \bar{B}$ is implicit in [12–14], whose approaches are not applicable to our case in Theorem 1.4 directly.

To prove Theorem 1.4, we obtain

Proposition 1.5 Let $\pi : \mathcal{X} \to Y$ be a holomorphic family of compact complex n-dimensional manifolds. Assume that there exists a holomorphic line bundle L on $\mathcal{X}$ such that for each t in an uncountable set of Y, $L|_{X_t}$ is big. Then for each $t \in Y$, $L|_{X_t}$ is also big and thus X_t is Moishezon.

The Hodge number deformation invariance or the existence of strongly Gauduchon metric in Theorem 1.4 is used to obtain such a holomorphic line bundle over the total space. Notice that the result [25, Examples 1 and 2] implies that 'uncountable' is an indispensable condition there. Moreover, Campana's counterexample in [4, Corollary 3.13] shows that the small deformation of a Moishezon manifold which is not of general type, is not necessarily Moishezon. Based on these, it is reasonable to propose

Question 1.6 Characterize those Moishezon manifolds which are still Moishezon after a small deformation.

Conjecture 1.7 If the fiber X_t is Moishezon for each t in an uncountable subset of Δ, then there exists a global holomorphic line bundle $\tilde{L}$ on $\mathcal{X}$ such that the restriction $\tilde{L}|_{X_t}$ is big for every $t \in \Delta$.

Actually, the proof of Theorem 1.4 shows that if Conjecture 1.7 holds true, then the family $\pi : \mathcal{X} \to \Delta$ is Moishezon, which in turn induces a bimeromorphic map on X_t for every $t \in \Delta \setminus \Sigma$ for some proper analytic set $\Sigma \subset \Delta$.

Theorem 1.4 can be considered as a new understanding of Popovici's remarkable result on deformation limit of projective manifolds from a global and algebraic point of view by a construction of a global holomorphic line bundle over the total space

Corollary 1.8 (see [14, Theorems 1.2, 1.4]) If for each $t \in \Delta^*$, the fiber X_t is projective and the reference fiber X_0 satisfies the local deformation invariance for Hodge number of type $(0, 1)$ or admits a strongly Gauduchon metric, then X_0 is Moishezon.

The work [17, Corollary 1.6] or the $q = 1$ case of [20, Theorem 1.4 (2)] shows that either the **sGG** condition on X_0 or the surjectivity of the natural mapping $\iota^{0,1}_{BC,\bar{\partial}}$ from the $(0, 1)$-Bott–Chern cohomology group of X_0 to the Dolbeault one, guarantees that the $(0, 1)$-type Hodge numbers of X_t are independent for small t. Notice that by [20, Remark 3.8] this surjectivity is equivalent to the **sGG** condition proposed by Popovici–Ugarte [15, 17]; see also [17, Theorem 2.1 (iii)]. Recall that the **sGG** condition for a complex manifold X means that every Gauduchon metric on X is automatically strongly Gauduchon.

After the completion of the paper [18], it came to our notice that another work [16] of Popovici just appeared in which he proposed a new approach to Conjecture 1.2.

The plurigenera are fundamental discrete invariants for the classification of complex varieties and deformation invariance of plurigenera is also another central topic in deformation theory. There is one interesting question proposed by Demailly.

Question 1.9 (Personal communication, 2017) Let $\pi : \mathcal{X} \to \Delta$ be a holomorphic family of compact complex manifolds over a unit disk in $\mathbb{C}$ such that each fiber $X_t := \pi^{-1}(t)$ is projective for any $t \in \Delta$. Then for each positive integer m, is the m-genus $\dim H^0(X_t, mK_{X_t})$ independent of $t \in \Delta$?

Recently, based on (the proof of) Theorem 1.4 and Takayama's work [24] on deformation invariance of plurigenera for algebraic family, we can answer Demailly's Question 1.9 in a more generality

Theorem 1.10 (see [19]) Let $\pi : \mathcal{X} \to \Delta$ be a family such that either of the following holds

(i) the family π is holomorphic and each fiber X_t at $t \in \Delta$ is a Moishezon manifold;

(ii) the family π is flat and each fiber X_t at $t \in \Delta$ is a projective variety of general type with only canonical singularities.

Then for each positive integer m, the m-genus $P_m(X_t)$ is independent of $t \in \Delta$.

2 Examples for Theorem 1.4

The goal of this section is to establish examples for Theorem 1.4. We first give a brief review of Siu–Demailly's solution of Grauert–Riemenschneider conjecture: if a compact complex manifold possesses a Hermitian holomorphic line bundle whose curvature is semi-positive everywhere and strictly positive at one point of the manifold, then this manifold is Moishezon.

Definition 2.1 A compact complex manifold is called semi-positive Moishezon if there exists a Hermitian holomorphic line bundle on this manifold, whose curvature is semi-positive everywhere and strictly positive at one point. By Siu's criterion [22], this manifold is Moishezon.

Let E be a holomorphic vector bundle of rank r and L a holomorphic line bundle on a compact complex manifold X of dimension n. If L is equipped with a smooth Hermitian metric h of Chern curvature form $\Theta_{L,h}$, we define the q-index set of L to be the open subset

$$X(L,h,q) = \left\{x \in X : \sqrt{-1}\Theta_{L,h} \text{ has } q \text{ negative eigenvalues and } n-q \text{ positive eigenvalues}\right\}$$

for $0 \le q \le n$. We also introduce

$$X(L,h,\le q) = \bigcup_{0\le j\le q} X(L,h,j).$$

Theorem 2.2 (see [7]) With the above setting, the cohomology groups $H^q(X, E\otimes L^{\otimes k})$ satisfy the asymptotic inequalities as $k \to +\infty$:

(1) (Weak Morse inequality)

$$h^q(X, E\otimes L^{\otimes k}) \le r\frac{k^n}{n!}\int_{X(L,h,q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi}\Theta_{L,h}\right)^n + o(k^n).$$

(2) (Strong Morse inequality)

$$\sum_{0\le j\le q} (-1)^{q-j} h^j(X, E\otimes L^{\otimes k}) \le r\frac{k^n}{n!}\int_{X(L,h,\le q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi}\Theta_{L,h}\right)^n + o(k^n).$$

Using the strong Morse inequality with $q = 1$, Demailly obtained

Theorem 2.3 (see [7]) Let X be a compact complex manifold with a Hermitian holomorphic line bundle (L,h) over X satisfying

$$\int_{X(L,h,\leq 1)}\left(\frac{\sqrt{-1}}{2\pi}\Theta_{L,h}\right)^n>0.$$

Then L is a big line bundle and thus X is a Moishezon manifold.

Obviously, a semi-positive Moishezon manifold (X,L,h) in the sense of Definition 2.1 satisfies

$$\int_{X(L,h,\leq 1)}\left(\frac{\sqrt{-1}}{2\pi}\Theta_{L,h}\right)^n>0$$

since $X(L,h,1)=\emptyset$ and

$$\int_{X(L,h,0)}\left(\frac{\sqrt{-1}}{2\pi}\Theta_{L,h}\right)^n=\int_{X}\left(\frac{\sqrt{-1}}{2\pi}\Theta_{L,h}\right)^n>0.$$

Under this type of integration conditions and assumptions on fibers X_t for all $t\in\Delta^*$, the proof for the deformation limit problem can be somewhat simplified.

Theorem 2.4 Let the fiber $X_t:=\pi^{-1}(t)$ be Moishezon for each $t\in\Delta^*$ and admit a Hermitian holomorphic line bundle (L_t,h_t) satisfying Demailly's integration condition

$$\int_{X(L_t,h_t,\leq 1)}\left(\frac{\sqrt{-1}}{2\pi}\Theta_{L_t,h_t}\right)^n>0.$$

Suppose that the reference fiber X_0 satisfies the local deformation invariance for Hodge number of type $(0,1)$ or admits a strongly Gauduchon metric. Then X_0 is still Moishezon.

Proof We deal with the Hodge number case first. Recall that any Moishezon manifold satisfies the $\partial\bar{\partial}$-lemma by [10] or [6, Theorem 5.22] and thus follows the degeneracy of Frölicher spectral sequence at E_1. So it satisfies the deformation invariance of all-type Hodge numbers by [27, Proposition 9.20] or also [20, Theorem 1.3]. By assumption, Grauert's continuity theorem [1, Theorem 4.12 (ii) of Chapter III] implies that $R^2\pi_*\mathcal{O}_{\mathcal{X}}$ over Δ^* is locally free. Then by using the fact that each of the Moishezon fiber X_t admits a big line bundle L_t, the Lebesgue negligibility argument in Subsection 3.2 of [18] leads to a section $s\in\Gamma(\Delta,R^2\pi_*\mathcal{O}_{\mathcal{X}})$ which arises from $c_1(L_t)$ and proves to be satisfying $s|_{\Delta^*}=0$, and thus $s=0$ by the torsion freeness of $R^2\pi_*\mathcal{O}_{\mathcal{X}}$. So by Lefschetz Theorem on $(1,1)$-classes, there exists a holomorphic line bundle L on $\mathcal{X}$ such that for some $t_0\in\Delta^*$, the Hermitian metric $(L|_{X_{t_0}},\tilde{h}_{t_0})$ satisfies

$$\int_{X(L|_{X_{t_0}},\tilde{h}_{t_0},\leq 1)}\left(\frac{\sqrt{-1}}{2\pi}\Theta_{L|_{X_{t_0}},\tilde{h}_{t_0}}\right)^n>0, \tag{2.1}$$

where the Hermitian metric $\tilde{h}_{t_0}$ is obtained by the $\partial\bar{\partial}$-lemma on X_{t_0} such that $\Theta_{L|_{X_{t_0}},\tilde{h}_{t_0}}=\Theta_{L_{t_0},h_{t_0}}$. Under the deformation invariance of $h^{0,1}$, we can also construct a holomorphic line bundle L' on $\mathcal{X}$ with $L'|_{X_{t_0}}=L_{t_0}$ for this $t_0\in\Delta^*$. However, for our purpose the equality $c_1(L|_{X_{t_0}})=c_1(L|_{t_0})$ is sufficient as far as (2.1) is concerned.

As for the second case, the argument of Theorem 1.4 with the assumption of strongly Gauduchon metric gives the desired holomorphic line bundle L on $\mathcal{X}$ with the same curvature integration property as (2.1).

Thus, one obtains a holomorphic Hermitian line bundle (L, h) on $\mathcal{X}$ and a hermitian metric on $L_{t_0} := L|_{X_{t_0}}$ for some $t_0 \in \Delta^*$ such that $(L_{t_0}, h_{t_0} := h|_{X_{t_0}})$ satisfies

$$\int_{X(L_{t_0},h_{t_0},\leq 1)} \left(\frac{\sqrt{-1}}{2\pi}\Theta_{L_{t_0},h_{t_0}}\right)^n > 0.$$

By Demailly's strong Morse inequality in Theorem 2.2, one has

$$h^0(X_{t_0}, L_{t_0}^{\otimes k}) \geq h^0(X_{t_0}, L_{t_0}^{\otimes k}) - h^1(X_{t_0}, L_{t_0}^{\otimes k}) \geq \frac{k^n}{n!}\int_{X(L_{t_0},h_{t_0},\leq 1)} \left(\frac{\sqrt{-1}}{2\pi}\Theta_{L_{t_0},h_{t_0}}\right)^n - o(k^n)$$

and thus L_{t_0} is big.

The difficulty here is that we have only one big line bundle L_{t_0} with $t_0 \in \Delta^*$ for the moment. Fortunately, for $|t - t_0| \leq \epsilon$ with some small constant $\epsilon > 0$, one still has, by continuity of smooth extension of the smooth Hermitian metric on $L|_{X_{t_0}}$, that

$$\int_{X(L_t,h_t,\leq 1)} \left(\frac{\sqrt{-1}}{2\pi}\Theta_{L_t,h_t}\right)^n > 0.$$

By Demailly's strong Morse inequality again, one obtains that L_t is big for $|t - t_0| \leq \epsilon$. So Proposition 1.5 completes the proof.

As a direct corollary of Theorem 2.4, one obtains the following result.

Corollary 2.5 If the fiber $X_t := \pi^{-1}(t)$ for each $t \in \Delta^*$ is semi-positive Moishezon and the $(0,1)$-Hodge number of X_0 satisfies the deformation invariance or admits a strongly Gauduchon metric, then X_0 is Moishezon.

Proof Here we give a second proof of Corollary 2.5, which seems of independent interest.

By the proof of Theorem 2.4, there exist a holomorphic line bundle L on $\mathcal{X}$ and some $\tau \in B$ such that $L_\tau := L|_{X_\tau}$ is semi-positive on the whole X_τ and strictly positive at one point of X_τ. The difficulty here is that the line bundle L_τ is big only at one τ for the moment. By Berndtsson's solution of Grauert–Riemenschneider conjecture [3], there exist $c_0, c_1, \cdots > 0$ and some positive integer N such that for all $k > N$, there hold

$$h^q(X_\tau, L_\tau^{\otimes k}) < c_q k^{n-q}$$

for all $1 \leq q \leq n$ and

$$h^0(X_\tau, L_\tau^{\otimes k}) \geq c_0 k^n.$$

For any $m > N$ and $1 \leq q \leq n$, let

$$V_{m,q} = \{t \in \Delta : h^q(X_t, L_t^{\otimes m}) \geq c_q m^{n-q}\}$$

and

$$V_m = \cup_{q=1}^n V_{m,q}.$$

Then V_m is an analytic subset of Δ but not equal to Δ since for $m > N$, $t = \tau$ is excluded from V_m. So for $m > N$, V_m is a discrete subset of Δ. Now set $V = \cup_{m>N} V_m$, which is a countable subset of Δ, and

$$\tilde{V} := \Delta \setminus V$$

which is non-empty and uncountable. So for $\tilde{\tau} \in \tilde{V}$, one has

$$h^q(X_{\tilde{\tau}}, L_{\tilde{\tau}}^{\otimes m}) < c_q m^{n-q}$$

for each $1 \leq q \leq n$ and $m > N$. Thus by asymptotic Riemann–Roch Theorem applied to $L_{\tilde{\tau}}^{\otimes m}$, one obtains

$$h^0(X_{\tilde{\tau}}, L_{\tilde{\tau}}^{\otimes m}) \geq c_0 m^n$$

for all $m > N$, giving that $L_{\tilde{\tau}}$ is also big on $X_{\tilde{\tau}}$ for each $\tilde{\tau} \in \tilde{V}$. We now apply Proposition 1.5 to complete the proof.

References

[1] Bănică C, Stănăşilă O. Algebraic methods in the global theory of complex spaces[M]. Translated from the Romanian. Editura Academiei, Bucharest. London, New York, Sydney: John Wiley Sons, 1976.

[2] Barlet D. Two semi-continuity results for the algebraic dimension of compact complex manifolds[J]. J. Math. Sci. Univ. Tokyo, 2015, 22(1): 39–54.

[3] Berndtsson B. An eigenvalue estimate for the $\bar{\partial}$-Laplacian[J]. J. Differential Geom., 2002, 60(2): 295–313.

[4] Campana F. The class $\mathcal{C}$ is not stable by small deformations[J]. Math. Ann., 1991, 290(1): 19–30.

[5] Campana F, Peternell T. Cycle spaces[M]. Several Complex Variables VII, 319–349, Encyclopaedia of Mathematical Sciences Volume 74, Berlin: Springer-Verlag, 1994.

[6] Deligne P, Griffiths P, Morgan J, Sullivan D. Real homotopy theory of Kähler manifolds[J]. Invent. Math., 1975, 29: 245–274.

[7] Demailly J P. Champs magnétiques et inégalités de Morse pour la d"-cohomologie (French)[J]. Magnetic fields and Morse inequalities for the d"-cohomology. Ann. Inst. Fourier (Grenoble), 1985, 35(4): 189–229.

[8] Fujita T. Approximating Zariski decomposition of big line bundles[J]. Kodai Math. J., 1994, 17: 1–3.

[9] Kodaira K. Complex manifolds and deformations of complex structures[M]. Grundlehren der Math. Wiss. 283, Springer, 1986.

[10] Parshin A. A generalization of the Jacobian variety (Russ.)[J]. Isvestia, 1966, 30: 175–182.

[11] Popovici D. Regularization of currents with mass control and singular Morse inequalities[J]. J. Differential Geom., 2008, 80(2): 281–326.

[12] Popovici D. Limits of projective manifolds under holomorphic deformations[J]. http://arxiv.org/abs/0910.2032, arXiv:0910.2032.

[13] Popovici D. Limits of Moishezon manifolds under holomorphic deformations[J]. http://arxiv.org/abs/1003.3605v1, arXiv:1003.3605v1.

[14] Popovici D. Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics[J]. Invent. Math., 2013, 194(3): 515–534.

[15] Popovici D. Aeppli cohomology classes associated with Gauduchon metrics on compact complex manifolds[J]. Bull. Soc. Math. France, 2015, 143(4): 763–800.

[16] Popovici D. Adiabatic limit and deformations of complex structures[J]. http://arxiv.org/abs/1901.04087v1, arXiv:1901.04087v1.

[17] Popovici D, Ugarte L. Compact complex manifolds with small Gauduchon cone[J]. Proc. Lond. Math. Soc., 2018, 116(5): 1161–1186.

[18] Rao S, Tsai I-H. Deformation limit and bimeromorphic embedding of Moishezon manifolds[J]. https://arxiv.org/abs/1901.10627, arXiv:1901.10627.

[19] Rao S, Tsai I-H. Deformation invariance of plurigenera: fiberwise Moishezon case[J]. preprint.

[20] Rao S, Zhao Q. Several special complex structures and their deformation properties[J]. J. Geom. Anal., 2018, 28(4): 2984–3047.

[21] Remmert R. Holomorphe und meromorphe Abbildungen Komplexer Räume[J]. Math. Ann., 1957, 133: 328–370.

[22] Siu Y T. A vanishing theorem for semipositive line bundles over non-Kähler manifolds[J]. J. Differential Geom., 1984, 19(2): 431–452.

[23] Stein K. Meromorphic mappings[J]. L'Ens. Math. Ser. II, 1968, 14: 29–46.

[24] Takayama S. On the invariance and lower semi-continuity of plurigenera of algebraic varieties[J]. J. Algebraic Geom., 2007, 16: 1–18.

[25] Tjurin G N. The space of moduli of a complex surface with $q = 0$ and $K = 0$[M]. Chapter IX of 'Algebraic surfaces', by the members of the seminar of I. R. Shafarevich. Translated from the Russian by Susan Walker. Translation edited, with supplementary material, by Kodaira K and Spencer D C. Proceedings of the Steklov Institute of Mathematics, No. 75, Providence, RI: American Mathematical Society, 1965.

[26] Ueno K. Classification theory of algebraic varieties and compact complex spaces[M]. Notes written in collaboration with Cherenack P. Lecture Notes in Mathematics, Vol. 439, Berlin, New York: Springer-Verlag, 1975.

[27] Voisin C. Hodge theory and complex algebraic geometry I[M]. Translated from the French original by Leila Schneps. Cambridge Studies in Advanced Mathematics 76, Cambridge: Cambridge University Press, 2002.

[28] Yau S T. On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I[J]. Comm. Pure. Appl. Math, 1978, 31: 339–411.