Vol. 40 (2020)
No. 2

数 学 杂 志
J. of Math. (PRC)

BERS EMBEDDING OF Q_K-TEICHMÜLLER SPACE

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Abstract: This paper study the Bers projection and pre-projection of Q_K-Teichmüller space. By means of quasiconformal mapping theory, we prove that the Bers projection and pre-projection of Q_K-Teichmüller space are holomorphic.

Keywords: quasisymmetric homeomorphism; universal Teichmüller space; Q_k-Teichmüller space; Bers projection

2010 MR Subject Classification: 30C62, 30H25

Document code: A **Article ID:** 0255-7797(2020)02-0210-09

1 Introduction

The Bers embedding plays an important role in Teichmüller theory. In terms of Schwarzian derivative, Teichmüller space is embedded onto an open subset of some complex Banach space of holomorphic functions. We emphasize that the Bers projection in Teichmüller space is a holomorphic split submersion (see [1], [2] for more details). The main purpose of this paper is to investigate the Bers projection and pre-projection of Q_K-Teichmüller space. We begin with some notations and definitions.

Let $\Delta = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, $\Delta^* = \mathbb{C}\backslash\overline{\Delta}$ be the outside of the unit disk and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. For $z, a \in \Delta$, we set $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ and denote by $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$ the Green function of with a pole at a. For a nonnegative and nondecreasing function K on $[0,\infty)$, the space Q_K consists of all analytic functions with the following finite norm

$$\|f\|^2_{Q_K} = \sup_{a\in\Delta}\iint_\Delta |f'(z)|^2K(g(z,a))dxdy. \tag{1.1}$$

We say f belongs to $Q_{K,0}$ space if $f \in Q_K$ and

$$\lim_{|a|\to 1}\iint_\Delta |f'(z)|^2K(g(z,a))dxdy = 0. \tag{1.2}$$

Wulan and Wu introduced in [3] the space Q_K which was investigated in recent years (see [4–9]). Wulan and Wu [3] proved that Q_K spaces are always contained in the Bloch

* **Received date:** 2018-10-23 **Accepted date:** 2019-03-05

Foundation item: Supported by National Natural Science Foundation of China (11601100); the Joint Foundation of Guizhou Provincial Science and Technology Department ([2017]7337; [2017]5726).

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space $\mathfrak{B}$, which consists of holomorphic functions f on Δ such that

$$\|f\|_B = \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty$$

and $Q_{K,0} \subset \mathfrak{B}_0$, which consists of all functions $f \in \mathfrak{B}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

If $K(t) = t^p$ for $0 < p < \infty$, the space Q_K gives a Q_p space (see [10, 11]). In particular, If $K(t) = t$, then Q_K =BMOA. Q_K space is nontrivial if and only if

$$\sup_{t\in(0,1)} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K\left(\log\frac{1}{r}\right) rdr < \infty, \tag{1.3}$$

here and in what follows we always assume that $K(0) = 0$ and condition (1.3) is satisfied. Furthermore, we require two more conditions on K as follows

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.4}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^{1+p}} ds < \infty,\ 0 < p < 2, \tag{1.5}$$

where $\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t)$, $0 < s < \infty$.

Let I be an arc of the unit circle S^1 with normalized arclength $\ell(I) < 1$, the Carleson box is defined by

$$S_\Omega(I) = \begin{cases} \{z \in \Delta : 1-\ell(I) \le |z| < 1, z/|z| \in I\}, & \Omega = \Delta, \\ \{z \in \Delta^* : 1 \le |z| < 1+\ell(I), z/|z| \in I\}, & \Omega = \Delta^*. \end{cases}$$

A positive measure λ on Ω is called a K-Carleson measure if

$$\|\lambda\|_{C,K} = \sup_{I\subset S^1} \iint_{S_\Omega(I)} K\left(\frac{1-|z|}{\ell(I)}\right) d\lambda(z) < \infty$$

and a compact K-Carleson measure if

$$\lim_{|I|\to 0} \iint_{S_\Omega(I)} K\left(\frac{1-|z|}{\ell(I)}\right) d\lambda(z) = 0.$$

We denote by $CM_K(\Omega)$ the set of all K-Carleson measures on Ω and $CM_{K,0}(\Omega)$ the set of all compact K-Carleson measures on Ω.

An orientation preserving homeomorphism f from domain Ω onto $f(\Omega)$ is quasiconformal if f has locally L^2 integrable distributional derivative on Ω and satisfies the following equation

$$f_{\overline{z}} = \mu(z) f_z$$

for some measurable functions μ with $\|\mu\|_\infty$. We say a sense preserving self-homeomorphism h is quasisymmetric if there exists some quasiconformal homeomorphism of Δ onto itself which has boundary value h (see [12]). Let $QS(S^1)$ be the group of quasisymmetric homeomorphisms of the unit circle S^1 and $\text{Möb}(S^1)$ the group of Möbius transformations mapping Δ onto itself. The universal Teichmüller space is defined as the right coset space $T = QS(S^1)/\text{Möb}(S^1)$.

The universal Teichmüller space T can also be described as $T = M(\Delta^*)/\sim$, where $M(\Delta^*)$ denotes the unit ball of the Banach space $L^\infty(\Delta^*)$ of bounded measurable functions on Δ^*. Let f_μ be the unique quasiconformal mapping whose complex dilatation is μ in Δ^* and is zero in Δ, normalized by

$$f_\mu(0) = f'_\mu(0) - 1 = f''_\mu(0) = 0.$$

We say that two Beltrami coefficients μ_1 and μ_2 in $M(\Delta^*)$ are Teichmüller equivalent and denote by $\mu_1 \sim \mu_2$ if $f_{\mu_1}(\Delta) = f_{\mu_2}(\Delta)$.

Let $\mathcal{B}_\infty(\Delta)$ denote the Banach space of holomorphic functions on Δ with norm

$$\|\varphi\|_{\mathcal{B}_\infty} = \sup_{z\in\Delta} |\varphi(z)|(1 - |z|^2)^2 < \infty.$$

For a conformal mapping f on Δ , its Schwarzian derivative S_f of is defined as

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Bers projection $\Phi : M(\Delta^*) \to \mathcal{B}_\infty(\Delta)$ is defined by $\mu \mapsto S_{f_\mu}$, which is a holomorphic split submersion onto its image and descends down to the Bers embedding $B : T \to \mathcal{B}_\infty(\Delta)$. Thus T carries a natural complex Banach manifold structure so that B is a holomorphic split submersion.

A quasisymmetric homeomorphism h is strongly quasisymmetric if h has a quasiconformal extension to Δ such that its complex dilatation $\mu(z)$ satisfies the condition

$$\frac{|\mu(z)|^2}{(1 - |z|^2)} dxdy$$

is a Carleson measure (see [13]). Strongly quasisymmetric homeomorphisms and its Teichmüller theory were studied in recent years (see [14, 15]).

Recently, the author, Feng and Huo [16] introduced the $F(p, q, s)$-Teichmüller space and showed that its pre-logarithmic derivative model is disconnected subset of the $F(p, q, s)$ space. It was also shown that the Bers projection of this space is holomorphic.

Let $\Omega = \Delta$ or $\Omega = \Delta^*$. We denote by $M_{Q_K}(\Omega)$ the Banach space of all essentially bounded measurable functions μ with

$$\lambda_\mu(z) := \frac{|\ \mu(z)\ |^2}{||z| - 1|^2} dxdy \in CM_K(\Omega).$$

The norm of $M_{Q_K}(\Omega)$ is defined as

$$\| \mu \|_K = \| \mu \|_\infty + \|\lambda_\mu\|_{C,K}^{1/2}, \tag{1.6}$$

where $\|\lambda_\mu\|_{C,K}$ is the K Carleson norm of λ_μ on Ω. Let $M_{Q_{K,0}}(\Omega)$ be the subspace of $M_{Q_K}(\Omega)$ consists of all elements μ such that $\lambda_\mu(z) \in CM_{K,0}(\Omega)$. Set $M^1_{Q_K}(\Delta^*) = M_{Q_K}(\Delta^*) \cap M(\Delta^*)$ and $M^1_{Q_{K,0}}(\Delta^*) = M_{Q_{K,0}}(\Delta^*) \cap M(\Delta^*)$. The Q_K-Teichmüller space T_{Q_K} is defined as $T_{Q_K} = M^1_{Q_K}(\Delta^*)/\sim$ and $Q_{K,0}$-Teichmüller space $T_{Q_{K,0}} = M^1_{Q_{K,0}}(\Delta^*)/\sim$.

Let T_K be the set of all functions $\log f'$, where f is conformal in Δ with the normalized condition $f(0) = f'(0) - 1 = 0$ and admits a quasiconformal extension to the whole plane such that its complex dilatation μ satisfies $\lambda_\mu(z) \in CM_K(\Delta^*)$. Then T_K gives another model of T_{Q_K}. Wulan and Ye [6] proved the following.

Theorem 1.1 [6] Let K satisfy (1.4) and (1.5). Then T_K is a disconnected subset of Q_K. Furthermore, $T_{K,b} = \{\log f' \in T_K : f(\Delta) \text{is bounded}\}$ and $T_{K,\theta} = \{\log f' \in T_K : f(e^{i\theta}) = \infty, \theta \in [0, 2\pi]$, are the connected components of T_K.

The authors also obtained some more characterizations of Q_K-Teichmüller space by using Grunsky kernel functions (see [17]).

The main purpose of this paper is to deal with the holomorphy of the Bers projection and the pre-Bers projection in Q_K-Teichmüller space. In what follows, C will denote a positive universal constant which may vary from line to line and $C(\cdot)$ will denote constant that depends only on the elements put in the brackets.

2 Holomorphy of Bers Projection

In this section, we prove the Bers projection is holomorphic in Q_K-Teichmüller space. Noting that K satisfies condition (1.5), we conclude that there exists $c > 0$ such that $K(t)/t^{p-c}$ is non-increasing and $K(2t) \approx K(t)$ for $0 < t < \infty$. Wulan and Zhou proved the following.

Lemma 2.1 [8] Let K satisfy conditions (1.4) and (1.5). Set $b + \alpha \geq 1 + p$, $b \geq p$, $b - p + \beta + c > 1$, and $\alpha > 0$. Then there exists a constant C (independent of the length $\ell(I)$ of I) such that

$$\iint_\Delta \frac{K\left(\frac{1-|\zeta|}{\ell(I)}\right)(1-|w|^2)^{b-1+\beta}}{(1-|\zeta|)^{1-\alpha+\beta}|1-w\bar{\zeta}|^{b+\alpha}} dA(\zeta) \leq CK\left(\frac{1-|w|}{\ell(I)}\right) \tag{2.1}$$

for all $w \in \Delta$ and arc $I \subset S^1$.

Lemma 2.2 Let K satisfy conditions (1.4) and (1.5). Let $b + \alpha \geq 1 + p$, $b \geq p$, $b - p + \alpha + c > 2$ and $\alpha > 0$. Set

$$\widetilde{\lambda}(\zeta) = \iint_\Delta \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) dA(w). \tag{2.2}$$

Then $\widetilde{\lambda} \in CM_K(\Delta)$ if $\lambda \in CM_K(\Delta)$ and $\|\widetilde{\lambda}\|_{C,K} \leq C\|\lambda\|_{C,K}$, while $\widetilde{\lambda} \in CM_{K,0}(\Delta)$ if $\lambda \in CM_{K,0}(\Delta)$.

Proof For any arc $I \subset S^1$, let $2I$ is the arc with the same center as I but with double length. We divide the following integral into two part as follows

$$\iint_{S(I)} \iint_{\Delta} \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) dA(w) K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta) = L_1 + L_2,$$

where

$$L_1 = \iint_{S(I)} \iint_{S(2I)} \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) dA(w) K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta)$$

and

$$L_2 = \iint_{S(I)} \iint_{\Delta \setminus S(2I)} \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) dA(w) K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta).$$

It follows from lemma that

$$\begin{aligned} L_1 &= \iint_{S(I)} \lambda(w) dA(w) \iint_{S(2I)} \frac{K\left(\frac{1-|\zeta|}{\ell(I)}\right)(1-|w|^2)^{b-2+\alpha}}{|1-w\bar{\zeta}|^{b+\alpha}} dA(\zeta) \\ &\leq C \sup_I \iint_{S(I)} \lambda(w) K\left(\frac{1-|w|}{\ell(I)}\right) dA(w). \end{aligned} \tag{2.3}$$

Now we estimate L_2. Set

$$S_n = S(2^n I) = \{r\zeta \in \Delta : 1 - 2^n \ell(I) < r < 1,\ \zeta \in 2^n I\}, \quad n = 1, 2, \cdots.$$

Let n_I be the minimum such that $2^n I \geq 1$. Then $S_n = \Delta$ when $n \geq n_I$. Denote by w_I the center of I and $w_1 = (1 - \ell(I)/2) w_I$. If $\zeta \in S(I)$ and $w \in S_n \setminus S_{n-1}$, $1 < n < n_I$, then

$$\frac{2^{n-1}}{\pi} \ell(I) \leq |\zeta - w_I| \leq \frac{3}{\pi} 2^n \ell(I).$$

This implies that

$$\begin{aligned} |\zeta - w_1| &\leq |\zeta - w_I| + |w_I - z_1| \leq 3 \cdot 2^n \ell(I), \\ |\zeta - w_1| &\geq |\zeta - w_I| - |w_I - z_1| \geq \frac{4-\pi}{8\pi} 2^n \ell(I). \end{aligned}$$

Consequently

$$1 - |\zeta| < \frac{8\pi}{4-\pi} 2^{-n} |\zeta - w_1|, \qquad 1 - |w| < 2^n \ell(I) \leq \frac{8\pi}{4-\pi} |\zeta - w_1|$$

and

$$\frac{1}{|1-w\bar{\zeta}|} \leq \frac{C}{2^{n-1}\ell(I)}.$$

Since $K(t)$ is a nondecreasing function, we have

$$\begin{aligned} L_2 &= \iint_{S(I)} \iint_{\Delta \setminus S(2I)} \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) dA(w) K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta) \\ &= \iint_{S(I)} \sum_{n=2}^{n_I} \iint_{S_n \setminus S_{n-1}} \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) dA(w) K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta) \\ &\leq C \iint_{S(I)} \sum_{n=2}^{\infty} \iint_{S_n \setminus S_{n-1}} \frac{(1-|w|^2)^{b+\alpha-2}}{|1-w\bar{\zeta}|^{b+\alpha}} \lambda(w) K\left(\frac{|\zeta - w_1|}{2^n \ell(I)}\right) dA(w) dA(\zeta). \end{aligned}$$

Noting that K satisfies (1.4) and (1.5), we conclude that $K(t)/t^{p-c}$ is non-increasing for some small $c>0$. Consequently,

$$\begin{aligned}
L_2 &\le C\iint_{S(I)}\sum_{n=2}^{\infty}\iint_{S_n\setminus S_{n-1}} K\left(\frac{1-|w|}{2^n\ell(I)}\right)\\
&\quad\times\frac{(1-|w|^2)^{b+\alpha-2+c-p}}{|\zeta-w_1|^{c-p}|1-w\overline{\zeta}|^{b+\alpha}}\lambda(w)dA(w)dA(\zeta)\\
&\le C\iint_{S(I)}\sum_{n=2}^{\infty}\iint_{S_n\setminus S_{n-1}} K\left(\frac{1-|w|}{2^n\ell(I)}\right)\\
&\quad\times\frac{(1-|w|^2)^{b+\alpha-2+c-p}(3\cdot 2^n\ell(I))^{p-c}}{|1-w\overline{\zeta}|^{b+\alpha}}\lambda(w)dA(w)dA(\zeta)\\
&= C\iint_{S(I)}\sum_{n=2}^{\infty}\iint_{S_n\setminus S_{n-1}} \frac{4\cdot 6^{p-c}}{2^{2n}\ell(I)^2}\frac{(1-|w|^2)^{b+\alpha-2+c-p}}{|1-w\overline{\zeta}|^{b+\alpha-2+c-p}}\\
&\quad\times\lambda(w)K\left(\frac{1-|w|}{2^n\ell(I)}\right)dA(w)dA(\zeta)\\
&\le C\sum_{n=2}^{\infty}\frac{4\cdot 6^{p-c}}{2^{2n}}\iint_{S_n}\lambda(w)K\left(\frac{1-|w|}{2^n\ell(I)}\right)dA(w)\\
&\le C(p,c)\sup_I\iint_{S(I)}\lambda(w)K\left(\frac{1-|w|}{\ell(I)}\right)dA(w). \qquad (2.4)
\end{aligned}$$

Combing (2.3) and (2.4), we deduce that $\|\widetilde{\lambda}\|_{C,K}\le C\|\lambda\|_{C,K}$, while $\widetilde{\lambda}\in CM_{K,0}(\Delta)$ if $\lambda\in CM_{K,0}(\Delta)$. The proof follows.

Let $N_K(\Delta)$ denote the space which consists of all analytic functions f in Δ with the following finite norm

$$\|f\|_{N_K}^2=\sup_{a\in\Delta}\iint_{\Delta}|f(z)|^2(|z|^2-1)^2K(1-|\varphi_a(z)|^2)dxdy. \qquad (2.5)$$

We say a analytic function f belongs to $N_{K,0}(\Delta)$ if $f\in N_K$ and

$$\lim_{|a|\to 1}\iint_{\Delta}|f(z)|^2(|z|^2-1)^2K(1-|\varphi_a(z)|^2)dxdy=0. \qquad (2.6)$$

The following is our main result.

Theorem 2.3 The Bers projection $\Phi: M^1_{Q_K}(\Delta^*)\to N_K(\Delta)$ is holomorphic.

Proof We first show that the Bers projection $\Phi: M^1_{Q_K}(\Delta^*)\to N_K(\Delta)$ is continuous. For any two elements μ and ν in $M^1_{Q_K}(\Delta^*)$, set $S_{f_\mu}=\Phi(\mu)$ and $S_{f_\nu}=\Phi(\nu)$. By an integral representation of the Schwarzian derivative by means of the representation theorem of quasiconformal mappings, Astala and Zinsmeister [13] proved that there exists some constant $C(\|\mu\|_\infty)$ such that

$$|S_{f_\mu}-S_{f_\nu}|^2\le\frac{C(\|\mu\|_\infty)}{(|z|^2-1)^2}\iint_{\Delta^*}\frac{|\mu(\zeta)-\nu(\zeta)|^2+\|\mu-\nu\|_\infty^2|\mu(\zeta)|^2}{|\zeta-z|^4}d\xi d\eta. \qquad (2.7)$$

Let $\widetilde{\mu}(z) = \mu(1/z)z^2/\overline{z}^2$ and $\widetilde{\nu}(z) = \nu(1/z)z^2/\overline{z}^2$. Then a change of variable gives

$$\|\lambda_{\mu-\nu}\|_{C,K} = \|\lambda_{\widetilde{\mu}-\widetilde{\nu}}\|_{C,K} \text{ and } \|\lambda_\mu\|_{C,K} = \|\lambda_{\widetilde{\mu}}\|_{C,K}, \tag{2.8}$$

where $d\lambda_\mu(z) = \frac{|\mu(z)|^2}{||z|^2-1|^2}dxdy$. Consequently, we have

$$|S_{f_\mu} - S_{f_\nu}|^2 \le \frac{C(\|\mu\|_\infty)}{(|z|^2-1)^2}\iint_\Delta \frac{|\widetilde{\mu}(\zeta) - \widetilde{\nu}(\zeta)|^2 + \|\mu-\nu\|_\infty^2|\widetilde{\mu}(\zeta)|^2}{|1-\zeta z|^4}d\xi d\eta. \tag{2.9}$$

From [5], the measure $\lambda_\mu \in CM_K(\Delta)$ if and only if

$$\sup_{a\in\Delta}\iint_\Delta K(1-|\varphi_a(z)|^2)d\lambda_\mu(z).$$

Furthermore, there exist two positive constant C_1 and C_2 such that

$$C_1\|\lambda_\mu\|_{C,K} \le \sup_{a\in\Delta}\iint_\Delta K(1-|\varphi_a(z)|^2)d\lambda_\mu(z) \le C_2\|\lambda_\mu\|_{C,K}.$$

Thus, it follows from Lemma 2.2 that

$$\begin{aligned}
\|S_{f_\mu} - S_{f_\nu}\|_{N_K}^2 &\le C(\|\mu\|_\infty)\sup_{a\in\Delta}\iint_\Delta K(1-|\varphi_a(z)|^2) \\
&\quad \times \iint_\Delta \frac{|\widetilde{\mu}(\zeta) - \widetilde{\nu}(\zeta)|^2 + \|\mu-\nu\|_\infty^2|\widetilde{\mu}(\zeta)|^2}{|1-\zeta z|^4}d\xi d\eta dxdy \\
&\le C(\|\mu\|_\infty)\sup_{a\in\Delta}\iint_\Delta K(1-|\varphi_a(z)|^2)\iint_\Delta \frac{(1-|\zeta|^2)^2}{|1-\zeta z|^4}d\lambda_{\widetilde{\mu}-\widetilde{\nu}}(\zeta)dxdy \\
&\quad + C(\|\mu\|_\infty)\|\mu-\nu\|_\infty^2\sup_{a\in\Delta}\iint_\Delta K(1-|\varphi_a(z)|^2)\iint_\Delta \frac{(1-|\zeta|^2)^2}{|1-\zeta z|^4}d\lambda_{\widetilde{\mu}}(\zeta)dxdy \\
&\le C(\|\mu\|_\infty)(\|\lambda_{\widetilde{\mu}-\widetilde{\nu}}\|_{C,K}^2 + \|\mu-\nu\|_\infty^2\|\lambda_{\widetilde{\mu}}\|_{C,K}^2) \\
&\le C(\|\mu\|_\infty)(1+\|\lambda_\mu\|_{C,K}^2)\|\mu-\nu\|_K^2.
\end{aligned}$$

This shows the Bers projection $\Phi : M_{Q_K}^1(\Delta^*) \to N_K(\Delta)$ is continuous.

We now prove that the mapping $\Phi : M_{Q_K}^1(\Delta^*) \to N_K(\Delta)$ is holomorphic. For each $z \in \Delta$, we define a continuous linear functional l_z on the Banach space $N_K(\Delta)$ by $l_z(\varphi) = \varphi(z)$ for $\varphi \in N_K(\Delta)$. Then the set $A = \{l_z : z \in \Delta\}$ is a total subset of the dual space of $N_K(\Delta)$. Now for each $z \in \Delta$, each pair $(\mu, \nu) \in M_{Q_K}^1(\Delta^*) \times M_{Q_K}(\Delta^*)$ and small t in the complex plane, it follows from the holomorphic dependence of quasiconformal mappings on parameters that $l_z(\Phi(\mu + t\nu)) = S_{f_{\mu+t\nu}}(z)$ is a holomorphic function of t (see [18, 2, 1]). From the infinite dimensional holomorphy theory (see Proposition 1.6.2 in [1]), we deduce that the Bers projection $\Phi : M_{Q_K}^1(\Delta^*) \to N_K(\Delta)$ is holomrphic.

Similarly, we can prove the following

Theorem 2.4 The Bers projection $\Phi : M_{Q_{K,0}}^1(\Delta^*) \to N_{K,0}(\Delta)$ is holomorphic.

3 Holomorphy of Pre-Bers Projection

Fix $z_0 \in \Delta^*$. For $\mu \in M^1_{Q_K}(\Delta^*)$, let $f^{z_0}_\mu$ be the quasiconformal mapping whose complex dilatation is μ in Δ^* and is zero in Δ, normalized by $f_\mu(0) = f'_\mu(0) - 1 = 0, f_\mu(z_0) = \infty$. The pre-Bers projection mapping L_{z_0} on $M^1_{Q_K}(\Delta^*)$ is defined by the correspondence $L_{z_0}(\mu) = \log f'_\mu$. Let Q^0_K be the space consisting of all functions $\varphi \in Q_K$ with $\varphi(0) = 0$, then $\cup_{z_0\in\Delta^*} L_{z_0}(M^1_{Q_K}(\Delta^*)) = T_{K,b} \cap Q^0_K$.

We prove the following.

Theorem 3.1 For $z_0 \in \Delta^*$, the pre-Bers projection mapping $L_{z_0} : M^1_{Q_K}(\Delta^*) \to Q^0_K$ is holomorphic.

Proof Since we can prove $L_{z_0} : M^1_{Q_K}(\Delta^*) \to Q^0_K$ is holomorphic by the same reasoning as the proof of the holomorphy of $\Phi : M^1_{Q_K}(\Delta^*) \to N_K(\Delta)$, we need only to show that $L_{z_0} : M^1_{Q_K}(\Delta^*) \to Q^0_K$ is continuous. For $\mu, \nu \in M^1_{Q_K}(\Delta^*)$, for simplicity of notation, we let f and g stand for $f^{z_0}_\mu$ and $f^{z_0}_\nu$, respectively. It follows from Theorem 3.1 in Chapter II in [2] that

$$\sup_{z\in\Delta}(1- \mid z \mid^2) \left|\frac{f''}{f'} - \frac{g''}{g'}\right| \le C\|\mu - \nu\|_\infty. \tag{3.1}$$

By theorem , we conclude that

$$\sup_{a\in\Delta} \iint_\Delta \quad (1- \mid z \mid^2)^2 \mid S_f(z) - S_g(z) \mid^2 K(1 - |\varphi_a(z)|^2)dxdy \quad \le C\|\mu - \nu\|^2_K. \tag{3.2}$$

It is known that a function of Q_K space can be characterized by its higher derivative [19]. Thus, for any $a \in \Delta$, we get

$$\begin{aligned} I(a) :=& \iint_\Delta \left|\frac{f''}{f'} - \frac{g''}{g'}\right|^2 K(1 - |\varphi_a(z)|^2)dxdy \\ \le& C \left|\frac{f''}{f'}(0) - \frac{g''}{g'}(0)\right|^2 \\ &+ C \iint_\Delta (1- \mid z \mid^2)^2 \left|\left(\frac{f''}{f'}\right)' - \left(\frac{g''}{g'}\right)'\right|^2 K(1 - |\varphi_a(z)|^2)dxdy. \end{aligned} \tag{3.3}$$

Combing (3.2) with (3.1) gives

$$\begin{aligned} I(a) \le& C\|\mu - \nu\|^2_\infty + C \iint_\Delta (1- \mid z \mid^2)^2 \mid S_f - S_g \mid^2 K(1 - |\varphi_a(z)|^2)dxdy \\ &+ C \iint_\Delta \left|\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right|^2 (1- \mid z \mid^2)^2 K(1 - |\varphi_a(z)|^2)dxdy \\ \le& C\|\mu - \nu\|^2_K \\ &+ C \iint_\Delta (1- \mid z \mid^2)^2 \left|\frac{f''}{f'} + \frac{g''}{g'}\right|^2 \left|\frac{f''}{f'} - \frac{g''}{g'}\right|^2 K(1 - |\varphi_a(z)|^2)dxdy \\ \le& C(\| \log g'_\mu\|^2_{Q_K} + \| \log g'_\nu\|^2_{Q_K})\|\mu - \nu\|^2_K. \end{aligned} \tag{3.4}$$

This implies that $L_{z_0} : M^1_{Q_K}(\Delta^*) \to Q^0_K$ is continuous. The proof follows.

Similar arguments apply to $Q_{K,0}$ space gives

Theorem 3.2 For $z_0 \in \Delta^*$, the pre-Bers projection mapping $L_{z_0}: M^1_{Q_{K,0}}(\Delta^*) \to Q^0_{K,0}$ is holomorphic, where $Q^0_{K,0}$ consists of all functions $\varphi \in Q_{K,0}$ with $\varphi(0) = 0$.

References

[1] Nag S. The complex analytic theory of Teichmüller space[M]. New York: Wiley-Interscience, 1988.

[2] Lehto O. Univalent functions and Teichmüller spaces[M]. New York: Springer-Verlag, 1987.

[3] Wulan H, Wu P. Characterizations of Q_T spaces[J]. J. Math. Anal. Appl., 2001, 254: 484–497.

[4] Essén M, Wulan H. On analytic and meromorphic functions and spaces of Q_K-type[J]. Illinois J. Math., 2002, 46: 1233–1258.

[5] Essén M, Wulan H, Xiao J. Several function theoretic characterizations of Möbius invariant Q_K space[J]. J. Funct. Anal., 2006, 230: 78–115.

[6] Wulan H, Ye F. Universal Teichmüller space and Q_K spaces[J]. Ann. Acad. Sci. Fenn. Math., 2014, 39: 691–709.

[7] Wulan H, Zhou J. Decomposition theorems for Q_K spaces and applications[J]. Forum Math., 2014, 26: 467–495.

[8] Wulan H, Zhou J. Q_K and Morrey type spaces[J]. Ann. Acad. Sci. Fenn. Math., 2013, 38: 193–207.

[9] Zhou J. Schwarzian derivative, geometric conditions and Q_K spaces[J]. Sci. Sin. Math. (in Chinese), 2012, 42: 939–950.

[10] Xiao J. Holomorphic Q classes[M]. Berlin: Springer-Verlag, 2001.

[11] Xiao J. Geometric Q Functions[M]. Basel: Birkhäuser, 2006.

[12] Beurling A, Ahlfors L V. The boundary correspondence under quasiconformal mappings[J]. Acta Math., 1956, 96: 125–142.

[13] Astala K, Zinsmeister M. Teichmüller spaces and BMOA[J]. Math. Ann., 1991, 289 : 613–625.

[14] Cui G, Zinsmeister M. BMO-Teichmüller spaces[J]. Illinois J. Math., 2004, 48: 1223–1233.

[15] Shen Y, Wei H. Universal Teichmüller space and BMO[J]. Adv. Math., 2013, 234: 129–148.

[16] Feng X, Huo S, Tang S. Universal Teichmüller space and $F(p, q, s)$ space[J]. Ann. Acad. Sci. Fenn. Math., 2017, 42: 1–14.

[17] Jin J, Tang S. On Q_K-Teichmüller spaces[J]. J. Math. Anal. Appl., 2018, 467: 622–637.

[18] Ahlfors L V. Lecture on quasiconformal mappings[M]. Princeton: D Van Nostrand, 1966.

[19] Wulan H, Zhu K. Q_K spaces via higher order derivatives[J]. Rocky Mountain J. Math., 2008, 38: 329–350.

Q_K-Teichmüller空间的Bers嵌入

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摘要: 本文研究Q_K-Teichmüller空间的Bers投影和预投影. 利用拟共形映射理论, 证明了Q_K-Teichmüller空间的Bers投影和预投影是全纯的.

关键词: 拟对称映射; 万有Teichmuller空间; Q_K-Teichmüller空间; Bers投影

MR(2010)**主题分类号**: 30C62, 30H25 **中图分类号**: O174.55