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RESEARCH ANNOUNCEMENTS ON "ESTIMATES OF DIRICHLET EIGENVALUES FOR DEGENERATE $\triangle_\mu$-LAPLACE OPERATOR"

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1 Introduction and Main Results

For $n \geq 2$, we consider the following Dirichlet eigenvalue problem

$$\begin{cases} -\triangle_\mu u = \lambda u, & x \in \Omega; \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

on a bounded open domain $\Omega \subset \mathbb{R}^n$, with smooth boundary $\partial\Omega$, where $\triangle_\mu$ is a degenerate elliptic operator generated by a system of real vector fields $X = (X_1, X_2, \cdots, X_n)$, namely,

$$\triangle_\mu := \sum_{j=1}^{n} X_j^2. \tag{1.2}$$

We assume that the system of real vector fields $X = (X_1, X_2, \cdots, X_n)$ is defined in $\mathbb{R}^n$ by $X_j = \mu_j(x)\partial_{x_j}$, where $\mu_1, \cdots, \mu_n$ are real continuous nonnegative functions in $\mathbb{R}^n$ satisfying following assumptions:

(H1) $\mu_1 = 1$, and $\mu_j(x) = \mu_j(x_1, \cdots, x_{j-1})$ for $j = 2, \cdots, n$.

(H2) For each $j = 1, \cdots, n$, $\mu_j \in C^1(\mathbb{R}^n \backslash \Pi)$, where $\Pi = \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n \,\middle|\, \prod_{i=1}^{n} x_i = 0\right\}$.

(H3) $\mu_j(x) > 0$ and $x_k(\partial_{x_k}\mu_j)(x) \geq 0$ for all $x \in \mathbb{R}^n \setminus \Pi$, $1 \leq k \leq j-1$, $j = 2, \ldots, n$. Furthermore, $\mu_j(x_1, \cdots, -x_k, \cdots, x_{j-1}) = \mu_j(x_1, \cdots, x_k, \cdots, x_{j-1})$ for all $1 \leq k \leq j-1$, $j = 2, \cdots, n$.

(H4) There exists a constant $\sigma_{j,k} \geq 0$ such that $\sup\limits_{x \in \mathbb{R}^n \setminus \Pi} \frac{x_k(\partial_{x_k}\mu_j)(x)}{\mu_j(x)} = \sigma_{j,k}$ holds for all $1 \leq k \leq j-1$, $j = 2, \cdots, n$.

(H5) If $A := \{2 \leq k \leq n | \mu_k(x) \not\equiv \mu_k(0)\} \neq \varnothing$, then there exists a constant $c_0 \in (0, 1)$ such that for all $j \in A$, $\langle \nabla\mu_j(x), x\rangle_{\mathbb{R}^n} \geq c_0\mu_j(x)$ for all $x \in \mathbb{R}^n \setminus \Pi$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the inner product in $\mathbb{R}^n$.

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Then, we define some positive constants $\varepsilon_1, \ldots, \varepsilon_n$ as

$$\varepsilon_1 = 1, \ \ \frac{1}{\varepsilon_k} = 1 + \sum_{l=1}^{k-1} \frac{\sigma_{k,l}}{\varepsilon_l} \ \text{ for all } k \geq 2, \tag{1.3}$$

and an index Q as

$$Q = \sum_{j=1}^{n} \frac{1}{\varepsilon_j}. \tag{1.4}$$

We remark that assumption (H1) allows us to write the operator $\triangle_\mu$ in the form $\triangle_\mu = \sum\limits_{j=1}^{n} \mu_j^2(x) \frac{\partial^2}{\partial x_j^2}$. Recently, Kogoj and Lanconelli in [3] studied such degenerate elliptic operators under the following additional assumption:

(H6) There exists a group of dilations $(\delta_t)_{t>0}$,

$$\delta_t : \mathbb{R}^n \to \mathbb{R}^n, \ \ \delta_t(x) = \delta_t(x_1, \cdots, x_n) = (t^{\alpha_1} x_1, \cdots, t^{\alpha_n} x_n)$$

with $1 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ such that μ_i is δ_t homogeneous of degree $\alpha_i - 1$, i.e.,

$$\mu_i(\delta_t(x)) = t^{\alpha_i - 1} \mu_i(x), \ \ \forall x \in \mathbb{R}^n, \ \forall t > 0, i = 1, \cdots, n. \tag{1.5}$$

We denote by $\tilde{Q}$ the homogeneous dimension of $\mathbb{R}^n$ with respect to the group of dilations $(\delta_t)_{t>0}$, i.e., $\tilde{Q} := \alpha_1 + \cdots + \alpha_n$. The index $\tilde{Q}$ plays a crucial role in the geometry and the functional settings associated to the vector fields X.

We next introduce the following weighted Sobolev spaces $\mathcal{H}_X^1(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid X_j u \in L^2(\mathbb{R}^n); j = 1, 2, \cdots, n\}$ associated with the real vector fields $X = (X_1, X_2, \cdots, X_n)$, then $\mathcal{H}_X^1(\mathbb{R}^n)$ is a Hilbert space endowed with norm $\|u\|^2_{\mathcal{H}_X^1(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \|Xu\|^2_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + \sum\limits_{j=1}^{n} \|X_j u\|^2_{L^2(\mathbb{R}^n)}$. Now let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary such that $\Omega \bigcap \Pi \neq \emptyset$, we denote by $\mathcal{H}_{X,0}^1(\Omega)$ the closure of $C_0^1(\Omega)$ with respect to the norm $\|u\|^2_{\mathcal{H}_X^1(\Omega)} = \|Xu\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}$. We know that $\mathcal{H}_{X,0}^1(\Omega)$ is a Hilbert space as well.

In this paper, the Dirichlet eigenvalue problem (1.1) of degenerate elliptic operator $-\triangle_\mu$ will be considered in the weak sense in $\mathcal{H}_{X,0}^1(\Omega)$, namely,

$$(Xu, X\varphi)_{L^2(\Omega)} = \lambda(u, \varphi)_{L^2(\Omega)} \ \text{ for all } \varphi \in \mathcal{H}_{X,0}^1(\Omega). \tag{1.6}$$

Based on assumptions (H1)–(H5) above, we can show that the Dirichlet eigenvalue problem (1.6) has a sequence of discrete eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$, which satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and $\lambda_k \to +\infty$ as $k \to +\infty$.

By using the regularity results of Franchi and Lanconelli [2] we can prove that the problem (1.6) has discrete Dirichlet eigenvalues. Then, by using the process of refinement in Li-Yau [4], we obtain an explicit lower bound estimates of Dirichlet eigenvalues λ_k as follows.

Theorem 1.1 Let $X=(X_1,\cdots,X_n)$ be real continuous vector fields defined in $\mathbb{R}^n$ and satisfy assumptions (H1)–(H5). Assume that Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary such that $\Omega\bigcap\Pi\neq\emptyset$. If we denote by λ_k the k^{th} Dirichlet eigenvalue of operator $-\triangle_\mu$ on Ω, then for any $k\geq 1$, we have

$$\sum_{j=1}^{k}\lambda_j\geq C_1\cdot k^{1+\frac{2}{Q}}, \tag{1.7}$$

where Q is defined by (1.4) and $C_1=\frac{1}{C}\cdot\frac{Q}{Q+2}\cdot\left(\prod_{j=1}^{n}\varepsilon_j\right)^{\frac{2}{Q}}\left(\Gamma\left(\frac{Q}{2}+1\right)\right)^{\frac{2}{Q}}\left(\prod_{j=1}^{n}\Gamma\left(\frac{1}{2\varepsilon_j}\right)\right)^{-\frac{2}{Q}}\cdot\left(\frac{(2\pi)^n}{|\Omega|}\right)^{\frac{2}{Q}}$, and $\Gamma(x)$ is the Gamma function, $|\Omega|$ is the n-dimensional Lebesgue measure of Ω and $C=C(X,\Omega)$ is a positive constant.

Remark 1 Since $k\lambda_k\geq\sum\limits_{j=1}^{k}\lambda_j$, then Theorem 1.1 implies that the k^{th} Dirichlet eigenvalue λ_k satisfies $\lambda_k\geq C_1k^{\frac{2}{Q}}$, for all $k\geq 1$.

Remark 2 In general, for degenerate case we have $Q=\sum\limits_{j=1}^{n}\frac{1}{\varepsilon_j}>n\geq 2$. If $Q=n$, the operator will be non-degenerate and the positive constant C can be replaced by $\left(\min\limits_{1\leq j\leq n}\mu_j^2(0)\right)^{-1}$, thus estimate (1.7) will be the generalization as the Li-Yau's lower bound estimate in [4].

Moreover, if the vector fields satisfy assumption (H6), then we have the following sharper lower bounds.

Theorem 1.2 Let $X=(X_1,\cdots,X_n)$ be real continuous vector fields defined in $\mathbb{R}^n$ and satisfy assumptions (H1)–(H3) and (H6). Assume that Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary such that $\Omega\bigcap\Pi\neq\emptyset$. Denote by λ_k the k^{th} Dirichlet eigenvalue of operator $-\triangle_\mu$ on Ω, and $\tilde{Q}=\sum\limits_{j=1}^{n}\alpha_j$ is the homogeneous dimension of $\mathbb{R}^n$ with respect to $(\delta_t)_{t>0}$. Then for any $k\geq 1$, we have

$$\sum_{j=1}^{k}\lambda_j\geq C_2\cdot k^{1+\frac{2}{\tilde{Q}}}, \tag{1.8}$$

where $C_2=\frac{1}{\tilde{C}}\cdot\frac{\tilde{Q}}{\tilde{Q}+2}\cdot\left(\prod_{j=1}^{n}\alpha_j\right)^{-\frac{2}{\tilde{Q}}}\left(\Gamma\left(\frac{\tilde{Q}}{2}+1\right)\right)^{\frac{2}{\tilde{Q}}}\left(\prod_{j=1}^{n}\Gamma\left(\frac{\alpha_j}{2}\right)\right)^{-\frac{2}{\tilde{Q}}}\cdot\left(\frac{(2\pi)^n}{|\Omega|}\right)^{\frac{2}{\tilde{Q}}}$, and $\tilde{C}=\tilde{C}(X,\Omega)$ is a positive constant.

Remark 3 If the vector fields admit the homogeneous structure assumptions (H1)–(H3) and (H6), then assumptions (H4) and (H5) will be also satisfied. But we cannot deduce assumption (H6) from assumptions (H1)–(H5), for example, $X=(\partial_{x_1},\partial_{x_2},(|x_1|^\alpha+|x_2|^\beta)\partial_{x_3})$ with $\alpha>\beta>0$.

Remark 4 If the vector fields admit the homogeneous structure assumptions (H1)–(H3) and (H6), then the lower bounds in (1.8) is sharper than (1.7) in the sense of growth order.

Furthermore, by the same condition in Kröger [5], we obtain an upper bound for the Dirichlet eigenvalues of operator $-\triangle_\mu$.

Theorem 1.3 Let $X = (X_1, \ldots, X_n)$ be real continuous vector fields defined in $\mathbb{R}^n$ and satisfy assumptions (H1)–(H5). Suppose that Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ such that $\Omega \bigcap \Pi \neq \emptyset$. Moreover, we assume that there exists a constant $C_0 > 0$ such that the measure of inner neighbourhood of the boundary $\Omega_r = \{x \in \Omega | \mathrm{dist}(x, \partial\Omega) < \frac{1}{r}\}$ satisfies that $|\Omega_r| \leq \frac{C_0}{r}|\Omega|^{\frac{n-1}{n}}$ for any $r > |\Omega|^{-\frac{1}{n}}$, where $\mathrm{dist}(x, \partial\Omega) := \inf\limits_{y \in \partial\Omega} |x - y|$ is the distance function and $|\Omega|$ is the n-dimensional Lebesgue measure of Ω. Denote by λ_k the k^{th} Dirichlet eigenvalue of operator $-\triangle_\mu$ on Ω. Then for any $k \geq C_0^n$, we have

$$\sum_{j=1}^{k} \lambda_j \leq \frac{(2\pi)^2 \cdot M \cdot \left(1 + \frac{n}{n+2}\right)}{|B_1|^{\frac{2}{n}} \cdot |\Omega|^{\frac{2}{n}} \left(1 - \frac{C_0}{(C_0^n+1)^{\frac{1}{n}}}\right)^{\frac{n+2}{n}}} \cdot (k+1)^{\frac{n+2}{n}} \tag{1.9}$$

and

$$\lambda_{k+1} \leq \frac{2^{1+\frac{2}{n}} \cdot (2\pi)^2 \cdot M}{|B_1|^{\frac{2}{n}} \cdot |\Omega|^{\frac{2}{n}} \left(1 - \frac{C_0}{(C_0^n+1)^{\frac{1}{n}}}\right)^{1+\frac{2}{n}}} \cdot \left(1 + \frac{n}{n+2}\right) \cdot k^{\frac{2}{n}}, \tag{1.10}$$

where $M = \max\limits_{x \in \overline{\Omega}, 1 \leq j \leq n} \mu_j^2(x)$ is a positive constant depending on X and Ω.

Remark 5 For a bounded domain Ω, if the $(n-1)$-dimensional Lebesgue measure of $\partial\Omega$ is bounded, then $|\{x \in \Omega | \mathrm{dist}(x, \partial\Omega) < \frac{1}{r}\}| \approx \frac{1}{r} \cdot |\partial\Omega|$. Thus the condition in Theorem 1.3 holds for some positive constant C_0.

The details of proofs for Theorem 1.1, Theorem 1.2 and Theorem 1.3 have been given by [1].

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