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HOPF DENSE GALOIS EXTENSIONS OVER A RING

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Abstract: Let R be a commutative domain, let H be a Hopf R-algebra which is a finitely generated free R-module, and let A be an R-algebra which is also a H-comodule algebra. We will say that A/A^{coH} is a Hopf dense Galois extension if the cokernel of the associated canonical map $\beta : A\otimes_{A^{coH}} A \to A\otimes_R H$ is quotient finite. It is a generalization of Hopf dense Galois extension over a field. This paper shows that a weaker version of Auslander theorem holds for Hopf dense Galois extensions over R. It is also proved that if the algebra A is almost commutative such that $gr(A)$ is a domain, and the canonical map β is strict, then a Hopf dense Galois extension A/A^{coH} will imply that H is dual to a finite dimensional group algebra over an algebraic closed field containing R.

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1 Introduction

Motivated by the study of noncommutative isolated singularities, the He-Van Oystaeyen-Zhang introduced in [1] the concept of Hopf dense Galois extensions over a field. Hopf dense Galois extensions provide candidates of noncommutative resolutions of quotient isolated singularities. However, it is usually difficult to see when a Hopf action or coaction on an algebra results a Hopf dense Galois extension. When the algebra A under consideration has a big center, namely, A is finitely generated over its center, then the problem becomes relatively easy [2]. Indeed, we may use the mod-p method to reduce the problem to algebras over fields with positive characteristic. For example, if A is a universal enveloping algebra of a finite dimensional Lie algebra, or A is a Weyl algebra over a field of characteristic $p > 0$, then A is finitely generated over its center. One of the essential parts to use the mod-p method is to find orders of Hopf actions. Hence it is necessary to consider the Hopf (co) actions and Hopf dense Galois extensions over a commutative domain.

In this paper, we introduce the concept of Hopf dense Galois extensions over a commutative domain. The theory involves several torsion theories. We show that Hopf dense Galois extensions work well. In particular, we prove that a weaker version of Auslander theorem holds for Hopf dense Galois extensions over a commutative domain (cf. Theorem 3.7).

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Hopf dense Galois extensions depend on the Hopf algebra (co) actions on the algebra under consideration. It was shown in [3, 4] that not too many semisimple Hopf algebras act inner faithfully on a graded commutative domain or Weyl algebras. Let R be a commutative domain of characteristic zero and let $\Bbbk$ be an algebraically closed field containing R as its subring. Assume that H is a Hopf R-algebra which is a finitely generated free R-module. Suppose that H coacts on an almost commutative algebra A and the coaction preserves the filtration. If A is Hopf dense Galois over the invariant subalgebra $A^{\mathrm{co}H}$, then $H^* \otimes_R \Bbbk$ is isomorphic to a group algebra over $\Bbbk$ (cf. Theorem 4.10). Theorem 4.10 applies to Hopf algebra coactions on Sridharan enveloping algebras which including universal enveloping algebras of finite dimensional Lie algebras and Weyl algebras. In particular, if H is a finite dimensional Hopf algebra over an algebraically closed field of characteristic zero which acts on a Sridharan enveloping algebra $U_f(\mathfrak{g})$ such that the action preserves the filtration of $U_f(\mathfrak{g})$ and the associated graded algebra of $U_f(\mathfrak{g})$ is a Hopf dense Galois extension on its invariant subalgebra, then H is isomorphic to a group algebra (cf. Corollary 5.12). This result partially generalizes [3, Theorem 4.2].

2 Torsion Theories over a Ring

Let R be a commutative domain. Let Q be the quotient field of R. Given a noetherian R-algebra A, the localizing $A \otimes_R Q$ is a Q-algebra. For simplicity, we write A_Q for the Q-algebra $A \otimes_R Q$. Similarly, if M is a right A-module, then $M_Q := M \otimes_R Q$ is a right A_Q-module. We will say that M is R-torsion free if for any $x \in M, r \in R$, $xr = 0$ implies $r = 0$. The localizing functor $- \otimes_R Q$ induces an exact functor $(-)_Q : \operatorname{Mod} A \longrightarrow \operatorname{Mod} A_Q$. We will frequently use the following properties of localizations.

Lemma 2.1 (i) Let M and N be R-modules. Then $(M \otimes_R N)_Q \cong M_Q \otimes_Q N_Q$.

(ii) Let M be a right A-module and N be a left A-module. Then $(M \otimes_A N)_Q \cong M_Q \otimes_{A_Q} N_Q$.

Let us recall some settings in [1]. For a right A_Q-module M, an element $x \in M$ is called an A_Q-torsion element if xA_Q is a finite dimensional Q-vector space. Let $\Gamma_{A_Q}(M)$ be the subset of M consisting of all the A_Q-torsion elements. Then $\Gamma_{A_Q}(M)$ is a right A_Q-submodule of M. If $M = \Gamma_{A_Q}(M)$, then M is called an *A_Q-torsion module*. Let $\operatorname{Tor} A_Q$ be the full subcategory of $\operatorname{Mod} A_Q$ consisting of A_Q-torsion modules. Then $\operatorname{Tor} A_Q$ is a Serre subcategory of $\operatorname{Mod} A_Q$. Denote the quotient category

$$Q\operatorname{Mod} A_Q := \frac{\operatorname{Mod} A_Q}{\operatorname{Tor} A_Q}.$$

We refer to the book [5] for the properties of the torsion theory and quotient categories.

Consider the composition of exact functors

$$F : \operatorname{Mod} A \xrightarrow{(-)_Q} \operatorname{Mod} A_Q \xrightarrow{\pi} Q\operatorname{Mod} A_Q.$$

Let $\operatorname{Tor} A$ be the full subcategory of $\operatorname{Mod} A$ consisting of right A-modules M such that $F(M) = 0$. We will say that M is torsion if $M \in \operatorname{Tor} A$. Let $\varphi : M \to M_Q$ be the localizing map. We have the following easy observation.

Lemma 2.2 A right A-module M is in $\operatorname{Tor} A$ if and only if for each $x \in M$, $\varphi(x)A_Q$ is finite dimensional over Q.

For a right A-module M, let $\Gamma_A(M) = \{x \in M | \varphi(x)A_Q \text{ is finite dimensional}\}$. Then $\Gamma_A(M)$ is a torsion submodule of M.

Lemma 2.3 With the above notions, we have

(i) $\Gamma_A(M)$ is the largest torsion submodule of M and $M/\Gamma_A(M)$ is torsion free, which is to say, $\Gamma_A(M/\Gamma_A(M)) = 0$.

(ii) $\Gamma_A(M)_Q = \Gamma_{A_Q}(M_Q)$.

Proof Statement (i) is easy to check. We next prove statement (ii). For $x \in M$ and an nonzero element $s \in R$, we have $(x/s)A_Q = \varphi(x)A_Q$. It follows that $(x/s)A_Q$ is finite dimensional if and only if $\varphi(x)A_Q$ is finite dimensional. Hence $\Gamma_A(M)_Q = \Gamma_{A_Q}(M_Q)$.

The subcategory $\operatorname{Tor} A$ is a Serre subcategory of $\operatorname{Mod} A$. Denote the quotient category

$$Q \operatorname{Mod} A := \frac{\operatorname{Mod} A}{\operatorname{Tor} A}.$$

Then we obtain an exact functor (use the same notation)

$$(-)_Q : Q \operatorname{Mod} A \longrightarrow Q \operatorname{Mod} A_Q. \tag{2.1}$$

As usual conventions, for an object $M \in \operatorname{Mod} A$, the corresponding object in $Q \operatorname{Mod} A$ is denoted by $\mathcal{M}$, and the object in $Q \operatorname{Mod} A_Q$ corresponding to M_Q is denoted by $\mathcal{M}_Q$.

Let M and N be right A-modules. Assume that M is finitely generated. It is well known

$$\operatorname{Hom}_A(M, N)_Q \cong \operatorname{Hom}_{A_Q}(M_Q, N_Q). \tag{2.2}$$

We next show that the above isomorphism may be extended to the quotient categories.

Lemma 2.4 Let M be a right A-module. Let L be an A_Q-submodule of M_Q such that M_Q/L is finite dimensional. Then there is an A-submodule K of M such that $M_Q/L \cong M_Q/K_Q$ and M/K is R-torsion free.

Proof Let $\varphi : M \to M_Q$ be the localizing map, and $K = \{m \in M | \varphi(m) \in L\}$. Then $L = K_Q$. By the construction, we see that M/K is R-torsion free.

Proposition 2.1 Let M be a finitely generated right A-module. For every $N \in \operatorname{Mod} A$, we have

$$\operatorname{Hom}_{Q \operatorname{Mod} A_Q}(\mathcal{M}_Q, \mathcal{N}_Q) \cong \operatorname{Hom}_{Q \operatorname{Mod} A}(\mathcal{M}, \mathcal{N})_Q.$$

Proof We have the following computations

$$\begin{aligned} \operatorname{Hom}_{Q \operatorname{Mod} A_Q}(\mathcal{M}_Q, \mathcal{N}_Q) &= \varinjlim \operatorname{Hom}_{A_Q}(L, N_Q/\Gamma_{A_Q}(N_Q)) \\ &= \varinjlim \operatorname{Hom}_{A_Q}(K_Q, N_Q/\Gamma_{A_Q}(N_Q)), \end{aligned}$$

where the first limit runs over all the A_Q-submodules L of M_Q such that M_Q/L is finite dimensional, and the second limit runs over all the A-submodules K such that $M_Q/K_Q \cong (M/K)_Q$ is finite dimensional. Let $T = \Gamma_A(N)$. Then $T_Q = \Gamma_{A_Q}(N_Q)$ by Lemma 2.3. Hence we have

$$\begin{aligned} & \varinjlim \operatorname{Hom}_{A_Q}(K_Q, N_Q/\Gamma_{A_Q}(N_Q)) \\ \cong\ & \varinjlim \operatorname{Hom}_{A_Q}(K_Q, N_Q/T_Q) \cong \varinjlim \operatorname{Hom}_{A_Q}(K_Q, (N/T)_Q) \\ \cong\ & \varinjlim \operatorname{Hom}_A(K, N/T)_Q = \operatorname{Hom}_{\mathrm{QMod}\,A}(\mathcal{M}, \mathcal{N})_Q. \end{aligned}$$

3 Hopf Dense Galois Extensions

In this section, R is a noetherian commutative domain. Let Q be its quotient field. An R-module M is said to be quotient-finite if M_Q is finite dimensional.

Suppose that A is a noetherian R-algebra which is projective as an R-module. Let H be a Hopf R-algebra which is a finitely generated free R-module. Assume that H coacts on A so that A is a right H-comodule algebra through the coaction $\rho : A \to A \otimes_R H$. As the usual convention, we denote $A^{\mathrm{co}H} = \{a \in A | \rho(a) = a \otimes 1\}$ the coinvariant subalgebra of A.

We next extend the concept of Hopf dense Galois extension (cf. [1]) to algebras over a ring. Consider the following map

$$\beta : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_R H,\ a \otimes b \mapsto (a \otimes 1)\rho(b).$$

We call $A/A^{\mathrm{co}H}$ is a Hopf dense Galois extension if the cokernel of β is quotient-finite. Note that if β is an epimorphism, then $A/A^{\mathrm{co}H}$ is a classical Hopf Galois extension (cf. [6, 7]).

Applying the localizing functor $(-)_Q$ to the algebra A and the Hopf algebra H, we obtain a finite dimensional Hopf algebra H_Q and a right H_Q-comodule algebra A_Q. Note that the coaction of H_Q on A_Q is the map $\rho_Q : A_Q \to A_Q \otimes_Q H_Q$.

Lemma 3.5 With the notions as above, $(A^{\mathrm{co}H})_Q \cong (A_Q)^{\mathrm{co}H_Q}$.

Proof Let $\varphi : A \to A_Q$ and $\phi : H \to H_Q$ be the localizing maps. Applying $(-)_Q$ to the inclusion map $A^{\mathrm{co}H} \to A$, we obtain that $(A^{\mathrm{co}H})_Q$ is contained in $(A_Q)^{\mathrm{co}H_Q}$. On the other hand, assume $a \in A$ and $\rho_Q(\varphi(a)) = \varphi(a) \otimes_Q 1$. Since H is a finitely generated free R-module and R is a noetherian commutative domain, we extend the unit 1 of R to an R-basis $h_0 = 1, h_1, \ldots, h_n$ of H. Then we may write $\rho(a) = \sum\limits_{i=0}^{n} a_i \otimes_R h_i$. Then $\rho_Q(\varphi(a)) = \sum\limits_{i=0}^{n} \varphi(a_i) \otimes_Q \phi(h_i)$. Since H is free, $\phi(h_0), \ldots, \phi(h_n)$ is a Q-basis of H_Q. Comparing with the assumption $\rho_Q(\varphi(a)) = \varphi(a) \otimes_Q 1$, we obtain $\varphi(a_0) = \varphi(a)$ and $\varphi(a_i) = 0$ for $i = 1, \ldots, n$. Since A is projective as an R-module, it is R-torsion free, hence φ is injective. It follows that $a_0 = a$ and $a_i = 0$ for $i = 1, \ldots, n$. Hence $\rho(a) = a \otimes_R 1$.

Proposition 3.2 Let A be an R-algebra which is projective as an R-module, and let H be an R-Hopf algebra which is R-free. Assume A is a right H-comodule algebra. Then $A/A^{\mathrm{co}H}$ is Hopf dense Galois if and only if $A_Q/(A_Q)^{\mathrm{co}H_Q}$ is Hopf dense Galois.

Proof Applying $(-)_Q$ to the map $\beta : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_R H$, we obtain

$$\beta_Q : A_Q \otimes_{(A_Q)^{\mathrm{co}H_Q}} A_Q \to A_Q \otimes_Q H_Q$$

by Lemma 3.5. Then the condition that the cokernel of β_Q is finite dimensional implies both $A/A^{\mathrm{co}H}$ and $A_Q/(A_Q)^{\mathrm{co}H_Q}$ are Hopf dense Galois.

Since the torsion functor Γ_A is left exact, it has right derived functors. Let $R^i\Gamma_A$ $(i \geq 0)$ denotes the i-th right derived functor of Γ_A. Similarly, we have $R^i\Gamma_{A_Q}$.

Lemma 3.6 If $R^i\Gamma_A(M) = 0$ for all $i \leq k$, then $R^i\Gamma_{A_Q}(M_Q) = 0$ for all $i \leq k$.

Proof Let $0 \to M \to I^0 \to I^1 \to \cdots \to I^k \to \cdots$ be an injective resolution of M. Since the localizing functor $(-)_Q$ preserves injective modules, it follows that $0 \to M_Q \to I^0_Q \to I^1_Q \to \cdots \to I^k_Q \to \cdots$ is an injective resolution of M_Q. Let $I^\bullet$ be the complex $0 \to I^0 \to \cdots \to I^k \to \cdots$. By Lemma 2.3, $R^i\Gamma_{A_Q}(M_Q) = H^i\Gamma_{A_Q}(I^\bullet_Q) \cong H^i(\Gamma_A(I^\bullet)_Q) \cong H^i(\Gamma_A(I^\bullet))_Q = R^i\Gamma_A(M)_Q$. So $R^i\Gamma_{A_Q}(M_Q) = 0$ for all $i \leq k$ in case $R^i\Gamma_A(M) = 0$ for $i \leq k$.

Since by assumption H is a finitely generated R-algebra, then similar to equation (2.2), we have an isomorphism of Hopf algebras $\mathrm{Hom}_R(H, R) \otimes_R Q \cong \mathrm{Hom}_Q(H \otimes_R Q, Q)$. Thus we can write

$$H^*_Q = (H^*)_Q \cong (H_Q)^*, \tag{3.1}$$

where $H^* = \mathrm{Hom}_R(H, R)$ is the dual Hopf algebra of H.

An important feature of Hopf dense Galois extensions over a field is the truth of Auslander theorem (cf. [1, Theorem 3.10]). Note that Theorem 3.10 of [1] is still true if the characteristic is positive. Next result shows that a weaker version of Auslander theorem holds for Hopf dense Galois extensions over a commutative domain.

Theorem 3.7 Let A and H be the algebras as in the beginning of this section. Assume further that H_Q is cosemisimple. If $A/A^{\mathrm{co}H}$ is a Hopf dense Galois extension, and $R^i\Gamma_A(A) = 0$ for $i \leq 2$, then the natural map

$$\psi : A\#H^* \longrightarrow \mathrm{End}_{A^{\mathrm{co}H}}(A), a \otimes \alpha \mapsto [b \mapsto a(\alpha \cdot b)]$$

is injective. Moreover, for each $f \in \mathrm{End}_{A^{\mathrm{co}H}}(A)$, there exist $0 \neq r \in R$ and $\sum\limits_{i=1}^{n} a_i\#\alpha_i \in A\#H^*$ such that $rf(b) = \sum\limits_{i=1}^{n} a_i(\alpha_i \cdot b)$ for all $b \in A$.

Proof By Proposition 3.2, $A_Q/(A_Q)^{\mathrm{co}H_Q}$ is a Hopf dense Galois extension over the field Q. By Lemma 3.6, $R^i\Gamma_{A_Q}(A_Q) = 0$ for $i \leq 2$. Then [1, Theorem 3.10] insures that the natural map

$$\xi : A_Q\#H^*_Q \longrightarrow \mathrm{End}_{(A_Q)^{\mathrm{co}H_Q}}(A_Q)$$

is an isomorphism. By Lemma 3.5, we have $(A_Q)^{\mathrm{co}H_Q} = (A^{\mathrm{co}H})_Q$. It follows $\mathrm{End}_{(A_Q)^{\mathrm{co}H_Q}}(A_Q) = \mathrm{End}_{A^{\mathrm{co}H}}(A)_Q$. Moreover, since $A_Q\#H^*_Q = (A\#H^*)_Q$, it follows that $\xi = \psi_Q : (A\#H^*)_Q \longrightarrow \mathrm{End}_{A^{\mathrm{co}H}}(A)_Q$. We next show that ψ is a monomorphism. Let $K = \ker\psi$ and $M = \mathrm{coker}\,\psi$. Then $K_Q = \ker\xi = 0$ and $M_Q = \mathrm{coker}\,\xi = 0$. Since A is projective over R and H is R-free, it follows that $A\#H^*$ is projective over R. Since R is a domain, $A\#H^*$ is R-torsion free. Hence K is R-torsion free, which implies $K=0$. Therefore ψ is injective. Moreover, since $M_Q=0$, it follows that for each $f\in \mathrm{End}_{A^{\mathrm{co}H}}(A)$, there is an element $0\neq r\in R$ such that rf lies in the image of ψ, that is $rf=\psi(\sum\limits_{i=1}^{n} a_i\#\alpha_i)$ for some $\sum\limits_{i=1}^{n} a_i\#\alpha_i\in A\#H^*$. Hence $rf(b)=\sum\limits_{i=1}^{n} a_i(\alpha_i\cdot b)$ for all $b\in A$.

4 Hopf Dense Galois Extensions of Almost Commutative Algebras

In this section, R is a noetherian commutative domain of characteristic zero, and $\Bbbk$ is an algebraic closed field containing R as a subring. H is an R-Hopf algebra which is a finitely generated free R-module. The filtration of a filtered R-algebra A is an ascending filtration

$$0\subseteq F_0A\subseteq F_1A\subseteq\cdots\subseteq F_iA\subseteq\cdots,\ i\in\mathbb{N}.$$

We call an R-algebra A is almost commutative if A is a filtered R-algebra and the associated graded algebra $gr(A)$ is a graded commutative algebra. Similar to equation (3.1), we will write

$$H^*_{\Bbbk}=\mathrm{Hom}_R(H,R)\otimes_R\Bbbk\cong \mathrm{Hom}_{\Bbbk}(H\otimes_R\Bbbk,\Bbbk)$$

for simplicity.

Lemma 4.8 Let $B=B_0\oplus B_1\oplus\cdots$ be a graded R-algebra which is a commutative domain and is projective over R. Let $\rho: B\to B\otimes_R H$ be a right H-coaction on B which preserves the gradings. If $B/B^{\mathrm{co}H}$ is a Hopf dense Galois extension, then $H^*_{\Bbbk}$ is isomorphic to a group algebra.

Proof We write $B_{\Bbbk}=B\otimes_R\Bbbk$ and $H_{\Bbbk}=H\otimes_R\Bbbk$. Applying $-\otimes_R\Bbbk$ to the right coaction $\rho: B\to B\otimes_R H$, we obtain a coaction $\rho_{\Bbbk}: B_{\Bbbk}\to B_{\Bbbk}\otimes_{\Bbbk} H_k$. Consider the canonical map $\beta_{\Bbbk}: B_{\Bbbk}\otimes_{B^{coH_{\Bbbk}}} B_{\Bbbk}\longrightarrow B_{\Bbbk}\otimes_{\Bbbk} H_{\Bbbk}$. Since B is commutative, $\beta_{\Bbbk}$ is indeed an algebra homomorphism, where we view $B_{\Bbbk}\otimes_{\Bbbk} H_{\Bbbk}$ as the algebra by the usual multiplication of tensor products of algebras. The same proof of Proposition 3.2 shows that $B_{\Bbbk}/(B_{\Bbbk})^{coH_{\Bbbk}}$ is a Hopf dense Galois extension, then the cokernel of $\beta_{\Bbbk}$ is finite dimensional over $\Bbbk$. Then there is an integer $n\geq 0$ such that $(\oplus_{i\geq n}B_i)_{\Bbbk}\otimes_{\Bbbk} H_{\Bbbk}\subseteq \mathrm{im}\,\beta_{\Bbbk}$. Since by assumption B is commutative, which implies $B_{\Bbbk}\otimes_{B^{coH_{\Bbbk}}} B_{\Bbbk}$ is commutative, and thus $\mathrm{im}\,\beta_{\Bbbk}$ is commutative. For $g,h\in H$, taking nonzero elements $a,b\in(\oplus_{i\geq n}B_i)_{\Bbbk}$, then $(a\otimes_{\Bbbk} g),(b\otimes_{\Bbbk} h)\in \mathrm{im}\,\beta_{\Bbbk}$, which impies $(a\otimes_{\Bbbk} g)(b\otimes_{\Bbbk} h)=(b\otimes_{\Bbbk} h)(a\otimes_{\Bbbk} g)$. Then $ab\otimes_{\Bbbk} gh=ba\otimes_{\Bbbk} hg=ab\otimes_{\Bbbk} hg$. Since B is a domain and B is projective over R, $ab\neq 0$. Hence we have $gh=hg$, that is, $H_{\Bbbk}$ is commutative. Since H is finitely generated as an R-module, $H_{\Bbbk}$ is finite dimensional. Therefore the dual Hopf algebra $H^*_{\Bbbk}$ is cocommutative. Since $\Bbbk$ is algebraic closed with characteristic zero, $H^*_{\Bbbk}$ is isomorphic to a group algebra.

Let B be a filtered R-algebra. Let M be a filtered right B-module and let N be a left filtered B-module. The tensor product $M\otimes_B N$ has an induced filtration defined by $F_n(M\otimes_B N)$ to be the abelian subgroup of $M\otimes_B N$ generated by elements $x\otimes y$ for all

$x \in F_iM$ and $y \in F_jN$ such that $i + j = n$. There is a graded epimorphism (cf. [8, §6, Chapter I])

$$\varphi_{M,N} : gr(M) \otimes_{gr(B)} gr(N) \longrightarrow gr(M \otimes_B N),\ \overline{x} \otimes \overline{y} \mapsto \overline{x \otimes y}, \tag{4.1}$$

where $x \in F_iM \setminus F_{i-1}M$, $y \in F_jN \setminus F_{j-1}N$ and $\overline{x}, \overline{y}$ are corresponding elements in the associated graded modules, similarly $\overline{x \otimes y}$ is the corresponding element in the graded abelian group associated to $M \otimes_B N$.

Suppose that there is a right H-coaction $\rho : B \to B \otimes_R H$ which preserves the filtration, where the filtration of $B \otimes_R H$ is induced by the filtration of B. Then the induced map

$$gr(\rho) : gr(B) \longrightarrow gr(B) \otimes_R H$$

is a right H-coaction on the associated graded algebra $gr(B)$. Then the filtration of B induces a filtration on $B^{\mathrm{co}H}$. Associated to this filtration, there is a graded algebra $gr(B^{\mathrm{co}H})$. Then $gr(B^{\mathrm{co}H})$ is a graded subalgebra of $(gr(B))^{\mathrm{co}H}$. In general, $gr(B^{\mathrm{co}H})$ is not equal to $(gr(B))^{\mathrm{co}H}$.

Let X and Y be filtered R-modules. An R-module homomorphism $f : X \to Y$ is called a strict filtered map [8] if f preserves the filtration and $F_nY \cap \mathrm{im}\, f = f(F_nX)$ for all n.

Lemma 4.9 Keep the notations as above. If $B/B^{\mathrm{co}H}$ is a Hopf dense Galois extension and the canonical map $\beta : B \otimes_{B^{\mathrm{co}H}} B \to B \otimes_R H$ is strict, then $gr(B)/(gr(B))^{\mathrm{co}H}$ is a Hopf dense Galois extension.

Proof Let $gr(\rho) : gr(B) \to gr(B) \otimes_R H$ be the induced H-coaction on $gr(B)$. Let

$$\beta_{gr} : gr(B) \otimes_{(gr(B))^{\mathrm{co}H}} gr(B) \to gr(B) \otimes_R H,\ a \otimes b \mapsto (a \otimes 1)(gr(\rho)(b))$$

be the canonical map associated to $gr(\rho)$. Denote $K = \mathrm{coker}\, \beta$. Then K has a natural filtration inherits from $B \otimes_R H$. Since β is strict, by [8, Theorem 4.2.4, Chapter I], we have an exact sequence

$$gr(B \otimes_{B^{\mathrm{co}H}} B) \xrightarrow{gr(\beta)} gr(B \otimes_R H) \longrightarrow gr(K) \longrightarrow 0.$$

By equation (4.1), the map $\varphi_{B,B} : gr(B) \otimes_{gr(B^{\mathrm{co}H})} gr(B) \longrightarrow gr(B \otimes_{B^{\mathrm{co}H}} B)$ is an epimorphism. Note that $gr(B \otimes_R H) = gr(B) \otimes_R H$, we have the following commutative diagram

$$\begin{tikzcd} gr(B) \otimes_{gr(B^{\mathrm{co}H})} gr(B) \arrow[r, "\varphi_{B,B}"] \arrow[d, "p"'] & gr(B \otimes_{B^{\mathrm{co}H}} B) \arrow[d, "gr(\beta)"] \\ gr(B) \otimes_{(gr(B))^{\mathrm{co}H}} gr(B) \arrow[r, "\beta_{gr}"'] & gr(B) \otimes_R H, \end{tikzcd} \tag{4.2}$$

where p is an epimorphism induced by the fact that $gr(B^{\mathrm{co}H})$ is a graded subalgebra of $(gr(B))^{\mathrm{co}H}$. Since $\varphi_{B,B}$ and p are both epic, we have $\mathrm{coker}\, \beta_{gr} = \mathrm{coker}\, gr(\beta) \cong gr(K)$. By assumption, $B/B^{\mathrm{co}H}$ is a Hopf dense Galois extension, thus K_Q is finite dimensional over Q. Then $(gr(K))_Q = gr(K_Q)$ is also finite dimensional over Q. Therefore $gr(B)/(gr(B))^{\mathrm{co}H}$ is a Hopf dense Galois extension.

Theorem 4.10 Let A be an almost commutative R-algebra such that $gr(A)$ is a domain. Assume that A is a right H-comodule algebra such that the right H-coaction preserves the filtration. If $A/A^{\mathrm{co}H}$ is a Hopf dense Galois extension and the canonical map $\beta : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_R H$ is strict, then $H_{\Bbbk}^*$ is isomorphic to a group algebra over $\Bbbk$.

Proof As before, we write $A_{\Bbbk} = A \otimes_R \Bbbk$ and $H_{\Bbbk} = H \otimes_R \Bbbk$. Since A is a filtered R-algebra, $A_{\Bbbk}$ is also a filtered $\Bbbk$-algebra with the obvious induced filtration. Since the right H-coaction preserves the filtration, it induces a right H-coaction on the associated graded algebra $gr(A)$. Applying the functor $- \otimes_R \Bbbk$ to the right H-coaction $\rho : A \to A \otimes_R H$, we obtain that $A_{\Bbbk}$ is a right $H_{\Bbbk}$-comodule algebra and $A_{\Bbbk}/(A_{\Bbbk})^{coH_{\Bbbk}}$ is a Hopf dense Galois extension. Moreover, since β is strict, the induced canonical map $\beta_{\Bbbk}$ is also a strict filtered map. By Lemma 4.9, $gr(A_{\Bbbk})/(gr(A_{\Bbbk}))^{coH_{\Bbbk}}$ is a Hopf dense Galois extension. Since A is almost commutative, $gr(A)$ is a commutative domain. Then $gr(A_{\Bbbk}) = gr(A) \otimes_R \Bbbk$ is a commutative domain over $\Bbbk$. By Lemma 4.8, $H_{\Bbbk}^*$ is a group algebra.

For a filtered algebra A and a filtration preserving right H-coaction, the canonical map

$$\beta : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_R H$$

may be not a strict map. Hence the associated graded algebra $gr(A)$ may not be a Hopf dense Galois extension over $(gr(A))^{\mathrm{co}H}$. Some further discussions will be given in the next section.

5 Some Corollaries

In this section, $\Bbbk$ is an algebraically closed field of characteristic zero. All the algebras and modules in this section are over $\Bbbk$. Let H be a finite dimensional Hopf algebra.

The next result is a direct consequence of [2, Proposition 3.6] if A is noetherian and H is semisimple. We give a direct proof and drop the assumptions in [2, Proposition 3.6].

Proposition 5.3 Let A be a filtered algebra with an ascending filtration

$$0 \subseteq F_0A \subseteq F_1A \subseteq \cdots \subseteq F_iA \subseteq \cdots, \ i \in \mathbb{N}$$

such that F_iA is finite dimensional for all $i \geq 0$. Assume that A is a right H-comodule algebra and the coaction preserving the filtration. If the associated graded algebra $gr(A)$ is a Hopf dense Galois extension over $(gr(A))^{\mathrm{co}H}$, then $A/A^{\mathrm{co}H}$ is a Hopf dense Galois extension.

Proof Let $\beta : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_{\Bbbk} H$ be the canonical map. Similar to the diagram (4.2), we have the following commutative diagram

$$\begin{tikzcd} gr(A) \otimes_{gr(A^{\mathrm{co}H})} gr(A) \arrow[r, "\varphi_{A,A}"] \arrow[d, "p"'] & gr(A \otimes_{A^{\mathrm{co}H}} A) \arrow[d, "gr(\beta)"] \\ gr(A) \otimes_{(gr(A))^{\mathrm{co}H}} gr(A) \arrow[r, "\beta_{gr}"'] & gr(A) \otimes_{\Bbbk} H. \end{tikzcd} \tag{5.1}$$

Since $gr(A)/(gr(A))^{\mathrm{co}H}$ is a Hopf dense Galois extension, which is to say that coker β_{gr} is finite dimensional. It follows that there is a positive number n such that for all $k \geq n$, we have

$$\beta_{gr}(gr(A) \otimes_{(gr(A))^{\mathrm{co}H}} gr(A))_k = (F_k A/F_{k-1}A) \otimes_{\Bbbk} H.$$

We claim that $\beta(F_k(A \otimes_{A^{\mathrm{co}H}} A)) + F_{n-1}A \otimes_{\Bbbk} H = F_k A \otimes_{\Bbbk} H$ for all $k \geq n$. By the commutative diagram (5.1), for every $x \in F_n A \otimes_{\Bbbk} H$, there is an element $y \in F_n(A \otimes_{A^{\mathrm{co}H}} A)$ such that $\beta(y) + z = x$ for some $z \in F_{n-1}A \otimes_{\Bbbk} H$. Hence $\beta(F_n(A \otimes_{A^{\mathrm{co}H}} A)) + F_{n-1} \otimes_{\Bbbk} H = F_n A \otimes_{\Bbbk} H$. Now assume that $\beta(F_i(A \otimes_{A^{\mathrm{co}H}} A)) + F_{n-1}A \otimes_{\Bbbk} H = F_i A \otimes_{\Bbbk} H$ for $i \geq n$. By the commutative diagram (5.1), we have $\beta(F_{i+1}(A \otimes_{A^{\mathrm{co}H}} A)) + F_i A \otimes_{\Bbbk} H = F_{i+1}A \otimes_{\Bbbk} H$. Then

$$\begin{aligned} F_{i+1}A \otimes_{\Bbbk} H &= \beta(F_{i+1}(A \otimes_{A^{\mathrm{co}H}} A)) + \beta(F_i(A \otimes_{A^{\mathrm{co}H}} A)) + F_{n-1}A \otimes_{\Bbbk} H \\ &= \beta(F_{i+1}(A \otimes_{A^{\mathrm{co}H}} A)) + F_{n-1}A \otimes_{\Bbbk} H. \end{aligned}$$

Hence we have $\dim((A \otimes_{\Bbbk} H)/\operatorname{im}\beta) \leq \dim(F_{n-1}A \otimes_{\Bbbk} H) < \infty$. Therefore, $A/A^{\mathrm{co}H}$ is a Hopf dense Galois extension.

Combined with Lemma 4.9, we have the following corollary.

Corollary 5.11 With the same conditions in Proposition 5.3, if in addition the canonical map $\beta : A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_{\Bbbk} H$ is strict, then $A/A^{\mathrm{co}H}$ is a Hopf dense Galois extension if and only if $gr(A)/(gr(A))^{\mathrm{co}H}$ is a Hopf dense Galois extension.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and let $f : \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ be a 2-cocycle, that is, for every $x, y, z \in \mathfrak{g}$, $f(x, x) = 0$, $f(x, [y, z]) + f(z, [x, y]) + f(y, [z, x]) = 0$. Then a Sridharan enveloping algebra [9] of $\mathfrak{g}$ is defined to be the associative algebra

$$U_f(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ over $\Bbbk$ and I is the ideal of $T(\mathfrak{g})$ generated by elements

$$x \otimes y - y \otimes x - [x, y] - f(x, y) \quad \text{for all } x, y \in \mathfrak{g}.$$

Assume that $\{x_1, \ldots, x_n\}$ is an R-basis of $\mathfrak{g}$. Then $U_f(\mathfrak{g})$ is a free R-module and it has a basis $\{x_1^{i_1} x_2^{x_2} \cdots x_n^{i_n} | i_1, i_2, \ldots, i_n \geq 0\}$ (cf. [9, Theorem 2.6]). And $U_f(\mathfrak{g})$ is a filtered algebra with an ascending filtration defined by

$$F_k U_f(\mathfrak{g}) = \operatorname{span}\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} | i_1 + i_2 + \cdots + i_n \leq k\}$$

for all $k \geq 0$. The associated graded algebra of $U_f(\mathfrak{g})$ is the commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$.

Suppose that there is a right H-coaction $\rho_{\mathfrak{g}} : U_f(\mathfrak{g}) \to U_f(\mathfrak{g}) \otimes_{\Bbbk} H$ which preserves the filtration defined as above. Then the associated graded map $gr(\rho_{\mathfrak{g}}) : grU_f(\mathfrak{g}) \to grU_f(\mathfrak{g}) \otimes_{\Bbbk} H$ is a right H-coaction on $grU_f(\mathfrak{g})$.

Corollary 5.12 With the notions as above, if $grU_f(\mathfrak{g})$ is a Hopf dense Galois extension over $(grU_f(\mathfrak{g}))^{\mathrm{co}H}$, then $U_f(\mathfrak{g})$ is a Hopf dense Galois extension over $U_f(\mathfrak{g})^{\mathrm{co}H}$ and H^* is isomorphic to a group algebra.

Proof The first part follows from Proposition 5.3. The second part follows from Lemma 4.8 since $grU_f(\mathfrak{g}) \cong \Bbbk[x_1, \ldots, x_n]$.

As we know, a right H-coaction on an algebra A is equivalent to a left H-action on A. Moreover, we have $A^{\mathrm{co}H} = A^H$, where $A^H = \{a \in A | ha = \varepsilon(h)a, \text{ for all } h \in H\}$. Hence we may rewrite Corollary 5.12 in the Hopf action version.

Corollary 5.13 Suppose that H acts on a Sridharan enveloping algebra $U_f(\mathfrak{g})$. If the H-action preserves the filtration of $U_f(\mathfrak{g})$ and $grU_f(\mathfrak{g})$ is a right H^*-Hopf dense Galois extension over $(grU_f(\mathfrak{g}))^H$, then $U_f(\mathfrak{g})$ is a right H^*-Hopf dense Galois extension over $U_f(\mathfrak{g})^H$, and H is isomorphic to a group algebra.

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环上的Hopf稠密伽罗瓦扩张

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摘要: 给定交换整环R, 以及Hopf R-代数H, 且H是一个有限生成的自由R-模. 设A是一个R-代数, 并且A是H-余模代数. 如果自然映射$\beta: A \otimes_{A^{\mathrm{co}H}} A \to A \otimes_R H$的余核是商有限的, 则称$A/A^{\mathrm{co}H}$是一个Hopf稠密伽罗瓦扩张. 它是域上Hopf稠密伽罗瓦扩张的推广. 本文证明了R上的Hopf稠密伽罗瓦扩张隐含了一个弱化的Auslander定理. 此外, 假设A是几乎可交换代数, 且$gr(A)$是一个整环. 如果$A/A^{\mathrm{co}H}$是Hopf稠密伽罗瓦扩张, 且自然映射β是严格的, 本文证明了在此情形下, H在一个包含R的代数闭域上对偶于一个有限群代数.

关键词: Hopf稠密伽罗瓦扩张; 局域化; 商范畴; 滤过代数

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