RESEARCH ANNOUNCEMENTS ON "RIGIDITY OF THURSTON'S SPHERE PACKINGS ON 3-DIMENSIONAL MANIFOLDS"

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1 Introduction and Main Results

In the study of hyperbolic metrics on 3-dimensional manifolds, Thurston (see [9], Chapter 13) introduced Thurston's circle packing on surfaces, which allows the circles to intersects with non-obtuse angles and is a natural generalization of the tangential circle packing studied by Andreev [1, 2] and Koebe [6]. Thurston further proved the Andreev-Thurston theorem, the rigidity of which states that Thurston's circle packing metric is determined by the discrete curvature defined at the vertices of the triangulated surfaces.

A longstanding problem since Thurston's circle packing on surfaces was introduced is whether Thurston's circle packing could be generalized to three or higher dimensional manifold with the properties of circle packing on surfaces preserved, specially the rigidity and the existence. The first attempt was made by Cooper-Rivin [3], who introduced the tangential sphere packing on 3-dimensional manifold. Suppose $(M, \mathcal{T})$ is a triangulated connected closed 3-dimensional manifold with a triangulation $\mathcal{T} = (V, E, F, T)$, where V, E, F, T denote the set of vertices, edges, faces and tetrahedra, respectively.

Definition 1.1 A tangential sphere packing metric is a map $r : V \to (0, +\infty)$ such that the length l_{ij} of the edge $\{ij\} \in E$ is $r_i + r_j$ and for any topological tetrahedron $\{ijkl\}$ the lengths $l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl}$ form the edge lengths of a nondegenerate Euclidean or hyperbolic tetrahedron.

If $r \in \mathbb{R}_{>0}^{V}$ is a nondegenerate sphere packing metric, then we can isometrically glue the Euclidean (hyperbolic respectively) tetrahedra isometrically along the faces to generate a cone 3-manifold. Cooper-Rivin introduced the combinatorial scalar curvature to characterize

the singularities at the vertices of the cone 3-manifold generated by tangential sphere packing metrics, which is defined to be $K_i = 4\pi - \sum\limits_{\{i,j,k,l\} \in T} \alpha_{ijkl}$, where α_{ijkl} is the solid angle at the vertex i of the tetrahedron $\{ijkl\} \in T$ and the summation is taken over all tetrahedra with i as a vertex. Using the combinatorial scalar curvature, Cooper, Rivin and Glickenstein proved the local rigidity of the tangential sphere packing on 3-dimensional manifolds.

Theorem 1.1 (see [3–5, 7])   Suppose $(M, \mathcal{T})$ is a closed 3-dimensional triangulated manifold. Then a Euclidean or hyperbolic tangential sphere packing on $(M, \mathcal{T})$ is locally determined by the combinatorial scalar curvature (up to scaling for the Euclidean background geometry).

The global rigidity of tangential sphere packing on 3-dimensional manifolds was conjectured by Cooper-Rivin [3] and proved by the second author [10].

Theorem 1.2 (see [10])   Suppose $(M, \mathcal{T})$ is a closed connected triangulated 3-manifold.

(1) A Euclidean tangential sphere packing metric on $(M, \mathcal{T})$ is determined by its combinatorial scalar curvature $K : V \to \mathbb{R}$ up to scaling.

(2) A hyperbolic tangential sphere packing metric on $(M, \mathcal{T})$ is determined by its combinatorial scalar curvature $K : V \to \mathbb{R}$.

Thurston's sphere packing on 3-dimensional manifolds is an analogy of Thurston's circle packing on surfaces and a natural generalization of Cooper-Rivin's tangential sphere packing on 3-manifolds with the edge length l_{ij} given by

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j \cos \Phi_{ij}}$$

for Euclidean background geometry and $l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos \Phi_{ij})$ for the hyperbolic background geometry, where $\Phi : E \to [0, \frac{\pi}{2}]$ is a weight on the edges corresponding to the intersection angle of the spheres attached to the vertices i and j. If $\Phi \equiv 0$, Thurston's sphere packing is reduced to Cooper-Rivin's tangential sphere packing. For Thurston's sphere packing, one can also define the combinatorial scalar curvature.

Definition 1.2   Suppose $(M, \mathcal{T}, \Phi)$ is a weighted triangulated closed 3-manifold with a nondegenerate sphere packing metric. The combinatorial scalar curvature at a vertex i for Thurston's sphere packing metric is defined to be

$$K_i = \sum_{j \sim i} K_{ij} \cos \tau_{ij},$$

where $K_{ij} = 2\pi - \sum\limits_{\{ijkl\} \in T} \beta_{ij,kl}$ is the combinatorial Ricci curvature along the edge $\{ij\}$ with $\beta_{ij,kl}$ being the dihedral angle along the edge $\{ij\}$ in the tetrahedron $\{ijkl\}$ and τ_{ij} is the inner angle opposite to the edge with length r_j in the triangle formed by three edges with lengths r_i, r_j, l_{ij}.

Recently, we proved the local rigidity of a large class of Thurston's sphere packings, which has been open for many years. This generalizes Cooper-Rivin-Glickstein's local rigidity for tangential sphere packing [3–5, 7] and confirms a strong local Stoker conjecture for cone 3-manifolds generated by Thurston's Euclidean sphere packing metrics [8].

Theorem 1.3   Suppose $(M, \mathcal{T})$ is a triangulated closed connected 3-manifold with a weight $\Phi : E \to [0, \frac{\pi}{2}]$ satisfying the condition

$$\Phi_{ij} + \Phi_{ik} + \Phi_{jk} > \pi, \forall \{ijk\} \in F \tag{1.1}$$

or

$$\Phi \equiv C \in [0, \frac{\pi}{2}]. \tag{1.2}$$

If $\bar{r}$ is a nondegenerate Thurston's Euclidean sphere packing metric on $(M, \mathcal{T}, \Phi)$ with $K_{ij}(\bar{r}) \sin^2 \Phi_{ij} \geq 0, \forall \{ij\} \in E$, then there exists a neighborhood U of $\bar{r}$ such that the sphere packing metrics in U are determined by combinatorial scalar curvatures up to scaling.

We also proved the infinitesimal rigidity of Thurston's Euclidean sphere packing, which confirms a strong infinitesimal Stoker conjecture for cone 3-manifolds generated by Thurston's Euclidean sphere packing metrics [8].

Theorem 1.4   Suppose $(M, \mathcal{T})$ is a triangulated closed connected 3-manifold with a weight $\Phi : E \to [0, \frac{\pi}{2}]$ satisfying (1.1) or (1.2). Then Thurston's Euclidean sphere packing on $(M, \mathcal{T}, \Phi)$ can not be deformed (except by scaling) while keeping the combinatorial Ricci curvatures along the edges fixed.

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