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RESEARCH ANNOUNCEMENTS ON "THE LIMIT FROM VLASOV-MAXWELL-BOLTZMANN SYSTEM TO NAVIER-STOKES-MAXWELL SYSTEM WITH OHM'S LAW"

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1 Introduction and Main Results

Two-species Vlasov-Maxwell-Boltzmann system (in brief, VMB) describes the evolution of a gas of two species of oppositely charged particles (the positively charged ions, i.e., cations of charge $q^+ > 0$ and mass $m^+ > 0$, and the negatively charged ions, i.e. anions of charge $-q^- < 0$ and mass $m^- > 0$), subject to auto-induced electromagnetic forces. Such a gas of charged particles, under a global neutrality condition, is called a plasma. The particle number densities $F^+(t,x,v) \geq 0$ and $F^-(t,x,v) \geq 0$ represent the distributions of cations, and anions at time $t \geq 0$, position $x \in \mathbb{T}^3$, with velocity $v \in \mathbb{R}^3$, respectively. Precisely, VMB system consists the following equations:

$$\begin{cases} \partial_t F^+ + v\cdot\nabla_x F^+ + \frac{q^+}{m^+}(E + v\times B)\cdot\nabla_v F^+ = \mathcal{B}(F^+,F^+) + \mathcal{B}(F^+,F^-)\,, \\ \partial_t F^- + v\cdot\nabla_x F^- - \frac{q^-}{m^-}(E + v\times B)\cdot\nabla_v F^- = \mathcal{B}(F^-,F^-) + \mathcal{B}(F^-,F^+)\,, \\ \mu_0\varepsilon_0\partial_t E - \nabla_x\times B = -\mu_0\int_{\mathbb{R}^3}(q^+F^+ - q^-F^-)v\,\mathrm{d}v\,, \\ \partial_t B + \nabla_x\times E = 0\,, \\ \mathrm{div}_x E = \frac{1}{\varepsilon_0}\int_{\mathbb{R}^3}(q^+F^+ - q^-F^-)\,\mathrm{d}v \quad \text{and} \quad \mathrm{div}_x B = 0\,. \end{cases} \tag{1.1}$$

The evolutions of the densities $F^\pm$ are governed by the Vlasov-Boltzmann equations, which are the first two lines in (1.1). They tell that the variations of the densities $F^\pm$ along the trajectories of the particles are subject to the influence of a Lorentz force and inter-particel collisions in the gas. The Lorentz force acting on the gas is auto-induced. That is, the electric field $E(t,x)$ and the magnetic field $B(t,x)$ are generated by the motion of the particles in the plasma itself. Their motion is governed by the Maxwell's equations, which are the remaining equations in (1.1), namely Ampère equation, Faraday's equation and Gauss' laws respectively. In (1.1), the physical constants $\mu_0, \varepsilon_0 > 0$ are, respectively,

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the vacuum permeability (or magnetic constant) and the vacuum permittivity (or electric constant). Note that their relation to the speed of light is the formula $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. For the sake of mathematical convenience, we make the simplification that both kinds of particles have the same mass $m^{\pm} = m > 0$ and charge $q^{\pm} = q > 0$.

The Boltzmann collision operator, presented in the right-hand sides of the Vlasov-Boltzmann equations in (1.1), is the quadratic form, acting on the velocity variable, associated to the bilinear operator,

$$\mathcal{B}(F,H)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}(F'H'_* - FH_*)b(v-v_*,\cos\theta)\mathrm{d}\omega\mathrm{d}v_*\,,$$

where we have used the standard abbreviations

$$F = F(v)\,,\quad F' = F(v')\,,\quad H_* = H(v_*)\,,\quad H'_* = H(v'_*)$$

with (v', v'_*) given by

$$v' = v - [(v-v_*)\cdot\omega]\omega\,,\quad v'_* = v_* + [(v-v_*)\cdot\omega]\omega$$

for $\omega \in \mathbb{S}^2$. In this paper, we will assume that the Boltzmann collision kernel is of the following hard sphere form

$$b(v-v_*,\cos\theta) = |(v-v_*)\cdot\omega| = |v-v_*||\cos\theta|\,. \tag{1.2}$$

This hypothesis is satisfied for all physical model and is more convenient to work with but do not impede the generality of our results. Then the collisional frequency can be defined as

$$\nu(v) = \int_{\mathbb{R}^3}|v-v_*|M(v_*)\mathrm{d}v_*\,. \tag{1.3}$$

There have been extensive research on the well-posedness of the VMB. DiPerna-Lions developed a theory of global-in-time renormalized solutions with large initial data, in particular to the Boltzmann equation [3] and Vlasov-Maxwell equations [2]. But for VMB there are severe difficulties, among which the major one is that the a priori bounds coming from physical laws are not enough to prove the existence of global solutions, even in the renormalized sense. In a recent remarkable breakthrough [1], Arsènio and Saint-Raymond not only proved the existence of renormalized solutions of VMB, as mentioned above, more importantly, also justified various limits (depending on the scalings) towards incompressible viscous electro-magneto-hydrodynamics. Among these limits, the most singular one is from renormalized solutions of two-species VMB to dissipative solutions of the two-fluid incompressible Navier-Stokes-Fourier-Maxwell (in brief, NSFM) system with Ohm's law.

Our result [4] is about the same limit as in [1], but in the context of classical solutions. We prove the uniform estimates with respect to Knudsen number ε for the fluctuations. As consequences, the existence of the global in time classical solutions of VMB with all $\varepsilon \in (0,1]$ is established. Furthermore, the convergence of the fluctuations of the solutions of VMB to the classical solutions of NSFM with Ohm's law is rigorously justified. The key of our proof is using the hidden damping effect of the Ohm's law and macro-micro decomposition in a novel way.

We denote the Knudsen number by ε, and the global Maxwellian $M(v) = (2\pi)^{-\frac{3}{2}} \exp(-\frac{|v|^2}{2})$. Let $F_\varepsilon^\pm(t,x,v) = M(v) + \varepsilon\sqrt{M(v)}G_\varepsilon^\pm(t,x,v)$, then VMB (1.1) can be written as

$$\begin{cases} \partial_t G_\varepsilon + \frac{1}{\varepsilon} v\cdot\nabla_x G_\varepsilon + \frac{1}{\varepsilon}\mathsf{q}(\varepsilon E_\varepsilon + v\times B_\varepsilon)\cdot\nabla_v G_\varepsilon + \frac{1}{\varepsilon^2}\mathscr{L}G_\varepsilon - \frac{1}{\varepsilon}(E_\varepsilon\cdot v)\sqrt{M}\mathsf{q}_1 \\ = \frac{1}{2}\mathsf{q}(E_\varepsilon\cdot v)G_\varepsilon + \frac{1}{\varepsilon}\Gamma(G_\varepsilon, G_\varepsilon)\,, \\ \partial_t E_\varepsilon - \nabla_x\times B_\varepsilon = -\frac{1}{\varepsilon}\int_{\mathbb{R}^3} G_\varepsilon\cdot\mathsf{q}_1 v\sqrt{M}\mathrm{d}v\,, \\ \partial_t B_\varepsilon + \nabla_x\times E_\varepsilon = 0\,, \\ \mathrm{div}_x E_\varepsilon = \int_{\mathbb{R}^3} G_\varepsilon\cdot\mathsf{q}_1\sqrt{M}\mathrm{d}v\,,\ \mathrm{div}_x B_\varepsilon = 0\,, \end{cases} \tag{1.4}$$

where $G_\varepsilon = [G_\varepsilon^+, G_\varepsilon^-]$ represents the column vector in $\mathbb{R}^2$ with the components $G_\varepsilon^\pm$, the 2×2 diagonal matrix $\mathsf{q} = \mathrm{diag}(1,-1)$, the column vector $\mathsf{q}_1 = [1,-1]$, the two species linearized collision operator $\mathscr{L}$ is given as

$$\mathscr{L}G_\varepsilon = \left[\mathcal{L}G_\varepsilon^+ + \mathcal{L}(G_\varepsilon^+, G_\varepsilon^-)\,,\ \mathcal{L}G_\varepsilon^- + \mathcal{L}(G_\varepsilon^-, G_\varepsilon^+)\right]\,, \tag{1.5}$$

where

$$\mathcal{L}g = \sqrt{M}\int_{\mathbb{R}^3}\Big(\tfrac{g}{\sqrt{M}} + \tfrac{g_*}{\sqrt{M_*}} - \tfrac{g'}{\sqrt{M'}} - \tfrac{g'_*}{\sqrt{M'_*}}\Big)|v-v_*|M_*\mathrm{d}v_* \tag{1.6}$$

is the usual linearized Boltzmann collision operator, and

$$\mathcal{L}(g,h) = \sqrt{M}\int_{\mathbb{R}^3}\Big(\tfrac{g}{\sqrt{M}} + \tfrac{h_*}{\sqrt{M_*}} - \tfrac{g'}{\sqrt{M'}} - \tfrac{h'_*}{\sqrt{M'_*}}\Big)|v-v_*|M_*\mathrm{d}v_*\,. \tag{1.7}$$

Here we denote by

$$\mathcal{Q}(g,h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}(g'h'_* - gh_*)b(|v-v_*|,\cos\theta)\mathrm{d}\sigma\mathrm{d}v_* = \int_{\mathbb{R}^3}(g'h'_* - gh_*)|v-v_*|\sqrt{M_*}\mathrm{d}v_*\,. \tag{1.8}$$

We then define a bilinear symmetric operator $\Gamma(G,H)$ as

$$\begin{aligned} \Gamma(G,H) =& \tfrac{1}{2}[\mathcal{Q}(G^+,H^+) + \mathcal{Q}(H^+,G^+) + \mathcal{Q}(G^+,H^-) + \mathcal{Q}(H^+,G^-)\,, \\ & \mathcal{Q}(G^-,H^-) + \mathcal{Q}(H^-,G^-) + \mathcal{Q}(G^-,H^+) + \mathcal{Q}(H^-,G^+)] \end{aligned} \tag{1.9}$$

for vector-valued functions $G(v) = [G^+(v), G^-(v)]$ and $H(v) = [H^+(v), H^-(v)]$. Without loss of generality, the initial conditions of (1.4) shall be imposed on

$$G_\varepsilon(0,x,v) = G_\varepsilon^{in}(x,v)\in\mathbb{R}^2\,,\quad E_\varepsilon(0,x) = E_\varepsilon^{in}(x)\in\mathbb{R}^3\,,\quad B_\varepsilon(0,x) = B_\varepsilon^{in}(x)\in\mathbb{R}^3\,, \tag{1.10}$$

which satisfy the conservation laws.

To state our main theorems, we introduce the following energy functional and dissipation rate functional respectively

$$\begin{aligned} \mathbb{E}_s(G,E,B) =& \|G\|^2_{H^s_{x,v}} + \|E\|^2_{H^s_x} + \|B\|^2_{H^s_x}\,, \\ \mathbb{D}_s(G,E,B) =& \tfrac{1}{\varepsilon^2}\|\mathbb{P}^\perp G\|^2_{H^s_{x,v}(\nu)} + \|\nabla_x\mathbb{P}G\|^2_{H^{s-1}_x L^2_v} + \|E\|^2_{H^{s-1}_x} + \|\nabla_x B\|^2_{H^{s-2}_x}\,. \end{aligned} \tag{1.11}$$

Theorem 1.1 For the integer $s \geq 3$ and $0 < \varepsilon \leq 1$, there are constants $\ell_0 > 0$, $c_0 > 0$ and $c_1 > 0$, independent of ε such that if $\mathbb{E}_s(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in}) \leq \ell_0$, then the Cauchy problem (1.4)–(1.10) admits a global solution

$$\begin{aligned} &G_\varepsilon(t,x,v) \in L_t^\infty(\mathbb{R}^+; H^s_{x,v}), \mathbb{P}^\perp G_\varepsilon(t,x,v) \in L_t^2(\mathbb{R}^+; H^s_{x,v}(\nu)), \\ &E_\varepsilon(t,x), B_\varepsilon(t,x) \in L_t^\infty(\mathbb{R}^+; H_x^s) \end{aligned} \tag{1.12}$$

with the global uniform energy estimate

$$\sup_{t\geq 0} \mathbb{E}_s(G_\varepsilon, E_\varepsilon, B_\varepsilon)(t) + c_0 \int_0^\infty \mathbb{D}_s(G_\varepsilon, E_\varepsilon, B_\varepsilon)(t)\mathrm{d}t \leq c_1 \mathbb{E}_s(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in}). \tag{1.13}$$

The next theorem is about the limit to the two fluid incompressible Navier-Stokes-Fourier-Maxwell system with Ohm's law

$$\begin{cases} \partial_t u + u\cdot\nabla_x u - \mu\Delta_x u + \nabla_x p = \frac{1}{2}(nE + j\times B), & \mathrm{div}_x\, u = 0, \\ \partial_t\theta + u\cdot\nabla_x\theta - \kappa\Delta_x\theta = 0, & \rho + \theta = 0, \\ \partial_t E - \nabla_x \times B = -j, & \mathrm{div}_x\, E = n, \\ \partial_t B + \nabla_x \times E = 0, & \mathrm{div}_x\, B = 0, \\ j - nu = \sigma\big(-\frac{1}{2}\nabla_x n + E + u\times B\big), & w = \frac{3}{2}n\theta, \end{cases} \tag{1.14}$$

where the viscosity μ, the heat conductivity κ and the electrical conductivity σ are given by

$$\mu = \frac{1}{10}\int_{\mathbb{R}^3} A : \widehat{A} M \mathrm{d}v, \quad \kappa = \frac{2}{15}\int_{\mathbb{R}^3} B\cdot\widehat{B}M\mathrm{d}v \quad \text{and} \quad \sigma = \frac{2}{3}\int_{\mathbb{R}^3}\Phi\cdot\widetilde{\Phi}M\mathrm{d}v. \tag{1.15}$$

For the derivation of (1.15), i.e. the relation of μ, κ, σ with A, $\widehat{A}$, B, $\widehat{B}$, Φ and $\widetilde{\Phi}$, see [1].

Theorem 1.2 Let $0 < \varepsilon \leq 1$, $s \geq 3$ and $\ell_0 > 0$ be as in Theorem 1.1. Assume that the initial data $(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in})$ in (1.10) satisfy

1. $G_\varepsilon^{in} \in H^s_{x,v}$, E_ε^{in}, $B_\varepsilon^{in} \in H^s_x$;
2. $\mathbb{E}_s(G_\varepsilon^{in}, E_\varepsilon^{in}, B_\varepsilon^{in}) \leq \ell_0$;
3. there exist scalar functions $\rho^{in}(x)$, $\theta^{in}(x)$, $n^{in}(x) \in H^s_x$ and vector-valued functions $u^{in}(x)$, $E^{in}(x)$, $B^{in}(x) \in H^s_x$ such that

$$\begin{aligned} G_\varepsilon^{in} &\to G^{in} \quad \text{strongly in } H^s_{x,v}, \\ E_\varepsilon^{in} &\to E^{in} \quad \text{strongly in } H^s_x, \\ B_\varepsilon^{in} &\to B^{in} \quad \text{strongly in } H^s_x \end{aligned} \tag{1.16}$$

as $\varepsilon \to 0$, where $G^{in}(x,v)$ is of the form

$$\begin{aligned} G^{in}(x,v) = &(\rho^{in}(x) + \tfrac{1}{2}n^{in}(x))\tfrac{\mathsf{q}_1+\mathsf{q}_2}{2}\sqrt{M} + (\rho^{in}(x) - \tfrac{1}{2}n^{in}(x))\tfrac{\mathsf{q}_2-\mathsf{q}_1}{2}\sqrt{M} \\ &+ u^{in}\cdot v\mathsf{q}_2\sqrt{M} + \theta^{in}(\tfrac{|v|^2}{2} - \tfrac{3}{2})\mathsf{q}_2\sqrt{M}. \end{aligned} \tag{1.17}$$

Let $(G_\varepsilon, E_\varepsilon, B_\varepsilon)$ be the family of solutions to the perturbed two-species Vlasov-Maxwell-Boltzmann (1.4) with the initial conditions (1.10) constructed in Theorem 1.1. Then, as $\varepsilon \to 0$,

$$G_\varepsilon \to (\rho + \tfrac{1}{2}n)\tfrac{\mathsf{q}_1+\mathsf{q}_2}{2}\sqrt{M} + (\rho - \tfrac{1}{2}n)\tfrac{\mathsf{q}_2-\mathsf{q}_1}{2}\sqrt{M} + u \cdot v\mathsf{q}_2\sqrt{M} + \theta(\tfrac{|v|^2}{2} - \tfrac{3}{2})\sqrt{M} \tag{1.18}$$

weakly-$\star$ in $t \geq 0$, strongly in $H^{s-1}_{x,v}$ and weakly in $H^s_{x,v}$, and

$$E_\varepsilon \to E \quad \text{and} \quad B_\varepsilon \to B \tag{1.19}$$

strongly in $C(\mathbb{R}^+; H^{s-1}_x)$, weakly-$\star$ in $t \geq 0$ and weakly in H^s_x. Here

$$(u, \theta, n, E, B) \in C(\mathbb{R}^+; H^{s-1}_x) \cap L^\infty(\mathbb{R}^+; H^s_x)$$

is the solution to the incompressible Navier-Stokes-Fourier-Maxwell equations (1.14) with Ohm's law, which has the initial data

$$u|_{t=0} = \mathcal{P}u^{in}(x)\,,\ \theta|_{t=0} = \tfrac{3}{5}\theta^{in}(x) - \tfrac{2}{5}\rho^{in}(x)\,,\ E|_{t=0} = E^{in}(x)\,,\ B|_{t=0} = B^{in}(x)\,, \tag{1.20}$$

where $\mathcal{P}$ is the Leray projection. Moreover, the convergence of the moments holds:

$$\begin{aligned} &\mathcal{P}\langle G_\varepsilon, \tfrac{1}{2}\mathsf{q}_2 v\sqrt{M}\rangle_{L^2_v} \to u\,, \\ &\langle G_\varepsilon, \tfrac{1}{2}\mathsf{q}_2(\tfrac{|v|^2}{5} - 1)\sqrt{M}\rangle_{L^2_v} \to \theta \end{aligned} \tag{1.21}$$

strongly in $C(\mathbb{R}^+; H^{s-1}_x)$, weakly-$\star$ in $t \geq 0$ and weakly in H^s_x as $\varepsilon \to 0$.

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