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IMPROVE INEQUALITIES OF ARITHMETIC-HARMONIC MEAN

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Abstract: We study the refinement of arithmetic-harmonic mean inequalities. First, through the classical analysis method, the scalar inequalities are obtained, and then extended to the operator cases. Specifically, we have the following main results: for $0<\nu,\tau<1,\ a,b>0$ with $(b-a)(\tau-\nu)>0$, we have $\frac{a\nabla_\nu b-a!_\nu b}{a\nabla_\tau b-a!_\tau b}\leq\frac{\nu(1-\nu)}{\tau(1-\tau)}$ and $\frac{(a\nabla_\nu b)^2-(a!_\nu b)^2}{(a\nabla_\tau b)^2-(a!_\tau b)^2}\leq\frac{\nu(1-\nu)}{\tau(1-\tau)}$, which are generalizations of the results of W. Liao et al.

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1 Introduction

Let M_n denote the algebra of all $n\times n$ complex matrices, M_n^+ be the set of all the positive semidefinite matrices in M_n. For two Hermitian matrices A and B, $A\geq B$ means $A-B\in M_n^+$, $A>B$ means $A-B\in M_n^{++}$, where M_n^{++} is the set of all the strictly positive matrices in M_n. I stands for the identity matrix. The Hilbert-Schmidt norm of $A=[a_{ij}]\in M_n$ is defined by $||A||_2=\sqrt{\sum\limits_{i,j=1}^{n}|a_{ij}|^2}$. It is well-known that the Hilbert-Schmidt norm is unitarily invariant in the sense that $||UAV||=||A||$ for all unitary matrices $U,V\in M_n$. What's more, we use the following notions

$$\begin{aligned}&A\nabla_vB=(1-v)A+vB,\\&A\sharp_vB=A^{\frac12}(A^{-\frac12}BA^{-\frac12})^vA^{\frac12},\\&A!_vB=((1-v)A^{-1}+vB^{-1})^{-1}\end{aligned}$$

for $A,B\in M_n^{++}$ and $0\leq v\leq 1$. Usually we denote by $A\nabla B$, $A\sharp B$ and $A!B$ for brevity respectively when $v=\frac12$.

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In this paper, we set $a, b > 0$. As we all know, the scalar harmonic-geometric-arithmetic mean inequalities

$$a!_v b \leq a\sharp_v b \leq a\nabla_v b \tag{1.1}$$

hold and the second inequality is called Young inequality. Similarly, we also have the related operator version

$$A!_v B \leq A\sharp_v B \leq A\nabla_v B \tag{1.2}$$

for two strictly positive operators A and B.

The first refinements of Young inequality is the squared version proved in [1]

$$(a^v b^{1-v})^2 + \min\{v, 1-v\}^2 (a-b)^2 \leq (va + (1-v)b)^2. \tag{1.3}$$

Later, authors in [2] obtained the other interesting refinement

$$a^v b^{1-v} + \min\{v, 1-v\}(\sqrt{a} - \sqrt{b})^2 \leq va + (1-v)b. \tag{1.4}$$

Then many results about Young inequalities presented in recent years. wa can see [3], [4] and [5] for some related results. Also, in [3] authors proved that

$$A\nabla_v B \geq A!_v B + 2\min\{v, 1-v\}(A\nabla_v B - A!_v B) \tag{1.5}$$

for $A, B \in M_n^{++}$ and $0 \leq v \leq 1$.

Alzer [6] proved that

$$(\frac{\nu}{\tau})^\lambda \leq \frac{(a\nabla_\nu b)^\lambda - (a\sharp_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (a\sharp_\tau b)^\lambda} \leq (\frac{1-\nu}{1-\tau})^\lambda \tag{1.6}$$

for $0 < \nu \leq \tau < 1$ and $\lambda \geq 1$, which is a different form of (1.5). By a similar technique, Liao [7] presented that

$$(\frac{\nu}{\tau})^\lambda \leq \frac{(a\nabla_\nu b)^\lambda - (a!_\nu b)^\lambda}{(a\nabla_\tau b)^\lambda - (a!_\tau b)^\lambda} \leq (\frac{1-\nu}{1-\tau})^\lambda \tag{1.7}$$

for $0 < \nu \leq \tau < 1$ and $\lambda \geq 1$. Sababheh [8] generalized (1.6) and (1.7) by convexity of function f,

$$(\frac{\nu}{\tau})^\lambda \leq \frac{((1-\nu)f(0) + \nu f(1))^\lambda - f^\lambda(\nu)}{((1-\tau)f(0) + \tau f(1))^\lambda - f^\lambda(\tau)} \leq (\frac{1-\nu}{1-\tau})^\lambda, \tag{1.8}$$

where $0 < \nu \leq \tau < 1$ and $\lambda \geq 1$. In the same paper [8], it is proved that

$$(\frac{\nu}{\tau})^\lambda \leq \frac{(a\sharp_\nu b)^\lambda - (a!_\nu b)^\lambda}{(a\sharp_\tau b)^\lambda - (a!_\tau b)^\lambda}, \tag{1.9}$$

where $0 < \nu \leq \tau < 1$ and $\lambda \geq 1$.

Our main task of this paper is to improve (1.7) for scalar and matrix under some conditions. The article is organized in the following way: in Section 2, new refinements of harmonic-arithmetic mean are presented for scalars. In Section 3, similar inequalities for operators are presented. And the Hilbert-Schmidt norm and determinant inequalities are presented in Sections 4 and 5, respectively.

2 Inequalities for Scalars

In this part, we first give an improved version of harmonic-arithmetic mean inequality for scalar. It is also the base of this paper.

Theorem 2.1 Let ν, τ a and b be real positive numbers with $0 < \nu, \tau < 1$, then we have

$$\frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \le \frac{\nu(1-\nu)}{\tau(1-\tau)} \quad \text{for } (b-a)(\tau-\nu) > 0 \tag{2.1}$$

and

$$\frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \ge \frac{\nu(1-\nu)}{\tau(1-\tau)} \quad \text{for } (b-a)(\tau-\nu) < 0. \tag{2.2}$$

Proof Put $f(v) = \frac{1-v+vx-(1-v+vx^{-1})^{-1}}{v(1-v)}$, then we have $f'(v) = \frac{1}{v^2(1-v)^2}h(x)$, where

$$h(x) = v(1-v)[x-1+(1-v+vx^{-1})^{-2}(\frac{1}{x}-1)] - (1-2v)[1-v+vx-(1-v+vx^{-1})^{-1}].$$

By a carefully and directly computation, we have $h'(x) = \frac{v^2}{((1-v)x+v)^3}g(x)$, where $g(x) = 2(1-v)(1-x)-(1-v)x-v+((1-v)x+v)^3$, so we can get $g'(x) = 3(1-v)^2(x-1)((1-v)x+v+1)$ easily.

Now if $0 < x \le 1$, then $g'(x) \le 0$, which means $g(x) \ge g(1) = 0$, and then $h'(x) \ge 0$; and if $1 \le x < \infty$, then $g'(x) \ge 0$, which means $g(x) \ge g(1) = 0$, and then $h'(x) \ge 0$.

That is to say that $h'(x) \ge 0$ for all $x \in (0,\infty)$. Hence when $0 < x \le 1$, we have $h(x) \le h(1) = 0$, and so $f'(v) \le 0$, which means that $f(v)$ is decreasing on $(0,1)$; and when $1 \le x < \infty$, $h(x) \ge h(1) = 0$, and so $f'(v) \ge 0$, which means that $f(v)$ is increasing on $(0,1)$. Put $x = \frac{b}{a}$, we can get our desired results easily.

Remark 2.2 Let $0 < \nu \le \tau < 1$, then we have the following inequalities from (2.1),

$$\frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \le \frac{\nu(1-\nu)}{\tau(1-\tau)} \le \frac{1-\nu}{1-\tau} \quad \text{for } (b-a) > 0. \tag{2.3}$$

On the other hand, when $0 < \nu \le \tau < 1$ and $b - a < 0$, we also have by (2.1),

$$\frac{a^{-1}\nabla_\nu b^{-1} - a^{-1}!_\nu b^{-1}}{a^{-1}\nabla_\tau b^{-1} - a^{-1}!_\tau b^{-1}} \le \frac{\nu(1-\nu)}{\tau(1-\tau)}, \tag{2.4}$$

which implies

$$\begin{aligned} \frac{\nu}{\tau} &\le \frac{\nu(1-\nu)}{\tau(1-\tau)} \le \frac{a\nabla_\nu b - a!_\nu b}{a\nabla_\tau b - a!_\tau b} \\ &\le \frac{(a\nabla_\nu b)(a!_\nu b)}{(a\nabla_\tau b)(a!_\tau b)}\frac{\nu(1-\nu)}{\tau(1-\tau)} \quad \text{for } (b-a) < 0, \end{aligned} \tag{2.5}$$

by (2.2). Therefore, it is clear that (2.3) and (2.5) are sharper than (1.7) under some conditions for $\lambda = 1$. Next, we give a quadratic refinement of Theorem 2.1, which is better than (1.7) when $\lambda = 2$.

Theorem 2.3 Let ν, τ, a and b are real numbers with $0 < \nu, \tau < 1$, then we have

$$\frac{(a\nabla_\nu b)^2 - (a!_\nu b)^2}{(a\nabla_\tau b)^2 - (a!_\tau b)^2} \leq \frac{\nu(1-\nu)}{\tau(1-\tau)} \quad \text{for } (b-a)(\tau-\nu) > 0 \tag{2.6}$$

and

$$\frac{(a\nabla_\nu b)^2 - (a!_\nu b)^2}{(a\nabla_\tau b)^2 - (a!_\tau b)^2} \geq \frac{\nu(1-\nu)}{\tau(1-\tau)} \quad \text{for } (b-a)(\tau-\nu) < 0. \tag{2.7}$$

Proof Put $f(v) = \frac{(1-v+vx)^2-(1-v+vx^{-1})^{-2}}{v(1-v)}$, then we have $f'(v) = \frac{1}{v^2(1-v)^2}h(x)$, where

$$\begin{aligned} h(x) &= 2v(1-v)[(1-v+vx)(x-1) + (1-v+vx^{-1})^{-3}(\frac{1}{x}-1)] \\ &\quad -(1-2v)[(1-v+vx)^2 - (1-v+vx^{-1})^{-2}] \end{aligned}$$

by a direct computation, we have $h'(x) = \frac{2xv^2}{((1-v)x+v)^4}g(v)$, where $g(v) = 3(1-v)(1-x) - (1-v)x - v + ((1-v)x+v)^4$. By $g'(v) = -4(1-x)^2(1-v)[((1-v)x+v)^2 + 1 + (1-v)x + v] \leq 0$, so we have $g(v) \geq g(1) = 0$, which means $h'(x) \geq 0$. Now if $0 < x \leq 1$, then $h(x) \leq h(1) = 0$, and so $f'(v) \leq 0$, which means that $f(v)$ is decreasing on $(0, 1)$. On the other hand, if $1 \leq x < \infty$, then $h(x) \geq h(1) = 0$, and so $f'(v) \geq 0$, which means that $f(v)$ is increasing on $(0, 1)$. Put $x = \frac{b}{a}$, we can get our desired results directly.

3 Inequalities for Operators

In this section, we give some refinements of harmonic-arithmetic mean for operators, which are based on inequalities (2.1) and (2.2).

Lemma 3.1 Let $X \in M_n$ be self-adjoint and let f and g be continuous real functions such that $f(t) \geq g(t)$ for all $t \in Sp(X)$ (the spectrum of X). Then $f(X) \geq g(X)$.

For more details about this property, readers can refer to [9].

Theorem 3.2 Let $A, B \in M_n^{++}$ and $0 < \nu, \tau < 1$, then

$$\tau(1-\tau)(A\nabla_\nu B - A!_\nu B) \leq \nu(1-\nu)(A\nabla_\tau B - A!_\tau B) \tag{3.1}$$

for $(B-A)(\tau-\nu) \geq 0$; and

$$\tau(1-\tau)(A\nabla_\nu B - A!_\nu B) \geq \nu(1-\nu)(A\nabla_\tau B - A!_\tau B) \tag{3.2}$$

for $(B-A)(\tau-\nu) \leq 0$.

Proof Let $a = 1$ in (2.1), for $(b-1)(\tau-v) \geq 0$, then we have

$$\begin{aligned} &\tau(1-\tau)[1-\nu+\nu b - (1-\nu+\nu b^{-1})^{-1}] \\ \leq\ &\nu(1-\nu)[1-\tau+\tau b - (1-\tau+\tau b^{-1})^{-1}]. \end{aligned} \tag{3.3}$$

We may assume $0 < \tau < v < 1$ and $0 < b \leq 1$. For $(B - A)(\tau - \nu) \geq 0$, we have $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq I$. The operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum. By Lemma 3.1 and (3.3), we have

$$\begin{aligned} & \tau(1-\tau)[1-\nu+\nu X-(1-\nu+\nu X^{-1})^{-1}] \\ \leq\ & \nu(1-\nu)[1-\tau+\tau X-(1-\tau+\tau X^{-1})^{-1}]. \end{aligned} \tag{3.4}$$

Multiplying (3.4) by $A^{\frac{1}{2}}$ on the both sides, we can get the desired inequality (3.1).

Using the same technique, we can get (3.2) by (2.2). Notice that the inequalities of Theorem 3.2 provide a refinement and a reverse of (1.2).

4 Inequalities for Hilbert-Schmidt Norm

In this section, we present inequalities of Theorem 2.2 for Hilbert-Schmidt norm.

Theorem 4.1 Let $X \in M_n$ and $B \in M_n^{++}$ for $0 < v, \tau < 1$, then we have

$$\begin{aligned} & \frac{||(1-v)X+vXB||_2^2-||[(1-v)X^{-1}+vB^{-1}X^{-1}]^{-1}||_2^2}{v(1-v)} \\ \leq\ & \frac{||(1-\tau)X+\tau XB||_2^2-||[(1-\tau)X^{-1}+\tau B^{-1}X^{-1}]^{-1}||_2^2}{\tau(1-\tau)} \end{aligned} \tag{4.1}$$

for $(B - I)(\tau - v) \geq 0$; and

$$\begin{aligned} & \frac{||(1-v)X+\nu XB||_2^2-||[(1-v)X^{-1}+vB^{-1}X^{-1}]^{-1}||_2^2}{v(1-v)} \\ \geq\ & \frac{||(1-\tau)X+\tau XB||_2^2-||[(1-\tau)X^{-1}+\tau B^{-1}X^{-1}]^{-1}||_2^2}{\tau(1-\tau)} \end{aligned} \tag{4.2}$$

for $(B - I)(\tau - v) \leq 0$.

Proof Since B is positive definite, it follows by spectral theorem that there exist unitary matrices $V \in M_n$ such that $B = V\Lambda V^*$, where $\Lambda = \mathrm{diag}(\nu_1, \nu_2, \cdots, \nu_n)$ and ν_i are eigenvalues of B, so $\nu_l > 0,\ l = 1, 2, \cdots, n$. Let $Y = V^*XV = [y_{il}]$, then

$$(1-v)X+vXB = V[(1-v)Y+vY\Lambda]V^* = V[(1-v+v\nu_l)y_{il}]V^*$$

and

$$[(1-v)X^{-1}+vB^{-1}X^{-1}]^{-1} = V[(1-v)Y^{-1}+v\Lambda^{-1}Y^{-1}]^{-1}V^* = V[(1-v+v\nu_l^{-1})^{-1}y_{il}]V^*.$$

Now, by (2.6) and the unitarily invariant of the Hilbert-Schmidt norm, we have

$$
\begin{aligned}
& ||(1-v)X+vXB||_2^2-||[(1-v)X^{-1}+vB^{-1}X^{-1}]^{-1}||_2^2 \\
= & \sum_{i,l=1}^{n}(1-v+v\nu_l)^2|y_{il}|^2-\sum_{i,l=1}^{n}(1-v+v\nu_l^{-1})^{-2}|y_{il}|^2 \\
= & \sum_{i,l=1}^{n}[(1-v+v\nu_l)^2-((1-v)+v\nu_l^{-1})^{-2}]|y_{il}|^2 \\
\leq & \frac{v(1-v)}{\tau(1-\tau)}\sum_{i,l=1}^{n}[((1-\tau)+\tau\nu_l)^2-((1-\tau)+\tau\nu_l^{-1})^{-2}]|y_{il}|^2 \\
= & \frac{v(1-v)}{\tau(1-\tau)}[\sum_{i,l=1}^{n}((1-\tau)+\tau\nu_l)^2|y_{il}|^2-\sum_{i,l=1}^{n}((1-\tau)+\tau\nu_l^{-1})^{-2}|y_{il}|^2] \\
= & \frac{v(1-v)}{\tau(1-\tau)}[||(1-\tau)X+\tau XB||_2^2-||[(1-\tau)X^{-1}+\tau B^{-1}X^{-1}]^{-1}||_2^2].
\end{aligned}
$$

Here we completed the proof of (4.1). Using the same method in (2.7), we can get (4.2) easily. So we omit it.

It is clear that Theorem 4.1 provids a refinement of Corollary 4.2 in [7].

Remark 4.2 Theorem 4.1 is not true in general when we exchange I for A, where A is a positive definite matrix. That is: let $X \in M_n$ and $A, B \in M_n^{++}$ for $0 < \nu, \tau < 1$, then we can not have results as below

$$
\begin{aligned}
& \frac{||(1-\nu)AX+\nu XB||_2^2-||[(1-\nu)X^{-1}A^{-1}+\nu B^{-1}X^{-1}]^{-1}||_2^2}{\nu(1-\nu)} \\
\leq & \frac{||(1-\tau)AX+\tau XB||_2^2-||[(1-\tau)X^{-1}A^{-1}+\tau B^{-1}X^{-1}]^{-1}||_2^2}{\tau(1-\tau)} \qquad (4.3)
\end{aligned}
$$

for $(B-A)(\tau-\nu) \geq 0$ and

$$
\begin{aligned}
& \frac{||(1-\nu)AX+\nu XB||_2^2-||[(1-\nu)X^{-1}A^{-1}+\nu B^{-1}X^{-1}]^{-1}||_2^2}{\nu(1-\nu)} \\
\geq & \frac{||(1-\tau)AX+\tau XB||_2^2-||[(1-\tau)X^{-1}A^{-1}+\tau B^{-1}X^{-1}]^{-1}||_2^2}{\tau(1-\tau)} \qquad (4.4)
\end{aligned}
$$

for $(B-A)(\tau-\nu) \leq 0$.

Now we give the following example to state it.

Example 4.3 Let $B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $X = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, then (4.3) and (4.4) are not true for $\nu = \frac{1}{2}$ and $\tau = \frac{2}{3}$.

Proof we can compute that

$$||(1-\nu)AX+\nu XB||_2^2 = \frac{53}{36}+\frac{23}{9}\nu+\frac{53}{36}\nu^2$$

and

$$||((1-\nu)X^{-1}A^{-1}+\nu B^{-1}X^{-1})^{-1}||_2^2=\frac{1}{(6-6\nu+2\nu^2)^2}[53+9\nu^2-40\nu].$$

A careful calculation shows that

$$\begin{aligned}&\frac{||(1-\nu)AX+\nu XB||_2^2-||[(1-\nu)X^{-1}A^{-1}+\nu B^{-1}X^{-1}]^{-1}||_2^2}{\nu(1-\nu)}\\ =\;&\frac{1}{(6-6\nu+2\nu^2)^2}[-\frac{53}{9}\nu^4+\frac{173}{9}\nu^3-\frac{123}{9}\nu^2-\frac{231}{9}\nu+26].\end{aligned}\tag{4.5}$$

Let $\nu=\frac{1}{2}$ and $\tau=\frac{2}{3}$, then (4.5) implies

$$\frac{||(1-\nu)AX+\nu XB||_2^2-||[(1-\nu)X^{-1}A^{-1}+\nu B^{-1}X^{-1}]^{-1}||_2^2}{\nu(1-\nu)}=0.96\cdots,$$

and

$$\frac{||(1-\tau)AX+\tau XB||_2^2-||[(1-\tau)X^{-1}A^{-1}+\tau B^{-1}X^{-1}]^{-1}||_2^2}{\tau(1-\tau)}=0.88\cdots,$$

which implies that (4.3) is not true clearly. Similarly, we can also prove that (4.4) is not true by exchanging ν and τ.

5 Inequalities for determinant

In this section, we present inequalities of Theorem 2.1 and Theorem 2.2 for determinant. Before it, we should recall some basic signs. The singular values of a matrix A are defined by $s_j(A), j=1,2,\cdots,n$. And we denote the values of $\{s_j(A)\}$ as a non-increasing order. Besides, $\det(A)$ is the determinant of A. To obtain our results, we need a following lemma.

Lemma 5.1 [10] (Minkowski inequality) Let $a=[a_i],\ b=[b_i],\ i=1,2,\cdots,n$ such that $a_i,\ b_i$ are positive real numbers. Then

$$(\prod_{i=1}^n a_i)^{\frac{1}{n}}+(\prod_{i=1}^n b_i)^{\frac{1}{n}}\le(\prod_{i=1}^n(a_i+b_i))^{\frac{1}{n}}.$$

Equality hold if and only if $a=b$.

Theorem 5.2 Let $X\in M_n$ and $A,B\in M_n^{++}$ for $0<v,\tau<1$, then we have for $(B-A)(\tau-v)\le 0$,

$$\det(A!_vB)^{\frac{1}{n}}+\frac{v(1-\nu)}{\tau(1-\tau)}\det(A\nabla_\tau B-A!_\tau B)^{\frac{1}{n}}\le\det(A\nabla_v B)^{\frac{1}{n}}.\tag{5.1}$$

Proof We may assume $0<v<\tau<1$, then $0<s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\le 1$ for $(B-A)(\tau-v)\le 0$, so we have $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\le I$. By inequality (2.2) and we denote the positive definite matrix $T=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then we have

$$\frac{((1-v)+vs_j(T))-((1-v)+vs_j(T)^{-1})^{-1}}{((1-\tau)+\tau s_j(T))-((1-\tau)+\tau s_j(T)^{-1})^{-1}}\ge\frac{v(1-v)}{\tau(1-\tau)}$$

for $j = 1, 2, \cdots, n$. It is a fact that the determinant of a positive definite matrix is product of its singular values, by Lemma 5.1, we have

$$\begin{aligned}
& \det(I\nabla_v T)^{\frac{1}{n}} = \det[(1-v)I + vT]^{\frac{1}{n}} \\
= \ & [\prod_{i=1}^{n}(1-v+vs_i(T))]^{\frac{1}{n}} \\
\geq \ & [\prod_{i=1}^{n}(1-v+vs_i(T)-(1-v+vs_i(T)^{-1})^{-1})]^{\frac{1}{n}} + [\prod_{i=1}^{n}(1-v+vs_i(T)^{-1})^{-1}]^{\frac{1}{n}} \\
\geq \ & [\prod_{i=1}^{n}\frac{v(1-v)}{\tau(1-\tau)}(1-\tau+\tau s_i(T)-(1-\tau+\tau s_i(T)^{-1})^{-1})]^{\frac{1}{n}} \\
& +[\prod_{i=1}^{n}(1-v+vs_i(T)^{-1})^{-1}]^{\frac{1}{n}} \\
= \ & \frac{v(1-v)}{\tau(1-\tau)}\det[(I\nabla_\tau T)-(I!_\tau T)]^{\frac{1}{n}} + \det(I!_v T)^{\frac{1}{n}}.
\end{aligned}$$

Multiplying $(\det A^{\frac{1}{2}})^{\frac{1}{n}}$ on the both sides of the inequalities above, we can get (5.1).

Theorem 5.3 Let $X \in M_n$ and $A, B \in M_n^{++}$ for $0 < \nu, \tau < 1$, then we have for $(B-A)(\tau-\nu) \leq 0$,

$$\det(A!_\nu B)^{\frac{2}{n}} + \frac{\nu(1-\nu)}{\tau(1-\tau)}\det(A\nabla_\tau B - A!_\tau B)^{\frac{2}{n}} \leq \det(A\nabla_\nu B)^{\frac{2}{n}}. \tag{5.2}$$

Proof Using the same technique above to (2.4), we can easily get the proof of Theorem 5.3.

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算术-调和平均不等式的改进

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摘 要: 本文研究了算术-调和平均不等式的加细. 首先利用经典分析的方法给出了关于标量情形的不等式，进而推广到算子的情形, 得出了若$0<\nu,\tau<1$, $a,b>0$ 且使$(b-a)(\tau-\nu)>0$, 则有$\frac{a\nabla_\nu b-a!_\nu b}{a\nabla_\tau b-a!_\tau b}\leq\frac{\nu(1-\nu)}{\tau(1-\tau)}$ 及$\frac{(a\nabla_\nu b)^2-(a!_\nu b)^2}{(a\nabla_\tau b)^2-(a!_\tau b)^2}\leq\frac{\nu(1-\nu)}{\tau(1-\tau)}$. 推广了W.Liao等人的结果.

关键词: 算术-调和平均; 算子不等式; Hilbert-Schmidt范数

MR(2010)**主题分类号**: 15A15; 15A42; 15A60; 47A30 **中图分类号**: O177.1