Vol. 40 (2020)
No. 2

数 学 杂 志
J. of Math. (PRC)

ON COSET DECOMPOSITIONS OF THE COMPLEX REFLECTION GROUPS $G(M, P, R)$

XU Jing-lei[1,2], WANG Yan-jie[1,3], WANG Li[1]

(*1. School of Mathematics and Science, Shanghai Normal University, Shanghai 200234, China*)

(*2. The Affiliated High School of Shanghai University, Shanghai 200444, China*)

(*3. Changshu High School, Changshu 215516, China*)

Abstract: We study the decomposition of the imprimitive complex reflection group $G(m, p, r)$ into right coset, where m, p, r are positive integers, and p divides m. By use of the software GAP to compute some special cases when m, p, r are small integers, we deduce a set of complete right coset representatives of the parabolic subgroup $G(m, p, r-1)$ in the group $G(m, p, r)$ for general cases, which lays a foundation for further study the distinguished right coset representatives of $G(m, p, r-1)$ in $G(m, p, r)$.

Keywords: right coset representatives; imprimitive complex reflection groups

2010 MR Subject Classification: 20F05; 20F55

Document code: A **Article ID:** 0255-7797(2020)02-0149-06

1 Introduction

Let $\mathbb{N}$ (respectively, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$) be the set of all positive integers (respectively, integers, real numbers, complex numbers). Let V be a Hermitian space of dimension n. A reflection in V is a linear transformation of V of finite order with exactly $n-1$ eigenvalues equal to 1. A reflection group G on V is a finite group generated by reflections in V. A reflection group G is called a Coxeter group if there is a G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective, or G is called a complex group. A reflection group G on V is called imprimitive if V is a direct sum of nontrivial linear subspaces $V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ such that every element $w \in G$ is a permutation on the set $\{V_1, V_2, \cdots, V_t\}$.

For any $m, p, r \in \mathbb{N}$ with $p \mid m$ (read " p divides m "), let $G(m, p, r)$ be the group consisting of all $r \times r$ monomial matrices whose non-zero entries $a_1, a_2, \cdots, a_r$ are mth roots of unity with $\left(\prod_{i=1}^{r} a_i\right)^{m/p} = 1$, where a_i is in the i-th row of the monomial matrix. In [1], Shephard and Todd proved that any irreducible imprimitive reflection group is isomorphic to some $G(m, p, r)$. We see that $G(m, p, r)$ is a Coxeter group if either $m \leq 2$ or $(p, r) = (m, 2)$.

The imprimitive reflection group $G(m, p, r)$ can also be defined by a presentation (S, P), where S is a set of generators of $G(m, p, r)$, subject only to the relations in P. In the cases $p = 1$, $p = m$, and $1 < p < m$, we list their presentations as follows (see [2]).

* **Received date:** 2018-10-29 **Accepted date:** 2019-03-05

Biography: Xu Jinglei (1992–), female, born at Xinyang, Henan, postguaduate, major in reflection groups and its associated combinatorics.

Corresponding author: Wang Li

(1) When $p=1$, S contains r reflections s_0 and s_i for $i \in \{1,2,\cdots,r-1\}$, and P consists of the relations $s_0^m = s_i^2 = 1$ for $i \in \{1,2,\cdots,r-1\}$; $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $i \in \{1,2,\cdots,r-2\}$; $s_is_j = s_js_i$ for $i,j \in \{0,1,2,\cdots,r-1\}$ and $|i-j|>1$; $s_0s_1s_0s_1 = s_1s_0s_1s_0$.

(2) When $p=m$, S contains r reflections s_1' and s_i for $i \in \{1,2,\cdots,r-1\}$, and P consists of the relations ${s_1'}^2 = s_i^2 = 1$ for $i \in \{1,2,\cdots,r-1\}$; $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $i \in \{1,2,\cdots,r-2\}$; $s_is_j = s_js_i$ for $i,j \in \{1,2,\cdots,r-1\}$ and $|i-j|>1$; $s_1's_i = s_is_1'$ for $i>2$; $s_1's_2s_1' = s_2s_1's_2$; $\underbrace{s_1s_1's_1\cdots}_{m} = \underbrace{s_1's_1s_1'\cdots}_{m}$; $s_1's_1s_2s_1's_1s_2 = s_2s_1's_1s_2s_1's_1$.

(3) When $1<p<m$, S contains $r+1$ reflections s_0, s_1' and s_i for $i \in \{1,2,\cdots,r-1\}$, and P consists of the relations $s_0^{m/p} = {s_1'}^2 = s_i^2 = 1$ for $i \in \{1,2,\cdots,r-1\}$; $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $i \in \{1,2,\cdots,r-2\}$; $s_is_j = s_js_i$ for $i,j \in \{0,1,2,\cdots,r-1\}$ and $|i-j|>1$; $s_1's_i = s_is_1'$ for $i>2$; $s_1's_2s_1' = s_2s_1's_2$; $\underbrace{s_1s_1's_1\cdots}_{m} = \underbrace{s_1's_1s_1'\cdots}_{m}$; $s_1's_1s_2s_1's_1s_2 = s_2s_1's_1s_2s_1's_1$; $s_0s_1's_0s_1' = s_1's_0s_1's_0$; $s_0s_1s_0s_1 = s_1s_0s_1s_0$; $s_0s_1's_1 = s_1's_1s_0$; $\underbrace{s_1s_0s_1's_1s_1's_1\cdots}_{p+1} = \underbrace{s_0s_1's_1s_1's_1s_1'\cdots}_{p+1}$.

Let W be a Coxeter group and (S,P) be its presentation. Let $J \subset S$ and W_J be a subgroup of W generated by J. Then W_J is also a Coxeter group, which is called a parabolic subgroup of W. A set of distinguished right coset representatives of W_J in W is defined in [3] as $X_J := \{w \in W | l(sw) > l(w) \forall s \in J\}$. Then for any $w \in W$, it can be decomposed as $w = vd$ with $v \in W_J$ and $d \in X_J$, and $l(w) = l(v) + l(d)$. Assume (S,P) is a presentation of $G(m,p,r)$, and let $S' = S \setminus \{s_{r-1}\}$. The subgroup of $G(m,p,r)$ generated by S' is denoted by $G(m,p,r-1)$, which can also be thought of as a "parabolic" subgroup of $G(m,p,r)$. In [4], Mac gave a set of complete right coset representatives of $G(m,1,r-1)$ in $G(m,1,r)$, which is denoted by X_r. And she also proved that X_r is distinguished, according to which she can obtain a reduced expression for any element $w \in G(m,1,r)$ as $w = d_1d_2\cdots d_r$, where $d_i \in X_i$ and $G(m,1,0)$ is a trivial group.

We mean to give a set of distinguished right coset representatives of $G(m,p,r-1)$ in $G(m,p,r)$ when $1<p\le m$, so that we are able to get a reduced expression for any element $w \in G(m,p,r)$, like what Mac did in [4]. Well, it turns out to be not very easy. But as the first step, we can at least determine a set of complete right coset representatives of $G(m,p,r-1)$ in $G(m,p,r)$ (here $r>2$), which is the main result of this paper.

Note that from now on, we always assume $1<p\le m$ when $G(m,p,r)$ is cited except special explanation.

2 Main Results

Lemma 2.1 We have $s_1s_0 = s_0s_1'(s_1s_1')^{p-1}$ in $G(m,p,r)$ when $1<p<m$.

Proof By the presentation of $G(m,p,r)$ when $1<p<m$, we have relation

$$\underbrace{s_1s_0s_1's_1s_1's_1\cdots}_{p+1} = \underbrace{s_0s_1's_1s_1's_1s_1'\cdots}_{p+1}.$$

If p is odd, this relation is

$$s_1s_0\underbrace{s_1's_1\cdots s_1's_1}_{p-1}=s_0s_1'\underbrace{s_1s_1'\cdots s_1s_1'}_{p-1}.$$

So we have $s_1s_0=s_0s_1'(s_1s_1')^{p-1}$.

If p is even, this relation is

$$s_1s_0s_1'\underbrace{s_1s_1'\cdots s_1s_1'}_{p-2}=s_0\underbrace{s_1's_1\cdots s_1's_1}_{p}.$$

So we also have $s_1s_0=s_0s_1'(s_1s_1')^{p-1}$.

Lemma 2.2 we have $s_2s_1s_1's_2(s_1's_1)^k=(s_1's_1)^ks_2s_1s_1's_2$ for $1\le k\le m$ in $G(m,p,r)$.

Proof We prove by induction on k. When $k=1$, by the presentation of $G(m,p,r)$, we have relation $s_2s_1's_1s_2s_1's_1=s_1's_1s_2s_1's_1s_2$. So

$$s_2s_1s_1's_2s_1's_1=s_2s_1s_1's_1's_1s_2s_1's_1s_2s_1s_1's_2=s_1's_1s_2s_1s_1's_2.$$

Assume the conclusion is true for $k=l$, i.e., we have $s_2s_1s_1's_2(s_1's_1)^l=(s_1's_1)^ls_2s_1s_1's_2$. For $k=l+1$, we have

$$\begin{aligned}s_2s_1s_1's_2(s_1's_1)^{l+1} &= s_2s_1s_1's_2(s_1's_1)^ls_1's_1=(s_1's_1)^ls_2s_1s_1's_2s_1's_1\\ &= (s_1's_1)^ls_1's_1s_2s_1s_1's_2=(s_1's_1)^{l+1}s_2s_1s_1's_2.\end{aligned}$$

Lemma 2.3 we have $s_2(s_1s_1')^ks_2s_1'=s_1(s_1's_1)^{k-1}s_2(s_1s_1')^ks_2$ for $1\le k\le m$ in $G(m,p,r)$.

Proof We prove by induction on k. When $k=1$, since $s_1's_2s_1'=s_2s_1's_2$ and $s_2s_1s_2=s_1s_2s_1$, we have $s_2s_1s_1's_2s_1'=s_2s_1s_2s_1's_2=s_1s_2s_1s_1's_2$. Assume the conclusion is true for $k=l$, i.e., we have $s_2(s_1s_1')^ls_2s_1'=s_1(s_1's_1)^{l-1}s_2(s_1s_1')^ls_2$. For $k=l+1$, we have

$$\begin{aligned}s_2(s_1s_1')^{l+1}s_2s_1' &= s_2s_1s_1'(s_1s_1')^ls_2s_1'=s_2s_1s_1's_2s_1(s_1's_1)^{l-1}s_2(s_1s_1')^ls_2\\ &= s_1s_2s_1s_1's_2s_1's_1(s_1's_1)^{l-1}s_2(s_1s_1')^ls_2=s_1s_2s_1s_1's_2(s_1's_1)^ls_2(s_1s_1')^ls_2.\end{aligned}$$

By Lemma 2.2, the last relation equals $s_1(s_1's_1)^ls_2s_1s_1's_2s_2(s_1s_1')^ls_2=s_1(s_1's_1)^ls_2(s_1s_1')^{l+1}s_2$.

Lemma 2.4 we have $s_2(s_1s_1')^ks_2s_1=s_1(s_1's_1)^ks_2(s_1s_1')^ks_2$ for for $1\le k\le m$ in $G(m,p,r)$.

Proof We prove by induction on k. When $k=1$, since $s_2s_1s_1's_2s_1s_1'=s_1s_1's_2s_1s_1's_2$ and $s_1's_2s_1'=s_2s_1's_2$, we have

$$\begin{aligned}s_2s_1s_1's_2s_1 &= s_1s_1's_2s_1s_1's_2s_1's_1s_2s_2s_1=s_1s_1's_2s_1s_1's_2s_1'\\ &= s_1s_1's_2s_1s_2s_1's_2=s_1s_1's_1s_2s_1s_1's_2.\end{aligned}$$

Assume the conclusion is true for $k = l$, i.e., we have $s_2(s_1s_1')^l s_2 s_1 = s_1(s_1's_1)^l s_2(s_1s_1')^l s_2$. For $k = l+1$, we have

$$\begin{aligned} s_2(s_1s_1')^{l+1}s_2s_1 &= s_2s_1s_1'(s_1s_1')^l s_2s_1 = s_2s_1s_1's_2s_1(s_1's_1)^l s_2(s_1s_1')^l s_2 \\ &= s_1s_1's_1s_2s_1s_1's_2(s_1's_1)^l s_2(s_1s_1')^l s_2 = s_1s_1's_1(s_1's_1)^l s_2s_1s_1's_2s_2(s_1s_1')^l s_2 \\ &= s_1(s_1's_1)^{l+1}s_2(s_1s_1')^{l+1}s_2. \end{aligned}$$

Note that the fourth equation holds by Lemma 2.2.

Theorem 2.5 Assume $r > 2$. Let $D_i^r = \{s_{r-1}s_{r-2}\cdots s_2(s_1s_1')^i, s_{r-1}s_{r-2}\cdots s_2(s_1s_1')^i s_1, s_{r-1}s_{r-2}\cdots s_2(s_1s_1')^i s_2, s_{r-1}s_{r-2}\cdots s_2(s_1s_1')^i s_2s_3, \dots, s_{r-1}s_{r-2}\cdots s_2(s_1s_1')^i s_2s_3\dots s_{r-1}\}$. Let $D^r = \cup_{i=0}^{m-1} D_i^r$, then D^r is a set of complete right coset representatives of $G(m,p,r-1)$ in $G(m,p,r)$ when $1 < p \le m$.

Proof Let $W = G(m,p,r)$ and $L = G(m,p,r-1)$. We want to show $W = \bigcup_{d\in D^r} Ld$. It's obvious that $\bigcup_{d\in D^r} Ld \subset W$ and $|L||D^r| = |W|$, so we only need to show that $\forall s \in S = \{s_0, s_1', s_1, \dots, s_{r-1}\}$, and $\forall d \in D^r$, there exists $d' \in D^r$ such that $Lds = Ld'$. We discuss in the following cases.

(a) Assume $s = s_0$, note that this case happens only when $1 < p < m$. We have the relation $s_1's_1s_0 = s_0s_1's_1$ and $s_0s_j = s_js_0$ for $j > 1$.

(a.1) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i$,

$$ds_0 = s_{r-1}s_{r-2}\dots s_2(s_1's_1)^{m-i}s_0 = s_0s_{r-1}s_{r-2}\dots s_2(s_1's_1)^{m-i} = s_0s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i \in Ld.$$

(a.2) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1$, by lemma 2.1 we have $s_1s_0 = s_0s_1'(s_1s_1')^{p-1}$, then

$$\begin{aligned} ds_0 &= s_{r-1}s_{r-2}\dots s_2s_1(s_1's_1)^i s_0 = s_{r-1}s_{r-2}\dots s_2s_1s_0(s_1's_1)^i \\ &= s_{r-1}s_{r-2}\dots s_2s_0s_1'(s_1s_1')^{p-1}(s_1's_1)^i = s_0s_{r-1}s_{r-2}\dots s_2(s_1's_1)^{p-1}s_1'(s_1's_1)^i \\ &= s_0s_{r-1}s_{r-2}\dots s_2(s_1's_1)^{p-1}(s_1s_1')^{i-1}s_1. \end{aligned}$$

The last relation equals

$$s_0s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{i-p}s_1 \in Ld'$$

with $d' = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{i-p}s_1$ when $p \le i$; or equals

$$s_0s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{m-(p-i)}s_1 \in Ld'$$

with $d' = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{m-(p-i)}s_1$ when $i < p$.

(a.3) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_j$ $(j \ge 2)$, then

$$\begin{aligned} ds_0 &= s_{r-1}s_{r-2}\dots s_2(s_1's_1)^{m-i}s_0s_2\dots s_j = s_0s_{r-1}s_{r-2}\dots s_2(s_1's_1)^{m-i}s_2\dots s_j \\ &= s_0s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_j \in Ld. \end{aligned}$$

(b) Assume $s = s_1'$,

(b.1) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i$, then

$$ds_1' = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1' = \begin{cases} s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{i-1}s_1 \in D_{i-1}^r, \text{for } i > 0, \\ s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{m-1}s_1 \in D_{m-1}^r, \text{for } i = 0. \end{cases}$$

(b.2) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1$, then

$$ds_1' = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1 s_1' = \begin{cases} s_{r-1}s_{r-2}\dots s_2(s_1s_1')^{i+1} \in D_{i+1}^r, \ \text{for } i < m-1, \\ s_{r-1}s_{r-2}\dots s_2 \in D_0^r, \text{for } i = m-1. \end{cases}$$

(b.3) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_j$ $(j \geq 2)$, then

$$ds_1' = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2 s_1' s_3\dots s_j.$$

When $i = 0$, $ds_1' = s_1'd \in Ld$; when $i > 0$, by Lemma 2.3, we have $s_2(s_1s_1')^i s_2 s_1' = s_1(s_1's_1)^{i-1}s_2(s_1s_1')^i s_2$. Then

$$ds_1' \quad = \quad s_1(s_1's_1)^{i-1}s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2 s_3\dots s_j \in Ld.$$

(c) Assume $s = s_j, 1 \leq j \leq r-1$.

(c.1) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i$.

(c.1.1) When $j = 1$ or 2, $ds_j \in D_i^r$.

(c.1.2) When $j \geq 3$, $ds_j = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_j = s_{j-1}s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i \in Ld$.

(c.2) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1$.

(c.2.1) When $j = 1$, $ds_j \in D_i^r$.

(c.2.2) When $j = 2$, $ds_j = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1 s_2$. By Lemma 2.4, we have

$$s_2(s_1s_1')^i s_1 s_2 = s_1(s_1's_1)^i s_2(s_1s_1')^i s_1.$$

Then

$$ds_j = s_1(s_1's_1)^i s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1 \in Ld.$$

(c.2.3) When $j \geq 3$, $ds_j = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1 s_j = s_{j-1}s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_1 \in Ld$.

(c.3) If $d = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_k$ $(2 \leq k \leq r-1)$.

(c.3.1) When $j = 1$, $ds_j = s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2 s_1\dots s_k$. By Lemma 2.4, we have

$$s_2(s_1s_1')^i s_1 s_2 = s_1(s_1's_1)^i s_2(s_1s_1')^i s_1.$$

Then $ds_j = s_1(s_1's_1)^i s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_k \in Ld$.

(c.3.2) When $2 \leq j \leq k-1$,

$$\begin{aligned} ds_j &= s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_{j-1}s_j s_{j+1}\dots s_k s_j \\ &= s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_{j-1}s_j s_{j+1}s_j\dots s_k \\ &= s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_{j-1}s_{j+1}s_j s_{j+1}\dots s_k \\ &= s_{r-1}s_{r-2}\dots s_{j+1}s_j s_{j+1}s_{j-1}\dots s_2(s_1s_1')^i s_2\dots s_k \\ &= s_{r-1}s_{r-2}\dots s_j s_{j+1}s_j s_{j-1}\dots s_2(s_1s_1')^i s_2\dots s_k \\ &= s_j s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_k \in Ld. \end{aligned}$$

(c.3.3) When $j=k$ or $k+1$, $ds_j\in D_i^r$.

(c.3.4) When $k+2\le j\le r-1$,

$$\begin{aligned} ds_j &= s_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_k s_j \\ &= s_{r-1}s_{r-2}\dots s_{j+1}s_js_{j-1}s_j\dots s_2(s_1s_1')^i s_2\dots s_k \\ &= s_{r-1}s_{r-2}\dots s_{j+1}s_{j-1}s_js_{j-1}\dots s_2(s_1s_1')^i s_2\dots s_k \\ &= s_js_{r-1}s_{r-2}\dots s_2(s_1s_1')^i s_2\dots s_k\in Ld \end{aligned}$$

Up to now, we have discussed all the cases, so the theorem follows.

References

[1] Shephard G C, Todd J A. Finite unitary reflection groups[J]. Canadian Journal of Mathematics, 1954, 6: 274–304.

[2] Broue M, Malle G, Rouquier R. Complex reflection groups, braid groups, Hecke algebras[J]. Journal Für Die Reine Und Angewandte Mathematik, 1998, 500: 127–190.

[3] Geck M, Pfeiffer G. Characters of finite Coxeter groups and Iwahori-Hecke algebras[M]. Oxford: Oxford University Press, 2000.

[4] Mac C K. On complex reflection groups $G(m,1,r)$ and their Hecke algebras[D]. Sydney: University of New South Wales, 2004.

复反射群$G(m,p,r)$ 的陪集分解

徐静蕾[1,2], 王彦杰[1,3], 王 丽[1]

(1. 上海师范大学数理学院, 上海 200234)

(2. 上海大学附属中学, 上海 200444)

(3. 常熟中学, 江苏 常熟 215516)

摘要: 本文研究了非本原复反射群 $G(m,p,r)$ 的右陪集分解, 其中 m,p,r 是正整数, 且 p 整除 m. 通过使用GAP软件计算一些当 m,p,r 取较小自然数时的特例, 推导出了一般情形下, 非本原复反射群 $G(m,p,r)$ 的抛物型子群 $G(m,p,r-1)$ 的一个完全的右陪集代表元集, 这个结果为进一步研究 $G(m,p,r-1)$ 在 $G(m,p,r)$ 中的特异右陪集代表元集打下基础.

关键词: 右陪集代表; 非本原复反射群

MR(2010)**主题分类号**: 20F05; 20F55 **中图分类号**: O152.3