Vol. 40 (2020)
No. 2

数 学 杂 志
J. of Math. (PRC)

WEAK AND STRONG DYADIC MARTINGALE SPACES WITH VARIABLE EXPONENTS

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Abstract: In this paper, we study the atomic decompositions of weak and strong dyadic martingale spaces with variable exponents. By atomic decompositions, we prove that sublinear operator T is bounded from $wH^s_{p(\cdot)}$ to $wL_{p(\cdot)}$; Cesàro operator is bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ and from $L_{p(\cdot)}$ to $L_{p(\cdot)}$, which generalize the boundedness of operators in constant exponent case.

Keywords: atomic decompositions; variable exponents; Cesàro operator
2010 MR Subject Classification: 60G46
Document code: A **Article ID:** 0255-7797(2020)02-0165-10

1 Introduction

It's well known that variable exponent Lebesgue spaces have been got more and more attention in modern analysis and functional space theory. In particular, Fan and Zhao [1] investigated various properties of variable exponent Lebesgue spaces and Sobolev spaces. Diening [2,3] and Cruz-Uribe [4] proved the boundedness of Hardy-Littlewood maximal operator on variable exponent Lebesgue function spaces $L^{p(\cdot)}(R^n)$ under the conditions that the exponent $p(\cdot)$ satisfies so called log-Hölder continuity and decay restriction. Many other authors studied its applications to harmonic analysis and some other subjects.

The situation of martingale spaces is different from function spaces. For example, the good-λ inequality method used in classical martingale theory can not be used in variable exponent case. However, recently, variable exponent martingale spaces have been paid more attention too. Aoyama [5] proved that, if $p(\cdot)$ is $\mathcal{F}_0$-measurable, then there exists a positive constant c such that $\|M(f)\|_{L_{p(\cdot)}} \le c\|f\|_{L_{p(\cdot)}}$ for $f \in L_{p(\cdot)}$. Nakai and Sadasue [6] pointed out that the inverse is not true, namely, there exists a variable exponent $p(\cdot)$ such that $p(\cdot)$ is not $\mathcal{F}_0$-measurable, and the above inequality holds, under the assumption that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms. Zhiwei Hao [7] established an atomic decomposition of a predictable martingale Hardy space with variable exponents defined on probability spaces. Motivated by them, we research dyadic martingale Hardy space with variable exponents.

2 Preliminaries and Notations

* **Received date:** 2018-11-11 **Accepted date:** 2019-03-05
Foundation item: Supported by National Natural Science Foundation of China (11871195).
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In this paper the unit interval $[0, 1)$ and Lebesgue measure P are to be considered. Throughout this paper, $\mathbf{Z}$, $\mathbf{N}$ denote the integer set and nonnegative integer set. By a dyadic interval we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k \in \mathbf{N}, 0 \le k < 2^n$. Given $n \in \mathbf{N}$ and $x \in [0, 1)$, let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in [0, 1)\}$ will be denoted by $\mathcal{F}_n$, more precisely,

$$\mathcal{F}_n = \sigma\{[k2^{-n}, (k+1)2^{-n}) : 0 \le k < 2^n\}.$$

Obviously, $(\mathcal{F}_n)$ is regular. Define $\mathcal{F} = \sigma(\cup_n \mathcal{F}_n)$ and denote the set of dyadic intervals by $A(\mathcal{F}_n)$ and write $A = \cup_n A(\mathcal{F}_n)$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. For a complex valued martingale $f = (f_n)_{n\ge0}$, denote $df_i = f_i - f_{i-1}$ (with convention $df_{-1} = 0$) and

$$\begin{aligned} M_n(f) &= \sup_{0\le i\le n} |f_i|, \quad M(f) = \sup_{n\ge0} |f_n|, \\ s_n(f) &= (\sum_{i=1}^{n} E_{i-1}|df_i|^2)^{1/2}, s(f) = (\sum_{i=1}^{\infty} E_{i-1}|df_i|^2)^{1/2}. \end{aligned}$$

Remark 2.1 (see [8]) If $(\mathcal{F}_n)$ is regular, then for all nonnegative adapted processes $\gamma = (\gamma_n)$ and $\lambda \ge \|\gamma_0\|_\infty$, there exist a constant $c > 0$ and a stopping time τ_λ such that

$$\begin{aligned} &\{M(\gamma) > \lambda\} \subset \{\tau_\lambda < \infty\}, \;\; P(\tau_\lambda < \infty) \le cP(M(\gamma) > \lambda), \\ &\sup\nolimits_{n\le t_\lambda} \gamma_n = M_{\tau_\lambda}(\gamma) \le \lambda, \;\; \lambda_2 \ge \lambda_1 \ge \|\gamma_0\|_\infty \Rightarrow \tau_{\lambda_1} \le \tau_{\lambda_2}. \end{aligned}$$

Let $p(\cdot) : [0, 1) \to (0, \infty)$ be an $\mathcal{F}$-measurable function, we define $p_B^- = \operatorname{ess\,inf}\{p(x) : x \in B\}, p_B^+ = \operatorname{ess\,sup}\{p(x) : x \in B\},\ B \subset [0, 1)$. We use the abbreviations $p^+ = p^+_{[0,1)}$ and $p^- = p^-_{[0,1)}$.

We say that p is log-Hölder continuous if

$$|p(x) - p(y)| \le \frac{c}{-\log d(x, y)}, \tag{2.1}$$

when $d(x, y) \le 1/2$.

The Lebesgue space with variable exponent $p(\cdot)$ denoted by $L_{p(\cdot)}$ is defined as the set of all $\mathcal{F}$-measurable functions f satisfying

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(|f(x)|/\lambda) \le 1\right\} < \infty,$$

where $\rho_{p(\cdot)}(f) = \int_0^1 |f(x)|^{p(x)} dP$.

If $f = (f_n)$ is a martingale, we define $\|f\|_{p(\cdot)} = \sup_{n\ge1} \|f_n\|_{p(\cdot)}$.

Remark 2.2 (see [7]) If p is log-Hölder continuous, then we have

$$P(B)^{1/p_B^-} \approx P(B)^{1/p_B^+} \approx P(B)^{1/p(x)} \approx \|\chi_B\|_{p(\cdot)}, \quad x \in B \subset [0, 1).$$

The weak Lebesgue space with variable exponent $p(\cdot)$ denoted by $wL_{p(\cdot)}$ is defined as the set of all $\mathcal{F}$-measurable functions f satisfying

$$\|f\|_{wL_{p(\cdot)}} := \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{p(\cdot)} < \infty.$$

Then we define the strong and weak variable exponent dyadic martingale Hardy spaces as follows

$$\begin{aligned} H_{p(\cdot)} &= \left\{ f = (f_n) : M(f) \in L_{p(\cdot)} \right\}, \|f\|_{H_{p(\cdot)}} = \|M(f)\|_{p(\cdot)}, \\ wH^s_{p(\cdot)} &= \left\{ f = (f_n) : s(f) \in wL_{p(\cdot)} \right\}, \|f\|_{wH^s_{p(\cdot)}} = \|s(f)\|_{wLp(\cdot)}. \end{aligned}$$

We always denote by c some positive constant, but its value may be different in each appearance.

3 Weak Atomic Decompositions

Definition 3.1 A measurable a is called a weak- atom if there exists a stopping time ν such that

(1) $E_n(a) = 0, n \geq \nu$,

(2) $\|s(a)\|_\infty < \infty$.

Theorem 3.2 Suppose that p is log-Hölder continuous and $0 < p^- < p^+ \leq 1$. For any $f = (f_n) \in wH^s_{p(\cdot)}$, there exist (a^k) of weak atoms with the corresponding stopping times ν_k and (u_k) of nonnegative real numbers such that

$$f_n = \sum_{k\in\mathbf{Z}} u_k E_n a^k \quad \text{a.e.}, \tag{3.1}$$

$$s(a^k) \leq 3 \cdot 2^k, \tag{3.2}$$

$$\|f\|_{wH^s_{p(x)}} \sim \inf \sup_{k\in\mathbf{Z}} 2^k \|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)}, \tag{3.3}$$

where the infimum is taken over all preceding decompositions of f.

Proof Assume that $f = (f_n) \in wH^s_{p(\cdot)}$. Let us define the stopping times $\nu_k := \inf\{n \in \mathbf{N} : s_{n+1}(f) > 2^k\}$. Consequently, f_n can be written as

$$f_n^\nu = \sum_{m=0}^{n} \chi_{\{\nu\geq m\}}(f_m - f_{m-1}). \tag{3.4}$$

Now let $a_n^k := f_n^{\nu_{k+1}} - f_n^{\nu_k}$. Thus $s(a^k) \leq s(f^{\nu_{k+1}}) + s(f^{\nu_k}) \leq 3 \cdot 2^k < \infty$. Then there exists a^k such that $a_n^k = E_n(a^k)$. It is clear that a^k is really a weak atoms for $k \in \mathbf{Z}$.

Since $\{\nu_k \neq \infty\} = \{s(f) > 2^k\}$, by the definition we have

$$2^k \|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)} = 2^k \|\chi_{\{s(f)>2^k\}}\|_{p(\cdot)} \leq \|s(f)\|_{wL_{p(\cdot)}}. \tag{3.5}$$

Thus

$$\sup_{k\in\mathbf{Z}} 2^k\|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)} \le \|s(f)\|_{wL_{p(\cdot)}}. \tag{3.6}$$

Conversely, let $D := \sup\limits_{k\in\mathbf{Z}} 2^k\|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)}$. For a fixed $y > 0$ choose $j \in \mathbf{Z}$ such that $2^j \le y < 2^{j+1}$. Then

$$f_n = \sum_{k=-\infty}^{j-1} a_n^k + \sum_{k=j}^{\infty} := g_n + h_n$$

and $p(\cdot)/p^- \ge 1$ implies that $s(f) \le s(g) + s(h)$ and

$$\|\chi_{\{s(f)>6y\}}\|_{p(\cdot)} \quad \le \quad c(\|\chi_{\{s(g)>3y\}}\|_{p(\cdot)/p^-} + \|\chi_{\{s(h)>3y\}}\|_{p(\cdot)/p^-})^{1/p^-}. \tag{3.7}$$

Since $s(a^k) \le 3\cdot 2^k$, thus we get

$$s(g) \le \sum_{k=-\infty}^{j-1} s(a^k) \le 3\sum_{k=-\infty}^{j-1} 2^k \le 3\cdot 2^j \tag{3.8}$$

and so $\{s(g) > 3y\} \subset \{s(g) > 3\cdot 2^j\} = \emptyset$.

The inequality $s(h) \le \sum\limits_{k=j}^{\infty} s(a^k)$ implies that $\{s(h) \neq 0\} \subset \cup_{k=j}^{\infty}\{\nu_k \neq \infty\}$. Consequently,

$$\begin{aligned}
\|\chi_{\{s(f)>6y\}}\|_{p(\cdot)} &\le c(\|\chi_{\{s(g)>3y\}}\|_{p(\cdot)/p^-} + \|\chi_{\{s(h)>3y\}}\|_{p(\cdot)/p^-})^{1/p^-} \\
&\le c\|\chi_{\{s(h)>0\}}\|_{p(\cdot)/p^-}^{p^-} \le c\sum_{k=j}^{\infty}\|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)} \\
&\le c\sum_{k=j}^{\infty} D2^{-k} = 2^{-j}D \le cD/y,
\end{aligned} \tag{3.9}$$

which implies

$$\|f\|_{wH_{p(\cdot)}^s} = \sup_{y>0} 6y\|\chi_{\{s(f)>6y\}}\|_{p(\cdot)} \le cD = c\sup_{k\in\mathbf{Z}} 2^k\|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)}. \tag{3.10}$$

Thus we complete the proof.

Theorem 3.3 Suppose that p is log-Hölder continuous, $1/2 < p^- \le p^+ \le 1$ and suppose that sublinear T is bounded from L_2 to L_2. If

$$P(Ta > 0) \le cP(I) \tag{3.11}$$

for all weak atom a supported on the interval I, then

$$\|Tf\|_{wL_{p(\cdot)}} \le c\|f\|_{wH_{p(\cdot)}^s} \quad (f \in wH_{p(\cdot)}^s). \tag{3.12}$$

Proof We may suppose $\|f\|_{wH_{p(\cdot)}^s} = 1$. Taking the atomic decomposition and the martingales g and h given in the proof of Theorem 3.2 we get that for any given $y > 0$,

$$Tf \le Tg + Th \quad \text{and} \quad \chi_{\{Tf>2y\}} \le \chi_{\{Tg>y\}} + \chi_{\{Th>y\}}.$$

(I) If $0 < y \le 1$, we choose integer j such that $2^j \le y < 2^{j+1}$. We know that

$$\begin{aligned}\|g\|_2 &\le \sum_{k=-\infty}^{j-1}\|a^k\|_2 \le \sum_{k=-\infty}^{j-1}\|M(a^k)\|_2 = \sum_{k=-\infty}^{j-1}\|M(a^k)\chi_{\{\nu_k\neq\infty\}}\|_2 \\ &\le \sum_{k=-\infty}^{j-1} 2^k P(\nu_k\neq\infty)^{1/2}. \end{aligned}\tag{3.13}$$

Since

$$\|\chi_B\|_{p(\cdot)} \approx P(B)^{1/p_B^+} \approx P(B)^{1/p_B^-} \approx P(B)^{1/p(\omega)}, \omega\in B$$

for $B = \{\nu_k \neq \infty\}$, we have

$$\begin{aligned}\|g\|_2 &\le \sum_{k=-\infty}^{j-1} 2^k P(B)^{1/2} = \sum_{k=-\infty}^{j-1} 2^k (P(\nu_k\neq\infty)^{1/p_B^+})^{p_B^+/2} \\ &\le c\sum_{k=-\infty}^{j-1} 2^k(\|\chi_B\|_{p(\cdot)})^{p_B^+/2} \le c\sum_{k=-\infty}^{j-1} 2^{k(1-p_B^+/2)}\|f\|_{wH_{p(\cdot)}}^{p_B^+/2} \\ &\le cy^{(1-p_B^+/2)}\|f\|_{wH_{p(\cdot)}}^{p_B^+/2} \le cy^{1/2}. \end{aligned}\tag{3.14}$$

Define $B_1 = \{Tg > y\}$, thus we have

$$\begin{aligned}\|\chi_{\{Tg>y\}}\|_{p(\cdot)} &\le cP(Tg>y)^{1/p_{B_1}^-} \le c\big(y^{-1}E(Tg)\big)^{1/p_{B_1}^-} \le c\big(y^{-1}\|Tg\|_2\big)^{1/p_{B_1}^-} \\ &\le c\big(y^{-1}\|g\|_2\big)^{1/p_{B_1}^-} \le c\big(y^{-1/2}\big)^{1/p_{B_1}^-} \le cy^{-1}. \end{aligned}$$

(II) If $y > 1$. Thus

$$\begin{aligned}\|g\|_2 &\le \sum_{k=-\infty}^{j-1} 2^k P(B)^{1/2} = \sum_{k=-\infty}^{j-1} 2^k (P(\nu_k\neq\infty)^{1/p_B^-})^{p_B^-/2} \\ &\le c\sum_{k=-\infty}^{j-1} 2^k(\|\chi_B\|_{p(\cdot)})^{p_B^-/2} \le c\sum_{k=-\infty}^{j-1} 2^{k(1-p_B^-/2)}\|f\|_{wH_{p(\cdot)}}^{p_B^-/2} \\ &\le cy^{(1-p_B^-/2)}\|f\|_{wH_{p(\cdot)}}^{p_B^-/2} \le cy^{3/4}. \end{aligned}\tag{3.15}$$

We also have

$$\begin{aligned}\|\chi_{\{Tg>y\}}\|_{p(\cdot)} &\le cP(Tg>y)^{1/p_{B_1}^-} \le c\big(y^{-2}E(Tg)^2\big)^{1/p_{B_1}^-} \le c\big(y^{-2}\|g\|_2^2\big)^{1/p_{B_1}^-} \\ &\le c\big(y^{-2}y^{3/2}\big)^{1/p_{B_1}^-} \le c\big(y^{-1/2}\big)^{1/p_{B_1}^-} \le cy^{-1}. \end{aligned}$$

Combining (3.15) and (3.17) we get

$$\|Tg\|_{wL_{p(\cdot)}} = \sup_{y>0} y\|\chi_{\{Tg>y\}}\|_{p(\cdot)} \le c\|f\|_{wH^s_{p(\cdot)}}. \tag{3.16}$$

On the other hand, let $B_2=\{Ta^k>0\}$, I_k is support of a^k, we have

$$\begin{aligned}
\|\chi_{\{Th>y\}}\|_{p(\cdot)} &\le \|\chi_{\{Th>0\}}\|_{p(\cdot)}=\|\chi_{\{Th>0\}}\|_{p(\cdot)/p^-}^{p^-}\\
&\le c(\sum_{k=j}^{\infty}\|\chi_{\{Ta^k>0\}}\|_{p(\cdot)/p^-})^{p^-}\le c\sum_{k=j}^{\infty}\|\chi_{\{Ta^k>0\}}\|_{p(\cdot)}\\
&\le c\sum_{k=j}^{\infty}P(Ta^k>0)^{1/p^-_{B_2}}\le c\sum_{k=j}^{\infty}P(I_k)^{1/p^-_{B_2}}\le c\sum_{k=j}^{\infty}(P(I_k)^{1/p_I^+})^{p_I^+/p^-_{B_2}}\\
&\le c\sum_{k=j}^{\infty}(P(I_k)^{1/p_I^+})^2\le c\sum_{k=j}^{\infty}P(I_k)^{1/p_I^+}\le c\sum_{k=j}^{\infty}\|\chi_{I_k}\|_{p(\cdot)}\\
&\le \sum_{k=j}^{\infty}\|\chi_{\{\nu_k\neq\infty\}}\|_{p(\cdot)}\le\sum_{k=j}^{\infty}D2^{-k}=2^{-j}D\\
&\le cD/y,
\end{aligned}$$

which implies

$$\|Tg\|_{wL_{p(\cdot)}}\le cD\le c\|f\|_{wH^s_{p(\cdot)}}. \tag{3.17}$$

By (3.18) and (3.19), we have

$$\|Tf\|_{wL_{p(\cdot)}}\le c(\|Tg\|_{wL_{p(\cdot)}}+\|Th\|_{wL_{p(\cdot)}})\le c\|f\|_{wH^s_{p(\cdot)}}.$$

Thus we complete the proof of Theorem 3.3.

4 Boundedness of Cesàro Operator

First we introduce the Walsh system. Every point $x\in[0,1)$ can be written in the following way

$$x=\sum_{k=0}^{\infty}\frac{x_k}{2^{k+1}},0\le x_k<2,x_k\in\mathbf{N}.$$

In case there are two different forms, we choose the one for which $\lim\limits_{k\to\infty}x_k=0$.

For $x,y\in[0,1)$ we define$x\oplus y=\sum\limits_{k=0}^{\infty}\frac{|x_k-y_k|}{2^{k+1}}:=d(x,y)$, which is also called dyadic distance.

The functions $r_n(x):=\exp(\pi x_n\sqrt{-1})$ $(n\in\mathbf{N})$ are called Rademacher functions.

The product system generated by these functions is the Walsh system: $\omega_n(x):=\prod\limits_{k=0}^{\infty}r_k(x)^{x_R}$, where $n=\sum\limits_{k=0}^{\infty}n_k2^k,0\le n_k<2$ and $n_k\in\mathbf{N}$.

If $f\in L_1[0,1)$, then the number $\hat{f}(n):=E(f\omega_n)$ is said to be the n-th Walsh-Fourier coefficient of f.

Denote by s_nf the n-th partial sum of the Walsh-Fourier series of a martingale f, namely,

$$s_nf:=\sum_{k=0}^{n-1}\hat{f}(k)\omega_k.$$

Recall that the Walsh-Dirichlet kernels $D_n := \sum_{k=0}^{n-1} \omega_k$ satisfy

$$D_{2^n}(x) = \begin{cases} 0 & \text{if } x \in [2^{-n}, 1) \ , \\ 2^n & \text{if } x \in [0, 2^{-n}). \end{cases} \tag{4.1}$$

Moreover, for any measurable function f, the sequence $\{f * D_{2^n} = s_{2^n} f = f_n\}$ is a martingale sequence.

The Walsh-Fejér kernels are defined with $K_n := \frac{1}{n} \sum_{k=1}^{n} D_k$ and can be estimated by

$$|K_n(x)| \leq \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} (D_{2^i}(x) + D_{2^i}(x \oplus 2^{-j-1})), \tag{4.2}$$

where $x \in [0,1), n, N \in \mathbf{N}, 2^{N-1} \leq n < 2^N$ (see [9]).

Moreover,

$$K_{2^n}(x) = \frac{1}{2}\left(2^{-n} D_{2^n}(x) + \sum_{j=0}^{n-1} 2^{j-n} D_{2^n}(x \oplus 2^{-j-1})\right).$$

For $n \in \mathbf{N}$ and a martingale f, the Cesàro mean of order n of the Walsh-Fourier series of f is given by $\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} s_n f$.

It is simple to show that in case $f \in L_1[0,1)$ we have

$$\sigma_n f(x) = f * K_n = \int f(t) K_n(x \oplus t) dt.$$

Define the maximal operator $\sigma^* f = \sup_n |\sigma_n f|$.

Definition 4.1 A pair (a, B) of measurable function a and $B \in A(F_n)$ is called a $p(\cdot)$-atom if (1) $E_n(a) = 0$, (2) $\|M(a)\|_\infty \leq \|\chi_B\|_{p(\cdot)}^{-1}$, (3) $\{a \neq 0\} \subset B$.

Lemma 4.2 (see [10]) Suppose that p is log-Hölder continuous and $0 < p^- < p^+ \leq 1$. For any $f = (f_n) \in H_{p(\cdot)}$, there exist $(a^B, B)_{B \in A}$ of $p(\cdot)$-atoms and $(u_B)_{B \in A}$ of nonnegative real numbers such that

$$f_n = \sum_{B \in A} u_B E_n a^B \text{ a.e.,} \quad \text{and} \quad \inf \left\| \sum_{B \in A} \Big(\frac{u_B \chi_B}{\|\chi_B\|_{p(\cdot)}}\Big) \right\|_{p(\cdot)} \leq c \|f\|_{H_{p(\cdot)}}.$$

Lemma 4.3 (see [10]) Suppose that the operator T is sublinear and for each $p_0 < p(\cdot) \leq 1$, there exists a constant $c > 0$ such that

$$\int_{[0,1) \setminus B} |Ta|^{p(\cdot)} d\mu \leq c \tag{4.3}$$

for every $p(\cdot)$-atom (a, B). If T is bounded from L_∞ to L_∞, then $\|Tf\|_{p(\cdot)} \leq c \|f\|_{H_{p(\cdot)}}$ $(f \in H_{p(\cdot)})$

Theorem 4.4 Suppose that p is log–Hölder continuous and $1/2 < p^- \leq p^+ \leq 1$. Then for any $f \in H_p(\cdot)$, we have $\|\sigma^* f\|_{p(\cdot)} \leq c\|f\|_{H_{p(\cdot)}}$.

Proof By Lemma 4.3, Theorem 4.4 will be complete if we show that the operator σ^* satisfies (4.3) and is bounded from L_∞ to L_∞.

Obviously,

$$\begin{aligned} |\sigma_n f(x)| &= |\int_0^1 f(t)K_n(x\oplus t)dt| \leq \int_0^1 |f(t)(K_n(x\oplus t))|dt \\ &\leq \|f\|_\infty \int_0^1 |K_n(x\oplus t)|dt. \end{aligned} \tag{4.4}$$

Since $\|D_{2^n}\|_1 = 1$ and (4.2), we can show that $\|K_n\|_1 \leq c$ for all $n \in \mathbf{N}$, which verifies that σ^* is bounded on L_∞.

Let $a \neq 1$ be an arbitrary $p(\cdot)$-atom with support B and $P(B) = 2^{-\tau}$. Without loss of generality, we may suppose that $B = [0, 2^{-\tau})$.

For $k < 2^\tau$, ω_k is constant on B and so $\sigma_n a(x) = \int_0^1 a(t)d\mu = 0$. Therefore, we may suppose that $n > 2^\tau$.

If $j \geq \tau$ and $x \notin B$, then $x \oplus 2^{-j-1} \notin B$. Consequently, for $x \notin B$ and $i \geq j \geq \tau$ we have

$$\int_B |a(t)| D_{2^i}(x\oplus t)dt = \int_B |a(t)| D_{2^i}(x\oplus t\oplus 2^{-j-1})dt = 0. \tag{4.5}$$

Moreover for $2^{N-1} \leq n < 2^N$ and $n > 2^\tau$ (which implies $N-1 \geq \tau$),

$$|\sigma_n a(x)| \leq \int_B |a(t)K_n(x\oplus t)|dt \leq |a| * |K_n|(x) \tag{4.6}$$

$$\leq \sum_{j=0}^{\tau-1} 2^{j-N} \sum_{i=j}^{N-1} \int_0^1 |a(t)|(D_{2^i}(x\oplus t) + D_{2^i}(x\oplus t\oplus 2^{-j-1}))dt. \tag{4.7}$$

Since $x \in [0,1)\backslash B$, we have

$$\begin{aligned} &\int_B D_{2^i}(x\oplus t\oplus 2^{-j-1})dt = 2^{i-\tau}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x) \quad \text{if} \quad j \leq i \leq \tau-1, \\ &\int_B D_{2^i}(x\oplus t)dt = 2^{i-\tau}\chi_{[2^{-\tau},2^{-i})}(x) \quad \text{if} \quad i \in \mathbf{N}, \\ &\int_B D_{2^i}(x\oplus t\oplus 2^{-j-1})dt = \chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-\tau})}(x) \quad \text{if} \quad i \geq \tau. \end{aligned}$$

By the definition of an atom, for $n \geq 2^\tau, 2^{N-1} \leq n < 2^N$, we have

$$\begin{aligned} \sup |a| * |K_n|(x) &\leq \sum_{j=0}^{\tau-1} 2^{j-N} \sum_{i=j}^{N-1} \int_B |a(t)|(D_{2^i}(x\oplus t) + D_{2^i}(x\oplus t\oplus 2^{-j-1}))dt \\ &\leq \|a\|_\infty \sum_{j=0}^{\tau-1} 2^{j-\tau} \sum_{i=j}^{\tau-1} (2^{i-\tau}\chi_{[2^{-\tau},2^{-i})}(x) + 2^{i-\tau}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x)) \\ &\quad + \|a\|_\infty \sum_{j=0}^{\tau-1} 2^{j-N} \sum_{i=\tau}^{N-1} 2^{i-\tau}\chi_{[2^{-\tau},2^{-i})}(x) + \chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x). \end{aligned}$$

To verify (4.3) we have to investigate the integral of $(\sup_n |a|*|K_n|(x))^{p(\cdot)}$ over $[0,1)\backslash B$. Integrating over $[0,1)\backslash B$, we obtain

$$\begin{aligned}
&\int_{[0,1)\backslash B}(\sup_{n\geq 2^\tau}|a|*K_n(x))^{p(\cdot)}dx\\
\leq\ & c\int_{[0,1)\backslash B}(\|a\|_\infty\sum_{j=0}^{\tau-1}2^{j-\tau}\sum_{i=j}^{\tau-1}(2^{i-\tau}\chi_{[2^{-\tau},2^{-i})}(x)+2^{i-\tau}\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x)))^{p(\cdot)}dx\\
&+c\int_{[0,1)\backslash B}(\|a\|_\infty\sum_{j=0}^{\tau-1}2^{j-N}\sum_{i=\tau}^{N-1}2^{i-\tau}\chi_{[2^{-\tau},2^{-i})}(x)+\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x)))^{p(\cdot)}dx\\
\leq\ & c(1\vee\|a\|_\infty)^{p^+}(\sum_{j=0}^{\tau-1}2^{(j-\tau)p^-}\sum_{i=j}^{\tau-1}2^{(i-\tau)p^-}\left(\|\chi_{[2^{-\tau},2^{-i})}(x)\|_{p(\cdot)}+\|\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x)\|_{p(\cdot)}\right)\\
&+c(1\vee\|a\|_\infty)^{p^+}\sum_{j=0}^{\tau-1}2^{(j-N)p^-}\sum_{i=\tau}^{N-1}\left(2^{(i-\tau)p^+}\|\chi_{[2^{-\tau},2^{-i})}(x)\|_{p(\cdot)}+\|\chi_{[2^{-j-1},2^{-j-1}\oplus 2^{-i})}(x)\|_{p(\cdot)}\right)\\
\leq\ & c\|\chi_B\|_{p(\cdot)}^{-p^+}(\sum_{j=0}^{\tau-1}2^{(j-\tau)p^-}\sum_{i=j}^{\tau-1}2^{(i-\tau)p^-}2^{-i}+cP(B)^{-p^+}\sum_{j=0}^{\tau-1}2^{(j-N)p^-}\sum_{i=\tau}^{N-1}(2^{(i-\tau)p^+}2^{-i}+2^{-i})\\
\leq\ & c\|\chi_B\|_{p(\cdot)}^{-p^+}(\sum_{j=0}^{\tau-1}2^{j(2p^--1)}2^{-2\tau p^-}+cP(B)^{-p^+}\sum_{j=0}^{\tau-1}2^{(j-N)p^-}\sum_{i=\tau}^{N-1}(2^{(i-\tau)p^+}2^{-i}+2^{-i})\\
\leq\ & c2^{\tau(p^+-2p^-)}\sum_{j=0}^{\tau}2^{j(2p^--1)}+c\sum_{j=0}^{\tau-1}2^{(j-N)p^-}\sum_{i=\tau}^{N-1}2^{i(p^+-1)}+c2^{\tau p^+}\sum_{j=0}^{\tau-1}2^{(j-N)p^-}\sum_{i=\tau}^{N-1}2^{-i}\\
\leq\ & c_1+c_2+c_3=c.
\end{aligned}$$

By Lemma 4.3 , the proof is completed.

Theorem 4.5 Suppose that p is log-Hölder continuous and $1<p^-\leq p^+<\infty$. Then for any $f\in L_p(\cdot)$, we have $\|\sigma^*f\|_{p(\cdot)}\leq c\|f\|_{p(\cdot)}$.

Proof We assume $\|f\|_{p(\cdot)}=1/2$. If else, we let f replaced by $\frac{f}{2\|f\|_{p(\cdot)}}$. Since

$$|K_n(x)|\leq\sum_{j=0}^{N-1}2^{j-N}\sum_{i=j}^{N-1}(D_{2^i}(x)+D_{2^i}(x\oplus 2^{-j-1})),$$

where $x\in[0,1),n,N\in\mathbf{N},2^{N-1}\leq n<2^N$. Thus we have

$$\begin{aligned}
|\sigma_nf(x)| &\leq |f|*|K_n|(x)\\
&\leq \sum_{j=0}^{N-1}2^{j-N}\sum_{i=j}^{N-1}(|f|*D_{2^i}(x)+|f|*D_{2^i}(x\oplus 2^{-j-1}))\\
&\leq \sum_{j=0}^{N-1}2^{j-N}\sum_{i=j}^{N-1}(|f_n(x)|+|f_n(x\oplus 2^{-j-1})|). \qquad (4.8)
\end{aligned}$$

Thus by Doob's inequality of variable exponents martingale spaces , we have

$$\begin{aligned}\|\sigma^* f\|_{p(\cdot)} = \|\sup_{n\geq 1}|\sigma_n f(x)|\|_{p(\cdot)} &\leq \sup_{N\geq 1}\sum_{j=0}^{N-1}2^{(j-N)}\sum_{i=j}^{N-1}(\|\sup_{n\geq 1}|f_n|\|_{p(\cdot)}+\|\sup_{n\geq 1}|f_n|\|_{p(\cdot)})\\ &\leq c\sup_{N\geq 1}\sum_{j=0}^{N-1}2^{(j-N)}\sum_{i=j}^{N-1}\|f\|_{p(\cdot)}\leq c\sup_{N\geq 1}\sum_{j=0}^{N-1}2^{(j-N)}(N-j)\\ &\leq \sum_{k=0}^{\infty}2^{-k}k\leq c\|f\|_{p(\cdot)}.\end{aligned}$$

Thus we complete the proof.

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弱型和强型二进制变指数鞅空间

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摘 要: 本文研究了二进制变指数强型和弱型鞅空间的原子分解理论. 利用原子分解的方法, 给出次线性算子T 是$wH^s_{p(\cdot)}$ 到$wL_{p(\cdot)}$ 有界; Cesàro 算子是$H_{p(\cdot)}$ 到$L_{p(\cdot)}$ 有界以及是$L_{p(\cdot)}$ 到$L_{p(\cdot)}$ 有界. 上述结论推广了常指数情况下算子有界性的结果.

关键词: 原子分解; 变指数; Cesàro 算子

MR(2010)**主题分类号**: 60G46 **中图分类号**: O211.5