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广义 (2+1) 维浅水波类方程的有理解

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摘 要: 本文研究了广义 (2+1) 维浅水波类方程的有理解问题. 利用一般双线性算子, 求解该方程具有素数 $p=3$ 对应的一般双线性方程的多项式解, 并得到了该方程的 4 类有理解.

关 键 词: 一般双线性算子; 有理解; 广义浅水波类方程
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1 引言

近年来, 数学、物理等各个领域都在研究孤子, 其中求解孤子方程的精确解是应用数学领域中的热门话题之一. 目前已有了多种求解孤子方程精确解的方法, 如: 双线性导数法 [1]、反散射方法 [2,3]、Darboux 变换 [4,5]、tanh 方法 [6,7] 等.

这些方法中, 双线性导数法是由著名的日本数学、物理学家 Ryogo Hirota 提出, 他是在研究非线性偏微分方程的解的过程中, 利用摄动法得到一种双线性方程, 并定义出一种新的微分算子一一 Hirota 双线性算子

$$\begin{aligned} & D_x^m D_t^n f\cdot g \\ = & (\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^n f(x,t)g(x',t')\,|_{x'=x,t'=t} \\ = & \frac{\partial^m}{\partial x'^m}\frac{\partial^n}{\partial t'^n}f(x+x',t+t')g(x-x',t-t')\,|_{x'=0,t'=0}\ . \end{aligned}$$

如果某一方程具有双线性形式, 那么该方程就能具备可积性. 在 Hirota 双线性算子基础上, 马文秀 [8−11] 教授提出了一般的双线性微分算子

$$\begin{aligned} & D_{p,x}^m D_{p,t}^n f\cdot f \\ = & (\frac{\partial}{\partial x}+\alpha_p\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t}+\alpha_p\frac{\partial}{\partial t'})^n f(x,t)f(x',t')\,|_{x'=x,t'=t} \\ = & \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m-i}}{\partial x^{m-i}}\frac{\partial^i}{\partial x'^i}\frac{\partial^{n-j}}{\partial t^{n-j}}\frac{\partial^j}{\partial t'^j}f(x,t)f(x',t')\,|_{x'=x,t'=t} \\ = & \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m+n-i-j}f(x,t)}{\partial x^{m-i}\partial t^{n-j}}\frac{\partial^{i+j}f(x,t)}{\partial x^i\partial t^j},\quad m,n\ge 0\,, \end{aligned} \tag{1.1}$$

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其中 p 是素数且 $p \geq 2$, 并且 (1.1) 式中的 α_p^s 满足

$$\alpha_p^s = (-1)^{r_p(s)}, \quad s = r_p(s) \bmod p \tag{1.2}$$

且 $\alpha_p^i \alpha_p^j \neq \alpha_p^{i+j}, \quad i, j \geq 0$.

本文中, 借助这个一般双线性算子 (1.1), 从 (2+1) 维浅水波方程 [12,13]

$$u_{xxy} + 3u_x u_y - u_y - u_t = 0 \tag{1.3}$$

的双线性形式, 构造出 $p = 3$ 对应的一个广义浅水波类方程. 再通过求解该方程的一般双线性方程的多项式解, 构造了该广义浅水波类方程的有理解.

2 广义 (2+1) 维浅水波类方程

(2+1) 维浅水波方程 (1.3) 通过变换 $u = 2(\ln f)_x$ 得到其双线性形式

$$\begin{aligned} & (D_x^3 D_y - D_x D_y - D_x D_t) f \cdot f \\ = \;& 6f_{xx}f_{xy} - 6f_x f_{xxy} + 2f_{xxxy}f - 2f_{xxx}f_y - 2ff_{xy} + 2f_x f_y - 2ff_{xt} + 2f_x f_t \\ = \;& 0. \end{aligned} \tag{2.1}$$

通过计算可以证明, (2.1) 即为 $p = 2$ 时 $(D_{2,x}^3 D_{2,y} - D_{2,x}D_{2,y} - D_{2,x}D_{2,t}) f \cdot f$ 的形式.

根据一般的双线性算子 (1.1), 求得

$$\begin{aligned} & D_{3,x}^3 D_{3,y} f \cdot f = 6f_{xx}f_{xy} - 6f_{xxy}f_x, \quad D_{3,x}D_{3,y} f \cdot f = 2f_{xy}f - 2f_x f_y, \\ & D_{3,x}D_{3,t} f \cdot f = 2f_{xt}f - 2f_x f_t, \end{aligned}$$

其中 $\alpha_3 = -1, \ \alpha_3^2 = \alpha_3^3 = 1, \ \alpha_3^4 = -1, \ \alpha_3^5 = \alpha_3^6 = 1, \cdots$, 所以有

$$\begin{aligned} & (D_{3,x}^3 D_{3,y} - D_{3,x}D_{3,y} - D_{3,x}D_{3,t}) f \cdot f \\ = \;& 6f_{xx}f_{xy} - 6f_{xxy}f_x - 2f_{xy}f + 2f_x f_y - 2f_{xt}f + 2f_x f_t \\ = \;& 0. \end{aligned} \tag{2.2}$$

利用贝尔多项式理论 [14−16], 选取变换

$$u = (\ln f)_x, \tag{2.3}$$

可得

$$\frac{(D_{3,x}^3 D_{3,y} - D_{3,x}D_{3,y} - D_{3,x}D_{3,t}) f \cdot f}{f^2} = -2(3uu_{xy} + 3u^2 u_y - 3u_x u_y + u_y + u_t), \tag{2.4}$$

则有广义 (2+1) 维浅水波类方程

$$3uu_{xy} + 3u^2 u_y - 3u_x u_y + u_y + u_t = 0. \tag{2.5}$$

比较浅水波方程 (1.3) 和广义浅水波类方程 (2.5), 可以发现 (2.5) 的双线性形式 (2.2) 比 (1.3) 的双线性形式 (2.1) 更简单一些, 但 (2.5) 比 (1.3) 更具有非线性.

根据变换 (2.3), 若 f 是方程 (2.2) 的解, 则有 u 为方程 (2.5) 的解.

3 广义 (2+1) 维浅水波类方程的有理解

借助数学软件 Mathematica, 令

$$f(x,y,t)=\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{k=0}^{1}c_{i,j,k}x^iy^jt^k, \tag{3.1}$$

代入 (2.2) 式, 可以得到它的一系列的多项式解

$$\begin{aligned} f_1 \;=\;& \frac{tx^2yc_{2,0,1}\left(c_{2,0,1}+c_{2,1,0}\right)}{c_{2,0,0}}+\frac{txyc_{1,0,0}c_{2,0,1}\left(c_{2,0,1}+c_{2,1,0}\right)}{c_{2,0,0}^2}+tx^2c_{2,0,1}+\frac{txc_{1,0,0}c_{2,0,1}}{c_{2,0,0}}\\ &+\frac{tyc_{0,0,1}\left(c_{2,0,1}+c_{2,1,0}\right)}{c_{2,0,0}}-\frac{x^2y^2\left(c_{2,0,1}^2+c_{2,1,0}c_{2,0,1}\right)}{c_{2,0,0}}-\frac{xy^2c_{1,0,0}c_{2,0,1}\left(c_{2,0,1}+c_{2,1,0}\right)}{c_{2,0,0}^2}\\ &+x^2yc_{2,1,0}+\frac{xyc_{1,0,0}c_{2,1,0}}{c_{2,0,0}}+x^2c_{2,0,0}+xc_{1,0,0}-\frac{y^2c_{0,0,1}\left(c_{2,0,1}+c_{2,1,0}\right)}{c_{2,0,0}}\\ &+\frac{y\left(-c_{0,0,1}c_{2,0,0}+c_{0,0,0}c_{2,0,1}+c_{0,0,0}c_{2,1,0}\right)}{c_{2,0,0}}+tc_{0,0,1}+c_{0,0,0}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} f_2 \;=\;& tx^2yc_{2,1,1}+\frac{txyc_{1,1,0}c_{2,1,1}}{c_{2,1,0}}-tx^2c_{2,1,0}-txc_{1,1,0}-\frac{tyc_{0,0,1}c_{2,1,1}}{c_{2,1,0}}-x^2y^2c_{2,1,1}\\ &-\frac{xy^2c_{1,1,0}c_{2,1,1}}{c_{2,1,0}}+x^2yc_{2,1,0}+xyc_{1,1,0}+\frac{y^2c_{0,0,1}c_{2,1,1}}{c_{2,1,0}}-\frac{y\left(c_{0,0,1}c_{2,1,0}+c_{0,0,0}c_{2,1,1}\right)}{c_{2,1,0}}\\ &+tc_{0,0,1}+c_{0,0,0}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} f_3 \;=\;& \frac{txyc_{1,0,1}\left(c_{1,0,1}+c_{1,1,0}\right)}{c_{1,0,0}}+txc_{1,0,1}+\frac{tyc_{0,0,1}\left(c_{1,0,1}+c_{1,1,0}\right)}{c_{1,0,0}}-\frac{xy^2\left(c_{1,0,1}^2+c_{1,1,0}c_{1,0,1}\right)}{c_{1,0,0}}\\ &+xyc_{1,1,0}-\frac{y^2c_{0,0,1}\left(c_{1,0,1}+c_{1,1,0}\right)}{c_{1,0,0}}+\frac{y\left(-c_{0,0,1}c_{1,0,0}+c_{0,0,0}c_{1,0,1}+c_{0,0,0}c_{1,1,0}\right)}{c_{1,0,0}}\\ &+xc_{1,0,0}+tc_{0,0,1}+c_{0,0,0}, \end{aligned} \tag{3.4}$$

$$f_4 \;=\; tx^2y^2c_{2,2,1}+\frac{txy^2c_{1,2,0}c_{2,2,1}}{c_{2,2,0}}+\frac{ty^2c_{0,2,0}c_{2,2,1}}{c_{2,2,0}}+x^2y^2c_{2,2,0}+xy^2c_{1,2,0}+y^2c_{0,2,0}. \tag{3.5}$$

对应地, 根据变换 $u=(\ln f)_x$, 可以求得广义 (2+1) 维浅水波类方程 (2.5) 的 4 类有理解.

第一类有理解

$$u_1=\frac{p}{q}, \tag{3.6}$$

其中

$$\begin{aligned} p \;=\;& 2txc_{2,0,1}c_{2,0,0}-2xyc_{2,0,1}c_{2,0,0}+2xc_{2,0,0}^2-yc_{1,0,0}c_{2,0,1}+tc_{1,0,0}c_{2,0,1}+c_{1,0,0}c_{2,0,0},\\ q \;=\;& tx^2c_{2,0,0}c_{2,0,1}-x^2yc_{2,0,0}c_{2,0,1}-xyc_{1,0,0}c_{2,0,1}+txc_{1,0,0}c_{2,0,1}+x^2c_{2,0,0}^2+xc_{1,0,0}c_{2,0,0}\\ &-yc_{0,0,1}c_{2,0,0}+tc_{0,0,1}c_{2,0,0}+c_{0,0,0}c_{2,0,0}. \end{aligned}$$

第二类有理解

$$u_2=-\frac{2txc_{2,1,0}-2xyc_{2,1,0}-yc_{1,1,0}+tc_{1,1,0}}{x^2yc_{2,1,0}-tx^2c_{2,1,0}+xyc_{1,1,0}-txc_{1,1,0}-yc_{0,0,1}+tc_{0,0,1}+c_{0,0,0}}. \tag{3.7}$$

第三类有理解

$$u_3=\frac{tc_{1,0,1}-yc_{1,0,1}+c_{1,0,0}}{txc_{1,0,1}-xyc_{1,0,1}+xc_{1,0,0}-yc_{0,0,1}+tc_{0,0,1}+c_{0,0,0}}. \tag{3.8}$$

第四类有理解

$$u_4=\frac{2xc_{2,2,0}+c_{1,2,0}}{x\left(xc_{2,2,0}+c_{1,2,0}\right)+c_{0,2,0}}. \tag{3.9}$$

4 结论分析

本文中, 在 (2+1) 维浅水波方程 (1.3) 的基础上, 利用一般的双线性微分算子 (1.1), 当素数 $p=3$ 时, 得到了具有一般双线性形式的微分方程一一广义 (2+1) 维浅水波类方程 (2.5). 借助广义 (2+1) 维浅水波类方程 (2.5) 的一般双线性形式, 利用数学软件 Mathematica, 得到了方程 (2.5) 的 4 类有理解.

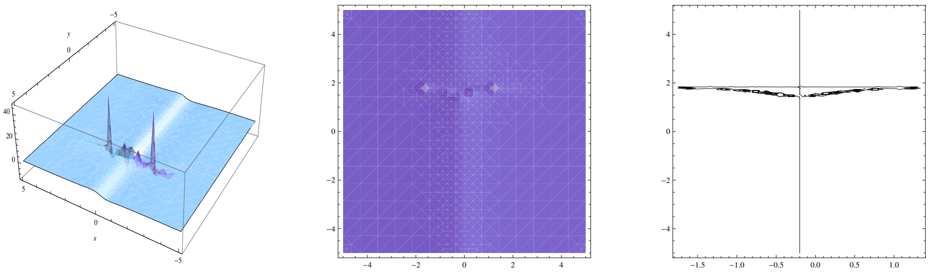

图 1：解 (4.2) 在 $t=1$ 时的三维图 (左), 密度图 (中) 和等高线图 (右)

当参数被选取为

$$c_{i,j,k}=1+i^2+j^2+k^2,\quad 0\le i,j\le 2,\quad 0\le k\le 1 \tag{4.1}$$

时, 解 (3.6) – (3.8) 分别为

$$u_1=\frac{60tx-60xy+12t+50x-12y+10}{30tx^2+12tx-30x^2y-12xy+25x^2+10x-10y+10t+5}, \tag{4.2}$$

$$u_2=-\frac{12tx-12xy+3t-3y}{6x^2y-6tx^2+3xy-3tx+2t-2y+1}, \tag{4.3}$$

$$u_3=\frac{3t-3y+2}{3tx-3xy+2x+2t-2y+1}. \tag{4.4}$$

即得到广义 (2+1) 维浅水波类方程 (2.5) 的 3 类特殊有理解. 解 (4.2), (4.3) 和 (4.4) 在 $t=1$ 时刻的三维图、密度图和等高线图如图 1–3 所示. 从这些图能看出, $y=0$ 直线附近解曲面变化很大, 且当 x 趋于无穷大时这些解都趋向于 0, 即解曲面趋向水平面.

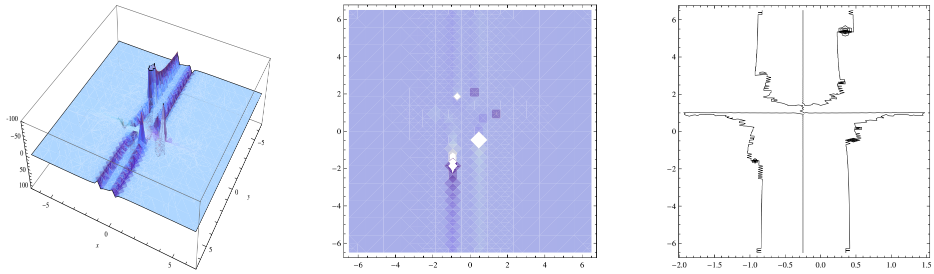

图 2: 解 (4.3) 在 $t=1$ 时的三维图 (左), 密度图 (中) 和等高线图 (右)

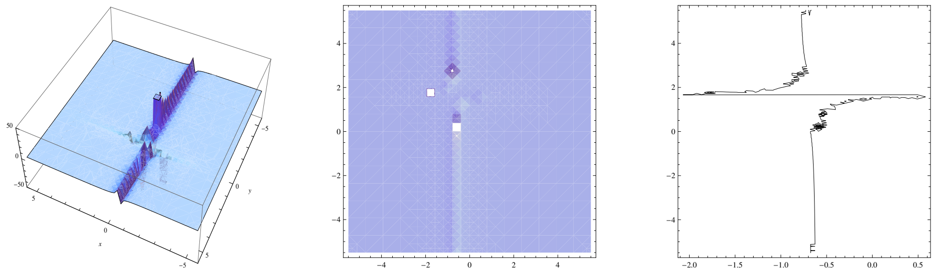

图 3: 解 (4.4) 在 $t=1$ 时的三维图 (左), 密度图 (中) 和等高线图 (右)

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RATIONAL SOLUTIONS TO A GENERALIZED (2+1)-DIMENSIONAL SHALLOW–WATER–WAVE–LIKE EQUATION

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Abstract: In this paper, we study the rational solutions of (2+1)-dimensional shallow-water-wave-like equation. By using the generalized bilinear operators, the polynomial solutions of the generalized bilinear equation with the prime number of $p = 3$ are solved, and four classes of rational solutions to the equation are obtained.

Keywords: generalized bilinear operator; rational solution; generalized shallow–water–wave–like equation

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