QUASI SURE LOCAL CHUNG’S FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR INCREMENTS OF A BROWNIAN MOTION

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Abstract: In this paper, we obtain the quasi sure local Chung’s functional law of the iterated logarithm for increments of a Brownian motion. As an application, a quasi sure Chung’s type functional modulus of continuity for a Brownian motion is also derived.

Keywords: Brownian motion; increment; local Chung’s law of the iterated logarithm; $(r,p)$-capacity

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1 Introduction and Main Result

Let $(B,H,\mu)$ be an abstract Wiener space. The capacity is a set function on $B$ with the property that it sometimes takes positive values even for $\mu$-null sets, while a set of capacity zero has always $\mu$-measure zero. As we know, capacity is much finer than probability. An important difference between the capacity and probability is that the second Borel-Cantelli’s lemma does not hold with respect to capacity $C_{r,p}$ while it holds with respect to probability. Therefore, an interesting problem is to find out what property holds not only almost sure but also quasi sure. In this paper, we discuss this topic.

Many basic properties of Wiener processes were studied by authors (see [1–6]), such as the functional law of iterated logarithm, the functional modulus of continuity and large increments hold not only for $\mu$-a.s. but also for the sense of $C_{r,p}$-a.s.

In recent paper [2], Gao and Liu established local functional Chung’s law for increments of Brownian motion. In the present paper, we discuss similar results, but the probability is

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replaced by \((r,p)\)-capacity. The exact approximation rate for the modulus of continuity of Brownian motion can be viewed as a special case of our results.

We use standard notation and concepts on the abstract Wiener space \((B, H, \mu)\), including the \(H\)-derivative \(D\), its adjoint \(D^*\) and the Ornstein-Uhlenbeck operator \(\mathcal{L} = -D^*D\). Let \(D^{r,p}, r > 0, 1 \leq p < \infty\) be Sobolev space of Wiener functionals, i.e.,

\[
D^{r,p} = (1 - \mathcal{L})^{-\frac{r}{2}} L^p, \quad \|F\|_{r,p} = \|(1 - \mathcal{L})^\frac{r}{2} F\|_{L^p}, \quad F \in L^p,
\]

where \(L^p\) denotes \(L^p\)-space of real-valued functions on \((B, \mu)\). For \(r > 0, p > 1\), \((r,p)\)-capacity is defined by \(C_{r,p}(O) = \inf\{\|F\|_{r,p}^p, F \geq 1, \mu\text{-a.s. on } O\}\), for open set \(O \subset W\), and for any set \(A \subset W\), \(C_{r,p}(A) = \inf\{C_{r,p}(O) : A \subset O \subset W, O \text{ is open}\}\).

Let us consider classical Wiener space \((W, H, \mu)\) as follows

\[
W = \{w \in C([0, \infty), \mathbb{R}^d); w(0) = 0, \lim_{t \to \infty} \frac{|w(t)|}{t} = 0\},
\]

\[
H = \{h \in W; \quad h(t) = \int_0^t \dot{h}(s) ds, \dot{h} \in L^2([0, \infty), \mathbb{R}^d)\},
\]

\[
\mu \quad \text{is Wiener measure on } W;
\]

\[
\|w\|_W = \sup_{t \geq 0} \frac{|w(t)|}{1 + t}, \quad w \in W; \quad \|h\|^2_H = \int_0^\infty (\dot{h}(t), \dot{h}(t)) dt.
\]

Let \(C^d\) denote the space of continuous functions from \([0, 1]\) to \(\mathbb{R}^d\) endowed with usual supnorm \(\|f\| := \sup_{0 \leq t \leq 1} |f(t)|\) and denote by \(C_0^d := \{f \in C^d; f(0) = 0\}\), by \(\mathcal{H}^d := \{f \in C_0^d; f(t) = \int_0^t \dot{f}(s) ds, \|f\|_{\mathcal{H}^d}^2 := \int_0^1 (\dot{f}(t))^2 dt < \infty\}\), \(K := \{f \in \mathcal{H}^d; 2I(f) \leq 1\}\), where

\[
I(f) = \begin{cases} \frac{1}{2} \int_0^1 (\dot{f}(t))^2 dt, & \text{if } f \text{ is absolutely continuous;} \\ +\infty & \text{otherwise.}
\end{cases}
\]

Throughout this paper, let \(a_u, b_u\) be two non-decreasing continuous functions from \((0, 1)\) to \((0, e^{-1})\) satisfying

(i) \(a_u \leq b_u\) for any \(u \in (0, 1)\) and \(\lim_{u \to 0} a_u = 0\),

(ii) \(\frac{b_u}{a_u}\) is non-increasing.

Let \(w \in W\), for \(u \in (0, 1), 0 \leq t \leq 1\), \(\Delta(t, u)\) denotes the following path

\[
\Delta(t, u)(s) = w(b_u t + a_u s) - w(b_u t), \quad s \in [0, 1].
\]

Set

\[
\ell_u = \log \frac{b_u \log b_u^{-1}}{a_u}, \quad \beta_u = (2a_u \ell_u)^{-1/2}, \quad u \in (0, 1)
\]

and

\[
K = \{\varphi \in C_0[0, 1] : 2I(\varphi) \leq 1\}.
\]

The following theorems are results of this paper.
Theorem 1.1 If conditions (i) and (ii) hold, then for any \( f \in K \) with \( 2I(f) < 1 \), we have
\[
\liminf_{u \to 0} \ell_u \inf_{t \in [0, 1 - \frac{u}{M + 1}]} \| \beta_u \Delta(t, u) - f \| = b(f), \ C_{r,p} \text{- q.s.,}
\]
where \( b(f) = \left( \frac{c_d}{2} \right)^{1/2} / \left( 1 - 2I(f) \right)^{-1/2} \), \( c_d \) is a positive constant.

Under the additional assumption (iii) \( \lim_{u \to 0} \frac{\log(b_u / a_u)}{\log \log b_u} = \infty \), we get

Theorem 1.2 If conditions (i), (ii) and (iii) hold, then for any \( f \in K \) with \( 2I(f) < 1 \), we have
\[
\liminf_{u \to 0} \ell_u \inf_{t \in [0, 1 - \frac{u}{M + 1}]} \| \beta_u \Delta(t, u) - f \| = b(f), \ C_{r,p} \text{- q.s.}
\]

Corollary 1.1 Let \( M_{t,h}(x) = \frac{u(t+h,h) - u(t)}{\sqrt{2h \log(2e - t - h)}} \), \( 0 \leq x \leq 1, 0 \leq t \leq 1 - h \). For any \( f \in K \) with \( 2I(f) < 1 \), we have
\[
\lim_{h \to 0} \frac{1}{h} \inf_{t \in [0, e^{-2h} - h]} \| M_{t,h}(\cdot) - f(\cdot) \| = b(f), \ C_{r,p} \text{- q.s.} \quad (1.1)
\]

2 Proof of Theorem 1.1

Proof of Theorem 1.1 is completed by the following lemmas.

Lemma 2.1 (see [3], Lemma 2.2) Let \( 1 \leq k \in Z, q_1, q_2, \in (1, \infty) \) be given so that
\[ \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}. \]
For any \( f \in K \), put
\[ F^{(i)}_\varepsilon(w) = \| \varepsilon \left( w(t_i + h_i) - w(t_i) \right) - f \|, \ i = 1, 2, \ldots, n, \ \varepsilon > 0, \]
where \( 0 \leq t_i < \infty, h_i > 0, i = 1, 2, \ldots, n. \) Then there exists a constant \( c = c(k, p, q_1, f, d) > 0 \), for any \( \delta \in (0, 1], \varepsilon \in (0, 1] \), we have
\[ C_{k,p} \left( \bigcap_{i=1}^n \{ z : a_i < F^{(i)}_\varepsilon(z) < b_i \} \right)^{1/p} \leq c_0 \left( 2 - k \right)^{1/q_2} \mu \left( \bigcap_{i=1}^n \{ z : a_i - \delta < F^{(i)}_\varepsilon(z) < b_i + \delta \} \right)^{1/q_2}.
\]

Lemma 2.2 (see [3], Lemma 2.4) There exists a positive number \( c_d \) such that for any \( h > 0, \tau > 0, f \in K \), we have
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \log C_{r,p} \left( \left\| \frac{w(t + h,h) - w(t)}{\sqrt{h}} - \frac{f}{\varepsilon} \right\| \leq \varepsilon \tau \right)
\]
\[
= \lim_{\varepsilon \to 0} \varepsilon^2 \mu \left( \left\| \frac{w(t + h,h) - w(t)}{\sqrt{h}} - \frac{f}{\varepsilon} \right\| \leq \varepsilon \tau \right) = -c_d \tau^{-2} - I(f).
\]

Lemma 2.3 For any \( f \in K \) with \( 2I(f) < 1 \), we have
\[
\liminf_{u \to 0} \ell_u \inf_{t \in [0, 1 - \frac{u}{M + 1}]} \| \beta_u \Delta(t, u) - f \| \geq b(f). \ C_{r,p} \text{- q.s.}
\]

Proof Case I \( \limsup_{u \to 0} \frac{\log b_u}{\log \log b_u} < \infty \). If \( \limsup_{u \to 0} \frac{\log b_u}{\log \log b_u} < \infty \), then \( b_u \to 0 \) as \( u \to 0 \) and there exists a \( 0 < M < \infty \) such that \( \frac{b_u}{a_u} \leq \left( \log b_u^{-1} \right)^M \). Let \( l(u) = a_u \left( \log \frac{b_u \log b_u^{-1}}{a_u} \right)^{-3} \), and
take $u_n$ such that $a_{u_n} = (\exp(n(\log n)^2))^{-1}$. Set $k_n = \lceil \frac{b_{u_n}}{\eta(u_{n+1})} \rceil$, $t_i = i{l(u_{n+1})}$, $i = 0, 1, \ldots, k_n$. Then

$$\min_{0 \leq i \leq k_n} \| \beta_{u_{n+1}}(w(t_i + a_{u_{n+1} \cdot}) - w(t_i)) - f \|
\leq 2 \sup_{t \in [0, b_{u_n}]} \sup_{s \in [0, \eta(u_{n+1})]} |w(t + s) - w(t)| \beta_{u_{n+1}}
+ \inf_{t \in [0, b_{u_n} - a_{u_{n+1}}]} \| \beta_{u_{n+1}}(w(t + a_{u_{n+1} \cdot}) - w(t)) - f \|.
$$

(2.1)

For any $0 < \varepsilon < 1$, choose $\delta > 0$ such that $\eta = -2\delta + 2f(f) + \frac{1 - 2f(f)}{(1 - \varepsilon)^2} > 1$. By Lemma 2.2, for $n$ large enough, we have

$$C_{r,p}(\ell_{u_{n+1}} \min_{0 \leq i \leq k_n} \| \beta_{u_{n+1}}(w(t_i + a_{u_{n+1} \cdot}) - w(t_i)) - f \| \leq (1 - \varepsilon)b(f))$$

$$\leq \sum_{0 \leq i \leq k_n} C_{r,p}(\ell_{u_{n+1}} \| \beta_{u_{n+1}}(w(t_i + a_{u_{n+1} \cdot}) - w(t_i)) - f \| \leq (1 - \varepsilon)b(f))$$

$$= \sum_{0 \leq i \leq k_n} C_{r,p}(\| w(t_i + a_{u_{n+1} \cdot}) - w(t_i) \| - \sqrt{2\ell_{u_{n+1}}f} \| \leq \frac{2}{\sqrt{2\ell_{u_{n+1}}}}(1 - \varepsilon)b(f))$$

$$\leq (1 + k_n) \exp\{(2 \log \frac{b_{u_{n+1}} \log b_{u_{n+1}^{-1}}}{a_{u_{n+1}}}(- \frac{c_d}{4(1 - \varepsilon)^2(b(f))^2} - I(f) + \delta))
= (1 + k_n) \exp\{(- \log \frac{b_{u_{n+1}} \log b_{u_{n+1}^{-1}}}{a_{u_{n+1}}})(2f(f) - 2\delta + \frac{1 - 2f(f)}{(1 - \varepsilon)^2})
\leq \frac{b_{u_n} + l(u_{n+1})}{l(u_{n+1})} \left( \frac{a_{u_{n+1}}}{b_{u_{n+1}} \log b_{u_{n+1}^{-1}}} \right)^\eta,$$

by Borel-Cantelli’s lemma,

$$\liminf_{n \to \infty} \ell_{u_{n+1}} \min_{0 \leq i \leq k_n} \| \beta_{u_{n+1}}(w(t_i + a_{u_{n+1} \cdot}) - w(t_i)) - f \| \geq b(f), \ C_{r,p} - q.s.. \quad (2.2)$$

On the other hand, for any $\delta_0 > 0$,

$$C_{r,p}(2\ell_{u_{n+1}} \sup_{0 \leq i \leq b_{u_{n}} \sup_{0 \leq s \leq \ell(u_{n+1})}} |w(t + s) - w(t)| \beta_{u_{n+1}} \geq \delta_0)$$

$$= C_{r,p}(\sqrt{\frac{2\ell_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{0 \leq i \leq b_{u_{n}} \sup_{0 \leq s \leq \ell(u_{n+1})}} |w(t + s) - w(t)| \geq \delta_0)$$

$$= C_{r,p}(\sqrt{\frac{2\ell_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{0 \leq i \leq b_{u_{n+1}} \sup_{0 \leq s \leq 1}} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0)$$

$$\leq \sum_{i = 0}^{\frac{b_{u_{n+1}} - 1}{\eta(u_{n+1})}} C_{r,p}(\sqrt{\frac{2\ell_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{0 \leq i \leq \frac{b_{u_{n+1}}}{\eta(u_{n+1})} \sup_{0 \leq s \leq 1}} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0),$$

moreover

$$\mu(\sqrt{\frac{2\ell_{u_{n+1}}}{a_{u_{n+1}}}} \sup_{0 \leq i \leq \frac{b_{u_{n+1}}}{\eta(u_{n+1})} \sup_{0 \leq s \leq 1}} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0)$$
Remark that thus inf

\[ \inf_{t \leq t + 1} |w(t + s) - w(t)| \geq \delta_0 \]

\[ = \mu(\sqrt{\frac{2}{(n+1)^2}} \sup_{t \leq t + 1} \sup_{0 \leq s \leq 1} |w(t + s) - w(t)|) \geq \delta_0 \]

where \( A = \{ f \in W; \sup_{0 \leq t \leq 1} \| f(\frac{1}{2} + t) - f(\frac{1}{2}) \| \geq \delta_0 \}. \) For \( f \in A, \sup_{0 \leq t \leq 1} \int_{\frac{1}{2}}^{\frac{1}{2} + t} |f(s)|^2 ds \geq \delta_0^2, \)

thus \( \inf_{f \in A} \frac{1}{2} \| f \|^2 \geq \frac{\delta_0^2}{2}. \) By Theorem 1.1 in [5], we have

\[
C_{r,p}(\sqrt{\frac{2\ell n_{u_{n+1}}}{a_{n+1}}} \sup_{0 \leq t \leq 1} \sup_{0 \leq \ell \leq 1} \sup_{0 \leq s \leq 1} |w(l(u_{n+1})t + l(u_{n+1})s) - w(l(u_{n+1})t)| \geq \delta_0) \\
\leq (\frac{a_{n+1}}{b_{n+1}} \log \frac{b_{n+1}^a b_{n+1}^l}{a_{n+1}}) \frac{\delta_0^2}{2} \log \frac{b_{n+1}^a b_{n+1}^l}{a_{n+1}}.
\]

Taking into account \( \log \frac{b_{n+1}^a b_{n+1}^l}{a_{n+1}} \to \infty, \) as \( n \to \infty, \) thus

\[
\sum_{n} b_{n+1} + l(u_{n+1}) < \infty,
\]

by Borel-Cantelli’s lemma

\[
\limsup_{n \to \infty} \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq 1} |w(t + s) - w(t)| = 0, \quad \text{C}_{r,p} \text{- q.s. (2.3)}
\]

By (2.1)–(2.3), we get

\[
\lim_{n \to \infty} \inf_{l(u_{n+1})} \| \beta_{u_{n+1}}(w(t + a_{n+1}) - w(t)) - f \| \geq b(f), \quad \text{C}_{r,p} \text{- q.s. (2.4)}
\]

Remark that \( a_n \to 0, \) so for any small enough \( u, \) there is a unique \( n \) such that \( u \in (u_{n} + u_{n+1}]. \) Let \( \phi_{l,u}(s) = \beta_{u}(w(t + a_{n}s) - w(t)), s \in [0,1], t \in [0, b_n - a_n]. \) We define \( X(u) = \ell_u \inf_{(u_{n+1}, u_n]} \| \phi_{l,u}(\cdot) - f(\cdot) \|, \)

\[
X_n = \inf_{u_{n+1} \leq u \leq u_n} X(u) \text{ by the definition of infimum, for any } \varepsilon > 0, \text{ there exists } T_n \in (u_{n+1}, u_n] \text{ such that } X_n \geq X(T_n) - \varepsilon.
\]

For any \( r \in [0,1], \) let \( x = \frac{r a_{n+1}}{a_{n+1}}. \) Then we have, for \( 0 \leq x \leq 1. \)

\[
\inf_{t \in [0, b_n - a_{n+1}]} \| \beta_{u_{n+1}}(w(t + a_{n+1}) - w(t)) - f \|
\leq \inf_{t \in [0, b_n - a_{n+1}]} \| \phi_{l,u}(r) - f(r) \|
\leq \inf_{t \in [0, b_n - a_{n+1}]} \sup_{0 \leq x \leq 1} \| \phi_{l,u}(x) - f(x) \|
\leq \inf_{t \in [0, b_n - a_{n+1}]} \sup_{0 \leq x \leq 1} \| \beta_{u_{n+1}} \phi_{l,T_n}(x) - f(x) \|
\leq \beta_{u_{n+1}} \beta_{T_n} \| \phi_{l,T_n}(\cdot) - f(\cdot) \| + \| \beta_{u_{n+1}} - 1 \| \| f(\cdot) \| + \| f(\cdot) - f(\frac{a_{n+1}}{a_{n+1}}) \|
\leq \beta_{u_{n+1}} \beta_{T_n} \frac{b_n \log b_n}{a_n} \sum_{i=0}^{n-1} X(T_i) + \| \beta_{u_{n+1}} - 1 \| \| f(\cdot) \| + \| f(\cdot) - f(\frac{a_{n+1}}{a_{n+1}}) \|.
\]
Noting that
\[
1 \geq \log \frac{b_{n+1} \log b_{n}^{-1}}{a_{n+1}} = \frac{a_{n} \log b_{n} \log b_{n}^{-1}}{a_{n+1}} \frac{a_{n+1}}{a_{n}} \geq \frac{a_{n+1}}{a_{n}} \to 1, \tag{2.6}
\]
\[
\frac{\beta_{n+1}}{\beta_{n}} \leq \frac{\beta_{n+1}}{\beta_{n}} \leq \frac{a_{n} \log \log a_{n}}{a_{n+1}} \leq \exp \left( \frac{1}{(\log n)^3} \right) = 1 + \frac{1}{(\log n)^3} + o(\frac{1}{(\log n)^3}), \tag{2.7}
\]
\[
\|f(\frac{a}{a_{n}}) - f(\cdot)\| \leq \sqrt{n} \sqrt{1 - \frac{a_{n+1}}{a_{n}}} \leq \sqrt{2} \left( \frac{1}{(\log n)^3} + o(\frac{1}{(\log n)^3}) \right)^{1/2}, \tag{2.8}
\]
\[
\log \frac{b_{n} \log b_{n}^{-1}}{a_{n}} \leq (M + 1) \log \log a_{n}^{-1}. \tag{2.9}
\]
by (2.4), (2.5), (2.7)–(2.9), we get
\[
\liminf_{n \to \infty} X(T_{n}) \geq b(f), \quad C_{r,p} \cdot \text{q.s.}
\]
Since
\[
\liminf_{u \to 0} X(u) \geq \liminf_{n \to \infty} X_{n} \geq \liminf_{n \to \infty} X(T_{n}) \geq b(f) - \varepsilon,
\]
which ends the proof.

**Case II** \( \limsup_{n \to \infty} \frac{\log \frac{b_n}{a_n}}{\log \log a_n} = \infty \). If \( \limsup_{n \to \infty} \frac{\log \frac{b_n}{a_n}}{\log \log b_n} = \infty \), then we can choose a non-increasing sequence \( \{u_{n}; n \geq 1\} \) with \( \frac{b_{u_{n}}}{a_{u_{n}}} = n^{d}, d > 1 \). Let \( h(n) = \frac{\log \frac{b_{u_{n}}}{a_{u_{n}}}}{\log \log b_{u_{n}}} = \frac{\log n^{d}}{\log \log b_{u_{n}}} \), then \( b_{u_{n}}^{-1} = \exp \left( \frac{n^{d}}{\log b_{u_{n}}} \right) \) and \( h(n) \to \infty \) as \( n \to \infty \). Let \( l(u), k_{n} \) and \( t_{i}, i = 1, 2, ..., k_{p} \) be defined by Case I. Then for some constant \( C > 0 \), if \( d \) is chosen in a suitable way, then
\[
\sum_{n=1}^{\infty} (1 + \frac{b_{u_{n}}}{l(u_{n+1})}) \left( \frac{a_{u_{n}}}{\log b_{u_{n}}} \right)^{q} \leq C \sum_{n=1}^{\infty} n^{-d(q-1)}(\log n)^{3} < \infty.
\]
Furthermore,
\[
\frac{a_{u_{n}}}{a_{u_{n+1}}} = \frac{b_{u_{n}}(n+1)^d}{n^d b_{u_{n+1}}} = (n+1)^d \frac{1}{n^d} \exp \left( (n+1)^{d/h(n+1)} - n^{d/h(n)} \right) \leq (1 + \frac{1}{n})^d \exp \left( n^{d/h(n)} - 1 \right) = \left(1 + \frac{1}{n}\right)^d \left(1 + \frac{1}{n^{1-(d/h(n))}} + o\left(\frac{1}{n^{1-(d/h(n))}}\right)\right),
\]
which implies that
\[
\log \frac{b_{u_{n}} \log b_{u_{n}^{-1}}}{a_{u_{n}}}(1 - \frac{a_{u_{n+1}}}{a_{u_{n}}}) = d \left( \log n + \log n \frac{\log n}{h(n)} \right) \left(1 - \frac{a_{u_{n+1}}}{a_{u_{n}}} \right) \to 0, \quad n \to \infty. \tag{2.10}
\]
Similarly to the proof of Case I, the proof of Lemma 2.3 is completed.

**Lemma 2.4** For any \( f \in K \) with \( 2I(f) < 1 \), we have
\[
\liminf_{u \to 0} \inf_{t \in [0,1-\frac{a_{u}}{a_{u}}]} \|\beta_{u} \Delta(t,u) - f\| \leq b(f), \quad C_{r,p} \cdot \text{q.s.}
\]
Proof Set \( \rho := \lim_{u \to 0} \frac{a_u}{b_u} \). If \( \rho < 1 \) and \( b_u \to b \neq 0 \) as \( u \to 0 \), then \( \lim_{u \to 0} \frac{\log \frac{b_u}{b_u}}{\log \log b_u} = \infty \).

In this case, see Lemma 3.2. Therefore, we only consider the following two cases:

(I) \( \rho < 1 \) and \( b_u \to 0 \) as \( u \to 0 \),

(II) \( \rho = 1 \).

Case I \( \rho < 1 \) and \( b_u \to 0 \) as \( u \to 0 \) If \( \rho < 1 \) and \( b_u \to 0 \) as \( u \to 0 \), then we can choose \( \{u_k, k \geq 1\} \) such that \( b_{u_{k+1}} = b_{u_k} - a_{u_k}, k \geq 1 \). For any \( \varepsilon > 0 \), choose \( \delta > 0 \) such that

\[
\frac{1}{(1+\varepsilon)^2} + 2I(f) + 2\delta < 1.
\]

Set \( k = [r] + 1 \), by Lemma 2.1, we have

\[
C_{r,p}(\ell_n \| \beta_n \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f \| \geq b(f)(1+2\varepsilon))^{1/p}
\]

\[
\leq c\delta^k(\ell_n f \varepsilon)^{-2k^2-k} \mu^{\frac{\Delta(1 - \frac{a_{u_n}}{b_{u_n}})}{\sqrt{2a_{u_n} \ell_n}} - f \| \geq b(f)(1+\varepsilon))^{1/q_2}
\]

\[
\leq c\delta^k(\ell_n f \varepsilon)^{-2k^2-k} \prod_{n=m_0}^{l} \{1 - \mu(\| \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f \| < b(f)(1+\varepsilon))^{1/q_2},
\]

moreover, by small deviation,

\[
\mu(\| \frac{1}{\sqrt{2f_{u_n}}} \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f \| < b(f)(1+\varepsilon)) = \mu(\| \frac{1}{\sqrt{2f_{u_n}}} w - \frac{2b(f)(1+\varepsilon)}{\sqrt{2f_{u_n}}} \| < (\frac{a_{u_n}}{b_{u_n}})^{\frac{1-2I(f)}{1+2I(f)+2\delta}},
\]

thus

\[
C_{r,p}(\ell_n \| \beta_n \Delta(1 - \frac{a_{u_n}}{b_{u_n}}, u_n) - f \| \geq b(f)(1+2\varepsilon))^{1/p}
\]

\[
\leq c\delta^k(\ell_n f \varepsilon)^{-2k^2-k} \exp\left( -\frac{1}{q_2} \sum_{n=m_0}^{l} (\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}})\left(\frac{1-2I(f)}{1+2I(f)+2\delta}\right)\right).
\]

Noting that, there exists \( A = A(m_0) > 0 \) such that

\[
\sum_{n=m_0}^{l} (\frac{a_{u_n}}{b_{u_n} \log b_{u_n}^{-1}})^{\frac{1-2I(f)}{1+2I(f)+2\delta}} > A(\log b_{u_1}^{-1})^{\varepsilon},
\]

where \( \varepsilon = 1 - \frac{1-2I(f)}{1+2I(f)+2\delta} > 0 \).

We discuss as follows:

(a) If \( \limsup_{u \to 0} \frac{\log(b_{u}/a_u)}{\log \log b_u} < \infty \), then for some \( 0 < M < \infty \), we have \( \frac{b_u}{a_u} \leq (\log b_u^{-1})^M \). Take \( \theta > 2/\varepsilon', u_0 = e^{-(\log l_0)\theta} \), there exists \( l_0 \) large enough, we can prove that

\[
\log b_{u_1}^{-1} \geq (\log l)^\theta, \quad l \geq l_0,
\]
we get

\[(\log b_n^{-1})^c \geq (\log l)^2.\]

When \(l\) is large enough, for constant \(c' > 0\), we can also prove that \(c' l \geq \log b_n^{-1}\). Thus we have

\[
C_{r,p}(\bigcap_{n=m_0}^l (\mathcal{C}_n ||\mathcal{C}_u \Delta (1 - \frac{a_n}{b_{u_n}}), u_n) - f|| \geq b(f)(1 + 2\varepsilon)))^{1/p} \leq c_{0} L 2^{2k+1} \exp(-\frac{A}{q_2} (\log b_n^{-1})) \leq c_{0} L 2^{2k+1} \exp(-\frac{A}{q_2} (\log l)^2) \to 0 (l \to \infty),
\]

where \(c_0\) is a constant. We get

\[
C_{r,p}(\bigcup_{l=1}^\infty (\mathcal{C}_n ||\mathcal{C}_u \Delta (1 - \frac{a_n}{b_{u_n}}), u_n) - f|| \geq b(f)(1 + 2\varepsilon))) = 0,
\]

consequently

\[
\lim_{n \to \infty} \inf_{t \to 1} \ell_{t} ||\mathcal{C}_u \Delta (1 - \frac{a_n}{b_{u_n}}), u_n) - f|| \leq b(f), \quad C_{r,p} \cdot q.s.,
\]

we have

\[
\lim_{u \to 0} \inf_{t \to 1} \ell_{t} ||\mathcal{C}_u \Delta (1 - \frac{a_n}{b_{u_n}}) - f|| \leq b(f), \quad C_{r,p} \cdot q.s..
\]

(b) If \(\lim_{u \to 0} \frac{\log b_n}{\log \log b_n} = \infty\), then see Lemma 3.2.

Case II

Set \(\rho = 1\). By applying Lemma 2.1, similarly to the corresponding that of (3.3) in [2].

3 Proof of Theorem 1.2

Proof of Theorem 1.2 is completed by the following lemmas.

Lemma 3.1 If condition (iii) also holds, then there exists an non-increasing \(\{u_n, n \geq 1\}\) for any \(f \in K\) with \(2f(f) < 1\), we have

\[
\limsup_{n \to \infty} \ell_{u_n} \inf_{t \in [0, b_{u_n+1} - a_{u_n}]} ||\mathcal{C}_u \Delta (B(t + a_{u_n}) - B(t)) - \varphi|| \leq b(f), \quad C_{r,p} \cdot q.s..
\]

Proof Owing to \(\lim_{u \to 0} \frac{\log b_n}{\log \log b_n} = \infty\), there exists a subsequence \(\{u_n; n \geq 1\}\) such that \(\frac{b_{u_n}}{a_{u_n}} = n^\alpha\). Let

\[
t_i = ia_n, i = 0, 1, \cdots, k_n = \lfloor b_{n+1}^{-1}\rfloor - 1 \quad \text{and} \quad h(n) = \frac{\log b_{u_n}}{\log a_{u_n}} = \frac{d \log n}{\log \log b_{u_n}},
\]

then \(b_{n+1}^{-1} = \exp\left(\frac{n}{\log b_{n+1}}\right)\) and \(h(n) \to \infty\) as \(n \to \infty\). And for any small enough \(\alpha > 0\),

\[
\frac{(n+1)^{\alpha}}{\log b_{n+1}} \to \infty, \quad 1 \leq \frac{b_{u_n}}{a_{u_n}} = \exp\left(\frac{(n+1)^{\alpha}}{\log b_{n+1}}\right) \leq \exp\left(\frac{n^{\alpha}}{\log b_{n+1}}\right) \to 1 \quad \text{and} \quad \frac{a_{u_n}}{a_{u_n+1}} \to 1
\]
as \( n \to \infty \). Choose \( \delta > 0 \) such that \( \eta_0 := \frac{1-2I(f)}{(1+\varepsilon)} + 2I(f) + 2\delta < 1 \). Take \( k = [r] + 1 \), by Lemma 2.1 and small deviation, for \( n \) large enough, we have

\[
C_{r,p}(\ell_{u_n}) \inf_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| \beta_{u_n}(w(t + a_{u_n} \cdot) - w(t)) - f \| \geq \frac{b(f)(1 + 2\varepsilon)^{1/p}}{\ell_{u_n}}
\]

\[
\leq C_{r,p}(\min_{0 \leq i \leq k_n} \| \beta_{u_n}(w(t_i + a_{u_n} \cdot) - w(t_i)) - f \| \geq \frac{b(f)(1 + \varepsilon)}{\ell_{u_n}}^{1/p}
\]

\[
\leq c(1 + k_n)^k \left( \frac{\ell_{u_n}}{b(f)\varepsilon} \right)^{2k^2 + k} \mu \left( \min_{0 \leq i \leq k_n} \| \beta_{u_n}(w(t_i + a_{u_n} \cdot) - w(t_i)) - f \| \geq \frac{b(f)(1 + \varepsilon)}{\ell_{u_n}} \right)^{1/q_2}
\]

\[
= c(1 + k_n)^k \left( \frac{\ell_{u_n}}{b(f)\varepsilon} \right)^{2k^2 + k} \mu \left( \| \frac{1}{\sqrt{a_{u_n}}}(w(a_{u_n} \cdot) - \\sqrt{2} \ell_{u_n} f) \| \geq \frac{2b(f)(1 + \varepsilon)}{\ell_{u_n}} \right)^{1/k_n}
\]

\[
\leq c(1 + k_n)^k \left( \frac{\ell_{u_n}}{b(f)\varepsilon} \right)^{2k^2 + k} \left( 1 - \left( \frac{a_{u_n}}{b_{a_{u_n} \log b_{a_{u_n}}^{-1}}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} \right)^{1 + \delta}
\]

\[
\leq c_0 n^{kd}(\log n)^{2(2k^2 + k)} \exp \left\{ - \frac{1}{q_2} \left( \frac{a_{u_n}}{b_{a_{u_n} \log b_{a_{u_n}}^{-1}}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} \right\}
\]

\[
= c_0 n^{kd}(\log n)^{2(2k^2 + k)} \exp \left\{ - \frac{1}{q_2} \left( \frac{a_{u_n}}{b_{a_{u_n} \log b_{a_{u_n}}^{-1}}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} \left( \frac{a_{u_{n+1}}}{a_{u_n}} \right) \right\},
\]

where \( c_0 \) is a constant. If \( d \) is chosen in a suitable way, then

\[
\sum_n c_0 n^{kd}(\log n)^{2(2k^2 + k)} \exp \left\{ - \frac{p}{q_2} \left( \frac{a_{u_n}}{b_{a_{u_n} \log b_{a_{u_n}}^{-1}}} \right)^{\frac{1-2I(f)}{(1+\varepsilon)^2} + 2I(f) + 2\delta} \left( \frac{a_{u_{n+1}}}{a_{u_n}} \right) \right\} < \infty,
\]

by Borel-Cantelli’s lemma.

\[
\limsup_{n \to \infty} \ell_{u_n} \inf_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| \beta_{u_n}(w(t + a_{u_n} \cdot) - w(t)) - f \| \leq b(f)(1 + 2\varepsilon), \quad C_{r,p} \text{- q.s.}
\]

**Lemma 3.2.** If conditions (i), (ii) and (iii) hold, then for any \( f \in K \) with \( 2I(f) < 1 \), we have

\[
\limsup_{n \to 0} \ell_u \inf_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| \beta_{u_n}(w(t + a_{u_n} \cdot) - w(t)) - f \| \leq b(f), \quad C_{r,p} \text{- q.s.}
\]

**Proof.** Let \( \phi_{t,u}(s) = \beta_u(w(t + a_{u} \cdot) - w(t)) \), \( u_n \) is defined as in Lemma 3.1. Since \( \phi_{t,u}(s) = \frac{\beta_{u_n}}{\beta_{u_{n+1}}} \phi_{t,u_n}(\frac{a_{u_{n+1}}}{a_{u_n}}s) \), we have

\[
\inf_{t \in [a_{u_{n+1}}^{-1} - a_{u_n}]} \| \phi_{t,u}(\cdot) - f(\cdot) \| \leq \inf_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| \phi_{t,u_n}(\frac{a_{u_{n+1}}}{a_{u_n}} \cdot) - f(\frac{a_{u_{n+1}}}{a_{u_n}} \cdot) \|
\]

\[
+ \sup_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| \beta_{u_n} - 1 \| \| \phi_{t,u_n}(\frac{a_{u_{n+1}}}{a_{u_n}} \cdot) \| + \sup_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| f(\frac{a_{u_{n+1}}}{a_{u_n}} \cdot) - f(\cdot) \|
\]

\[
\leq \inf_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| \phi_{t,u_n}(\cdot) - f(\cdot) \| + \sup_{t \in [0, b_{a_{u_{n+1}}^{-1}} - a_{u_n}]} \| 1 - \frac{a_{u_{n+1}}}{a_{u_n}} \| \| \phi_{t,u_n}(\frac{a_{u_{n+1}}}{a_{u_n}} \cdot) \|
\]

\[
+ \| f(\frac{a_{u_{n+1}}}{a_{u_n}} \cdot) - f(\cdot) \|.
\]
Moreover

\[ \| f(\frac{a_u}{a_{u_n}}) - f(\cdot) \| \leq 2\| 1 - \frac{a_{u_{n+1}}}{u_n} \|^1/2, \quad (3.2) \]

\[ (\log b_{u_n} \log b_{u_n}^{-1})(1 - \frac{a_{u_{n+1}}}{a_{u_n}}) \leq d(\log n + \frac{\log n}{h(n)})(1 - \frac{a_{u_{n+1}}}{a_{u_n}}). \quad (3.3) \]

We can conclude Lemma 3.2 from (3.1)–(3.3) and Lemma 3.1.

References