Vol. 39 (2019)
No. 6

数 学 杂 志
J. of Math. (PRC)

AN INVESTIGATION ON THE EXISTENCE AND ULAM STABILITY OF SOLUTION FOR AN IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATION

GUO Yu-chen, SHU Xiao-bao

(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*)

Abstract: In this paper, we investigate the existence and Ulam stability of solution for impulsive Riemann-Liouville fractional neutral function differential equation with infinite delay of order $1 < \beta < 2$. Firstly, the solution for the equation is proved. By using the fixed point theorem as well as Hausdorff measure of noncompactness, the existence results are obtained and the Ulam stability of the solution is proved.

Keywords: impulsive Riemann-Liouville fractional differential equation; the fixed point theorem; Hausdorff measure of noncompactness; Ulam stability

2010 MR Subject Classification: 35R10; 34K45; 26A33

Document code: A **Article ID:** 0255-7797(2019)06-0835-17

1 Introduction

Fractional differential equation, as an excellent tool for describing memory and hereditary properties of various materials and processes in natural sciences and engineering, received a great deal of attention in the literature [1–4] and there were some works on the investigation of the solution of fractional differential equation [5,6].

On the other hand, Riemann-Liouville fractional derivatives or integrals are strong tools for resolving some fractional differential problems in the real world. It is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals which were verified by Heymans and Podlubny[7], and such initial conditions are more appropriate than physically interpretable initial conditions. For another, they considered the impulse response with Riemann-Liouville fractional derivatives as widely used in the fields of physics, such as viscoelasticity.

In recent years, many authors investigated the existence and stability of solutions to fractional differential equations with Caputo fractional derivative, and there were a lot of interesting and excellent results on this fields. However, there is still little literature on the

* **Received date:** 2018-06-01 **Accepted date:** 2019-01-14

Biography: Guo Yuchen (1997–), male, born at Hebi, Henan, master, major in fractional differential equation. E-mail: guoyuchen1228@vip.qq.com .

Corresponding Author: Shu Xiaobao

existence and stability of solutions to Riemann-Liouville fractional differential equations. Three years ago, Weera Yukunthorn et al.[8] studied the existence and uniqueness of solutions to impulsive multiorders Riemann-Liouville fractional differential equations

$$\begin{cases} D^{\alpha_k}_{t_k}x(t) = f(t, x(t)), \quad t \in J, \quad t \neq t_k, \\ \tilde{\Delta}x(t_k) = \varphi_k(x(t_k)), \quad \Delta^* x(t_k) = \varphi^*_k(x(t_k)), \quad k = 1, 2, \cdots, m, \\ x(0) = 0, \quad D^{\alpha_0 - 1}x(0) = \beta, \end{cases}$$

where $\beta \in R, 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times R \to R$ is a continuous function, $\varphi_k, \varphi^*_k \in C(R, R)$ for $k = 1, 2, \cdots, m$, and $D^{\alpha_k}_{t_k}$ is the Riemann-Liouville fractional derivative of order $1 < \alpha_k < 2$ on intervals J_k for $k = 0, 1, 2, \cdots, m$. The notation $\tilde{\Delta}x(t_k)$ is defined by

$$\tilde{\Delta}x(t_k) = I^{1-\alpha_k}_{t_k}x(t^+_k) - I^{1-\alpha_{k-1}}_{t_{k-1}}x(t_k), \quad k = 1, 2, \cdots, m,$$

and $\Delta^* x(t_k)$ is defined by

$$\Delta^* x(t_k) = I^{2-\alpha_k}_{t_k}x(t^+_k) - I^{2-\alpha_{k-1}}_{t_{k-1}}x(t_k), \quad k = 1, 2, \cdots, m,$$

where $I^{2-\alpha_k}_{t_k}$ is the Riemann-Liouville fractional integral of order $2 - \alpha_k > 0$ on J_k. By using Banach's fixed point theorem, the authors developed the existence theorem for such equations.

Motivated by this work, we use Mönch's fixed point theorem via measure of noncompactness as well as the basic theory of Ulam stability to investigate the existence and stability of solution to the following impulsive Riemann-Liouville fractional neutral function differential equation with infinite delay in a Banach space X.

$$\begin{cases} D^{\beta}_{0^+}[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, T], \quad t \neq t_k, \\ \Delta I^{2-\beta}_{0^+}x(t_k) = I_k(x(t^-_k)), \quad \Delta I^{1-\beta}_{0^+}x(t_k) = J_k(x(t^-_k)), \\ I^{2-\beta}_{0^+}[x(0) - g(0, x_0)] = \varphi_1 \in B_v, I^{1-\beta}_{0^+}[x(0) - g(0, x_0)] = \varphi_2 \in B_v, \end{cases} \tag{1.1}$$

where $k = 1, 2, \cdots, m$ and $D^{\beta}_{0^+}$ is the Riemann-Liouville fractional derivative of order $1 < \beta < 2$. $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, Let $T_k = (t_k, t_{k+1}], k = 1, 2, \cdots, m, T_0 = [0, t_1]$. $f : J \times B_v$ and $g : J \times B_v$ are given functions, where B_v is the phase space defined in Section 2. The impulsive functions $I_k, J_k : X \to X$ $(k = 1, 2, \cdots, m)$ is an appropriate functions. The notation $\Delta I^{2-\beta}_{0^+}x(t_k), \Delta I^{1-\beta}_{0^+}x(t_k)$ is defined by

$$\Delta I^{2-\beta}_{0^+}x(t_k) = I^{2-\beta}_{0^+}x(t^+_k) - I^{2-\beta}_{0^+}x(t^-_k),$$

$$\Delta I^{1-\beta}_{0^+}x(t_k) = I^{1-\beta}_{0^+}x(t^+_k) - I^{1-\beta}_{0^+}x(t^-_k), \quad k = 1, 2, \cdots m,$$

where $I^{2-\beta}_{0^+}, I^{1-\beta}_{0^+}$ is the Riemann-Liouville fractional integral of order $2 - \beta$, $1 - \beta$. The histories $x_t : (-\propto, 0] \to X$, defined by $x_t(s) = x(t + s)$, $s \leq 0$, belong to some abstract phase space B_v.

The rest of the paper is organized as follows: in section 2, some basic definitions, notations and preliminary facts that are used throughout the paper are presented. In Section 3, we prove the solution of the equation and present the main results for problem (1.1).

2 Preliminaries

In this section, we mention some notations, definitions, lemmas and preliminary facts needed to establish our main results.

Let X be a complex Banach space, whose norm is denoted by $\|\cdot\|$. Let $J=[0,T], J_0=[0,t_1], J_k=(t_k,t_{k+1}]$ for $k=1,2,\cdots,m$. Let

$$PC(J,X):=\{x:J\to X, \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^+) \text{ and } x(t_k^-) \text{ exist, and } x(t_k^-)=x(t_k), k=1,2,\cdots,m\}.$$

We introduce the space $C_{2-\beta,k}(J_k,X):=\{x:J_k\to X: t^{2-\beta}x(t)\in C(J_k,X)\}$ with the norm $\|x\|_{C_{2-\beta,k}}=\sup\limits_{t\in J_k}|t^{2-\beta}\|x(t)\||$ and $PC_{2-\beta}=\{x:J\to X: \text{ for each } t\in J_k \text{ and } t^{2-\beta}x(t)\in C(J_k,X), k=0,1,2,\cdots,m\}$ with the norm

$$\|x\|_{PC_{2-\beta}}=\max\sup_{t\in J_k}|t^{2-\beta}\|x(t)\||:k=0,1,2,\cdots,m.$$

Clearly $PC_{2-\beta}$ is a Banach space. We use $B_r(x,X)$ to denote the closed ball in X with center at x and radius r.

Before introducing the fractional-order functional differential equation with infinite delay, we define the abstract phase space B_v. Let $v:(\infty,0]\to(0,\infty)$ be a continuous function that satisfies $l=\int_{-\infty}^{0}v(t)dt<+\infty$. The Banach space $(B_v,\|\cdot\|_{B_v})$ induced by v is then given by

$$B_v:=\{\varphi:(-\infty,0)\to X: \text{for any } c>0,\ \varphi(\theta)\ \text{ is a bounded and measurable function on [-c, 0], and } \int_{-\infty}^{0}v(s)\sup_{s\le\theta\le0}\|\varphi(\theta)\|ds<+\infty\}$$

endowed with the norm $\|\varphi\|_{B_v}:=\int_{-\infty}^{0}v(s)\sup\limits_{s\le\theta\le0}\|\varphi(\theta)\|ds$.

Define the following space

$$B_v':=\{\varphi:(-\infty,T]\to X:\varphi_k\in C^1(J_k,X),k=0,1,2,\cdots,m, \text{ and there exist } \varphi(t_k^-) \text{ and } \varphi(t_k^+) \text{ with } \varphi(t_k)=\varphi(t_k^-),\varphi_0=\phi\in B_v\},$$

where φ_k is the restriction of φ to J_k, $J_0=[0,t_1]$, $J_k=(t_k,t_{k+1}]$, $k=1,2,\cdots,m$.

We use $\|\cdot\|_{B_v'}$ to denote a seminorm in the space B_v' defined by

$$\|\varphi\|_{B_v'}:=\|\phi\|_{B_v}+\max\{\|\varphi_k\|_{J_k(2-\beta)},\ \ k=0,1,\cdots,m\},$$

where

$$\phi=\varphi_0,\|\varphi_k\|_{J_k(2-\beta)}=\sup_{s\in J_k}\{s^{2-\beta}\|\varphi(s)\|\}.$$

Now we consider some definitions about fractional differential equations.

Definition 2.1 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f;(a,b)\to X$ is defined by

$$D^{\alpha}_{a^+}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1}f(s)ds,\quad n-1<\alpha<n,\quad t\in(a,b),$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of number α, provided the right-hand side is pointwise defined on (a,b), Γ is the gamma function.

Definition 2.2 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f:(a,b)\to X$ is defined by

$$I^{\alpha}_{a^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,\quad t\in(a,b),$$

provided the right-hand side is pointwise defined on (a,b).

Lemma 2.1 (see [9]) Let $\alpha>0$. Then for $x\in C(a,b)\cap L(a,b)$, it holds

$$D^{\alpha}_{a^+}I^{\alpha}_{a^+}x(t)=x(t),$$

$$I^{\alpha}_{a}D^{\alpha}_{a}x(t)=x(t)-\sum_{j=1}^{n}\frac{(I^{n-\alpha}_a)^{(n-j)}x(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j},$$

where $n-1<\alpha<n$.

Lemma 2.2 (see [9]) If $\alpha\ge 0$ and $\beta>0$, then

$$I^{\alpha}_{a^+}(t-s)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1},$$

$$D^{\alpha}_{a^+}(t-s)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}.$$

Before investigating the solutions to equation (1.1), we consider a simplified version of (1.1), given by

$$\begin{cases} D^{\beta}_{0^+}x(t)=f(t),\quad t\in[0,T],\quad t\neq t_k,\\ \Delta I^{2-\beta}_{0^+}x(t_k)=y_k,\quad \Delta I^{1-\beta}_{0^+}x(t_k)=\overline{y_k},\\ I^{2-\beta}_{0^+}x(0^+)=x_0,\quad I^{1-\beta}_{0^+}x(0^+)=x_1,\end{cases}\tag{2.1}$$

where $k=1,2,\cdots,m,x_0,x_1,y_k,\overline{y_k}\in X$ and $D^{\beta}_{0^+}$ is the Riemann-Liouville fractional derivative of order $1<\beta<2$.

Theorem 2.1 Let $1<\beta<2$ and $f:J\to X$ be continuous. If $x\in PC_{2-\beta}(J,X)$ is a solution of (2.1) if and only if x is a solution of the following the fractional integral equation

$$x(t)=\begin{cases}\dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t(t-s)^{\beta-1}f(s)ds+x_1\frac{t^{\beta-1}}{\Gamma(\beta)}+x_0\frac{t^{\beta-2}}{\Gamma(\beta-1)},\quad t\in[0,t_1],\\ \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t(t-s)^{\beta-1}f(s)ds+x_1\frac{t^{\beta-1}}{\Gamma(\beta)}+x_0\frac{t^{\beta-2}}{\Gamma(\beta-1)}\\ +\displaystyle\sum_{i=1}^{k}\overline{y_i}\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)+\sum_{i=1}^{k}y_i\frac{t^{\beta-2}}{\Gamma(\beta-1)},\quad t\in(t_k,t_{k+1}],\end{cases}\tag{2.2}$$

where $k=1,2,\cdots,m$.

Proof For all $t\in(t_k,t_{k+1}]$ where $k=0,1,\cdots,m$ by Lemma 2.1 and 2.2, we obtain

$$\begin{aligned}
D^{\beta}_{0^+}x(t)=&D^{\beta}_{0^+}\left[\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}f(s)ds+x_1\frac{t^{\beta-1}}{\Gamma(\beta)}+x_0\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right.\\
&\left.+\sum_{i=1}^{k}\overline{y_i}\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)+\sum_{i=1}^{k}y_i\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right]\\
=&D^{\beta}_{0^+}I^{\beta}_{0^+}f(t)+D^{\beta}_{0^+}\left[x_1\frac{t^{\beta-1}}{\Gamma(\beta)}+x_0\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right.\\
&\left.+\sum_{i=1}^{k}\overline{y_i}\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)+\sum_{i=1}^{k}y_i\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right]\\
=&f(t).
\end{aligned}$$

Thus, expression (2.2) satisfies the first equation of problem (2.1). For $k=1,2,\cdots,m$, it follows from (2.1) that

$$\begin{aligned}
I^{2-\beta}_{0^+}x(t)=&I^{2-\beta}_{0^+}\left[I^{\beta}_{0^+}f(t))+x_1\frac{t^{\beta-1}}{\Gamma(\beta)}+x_0\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right.\\
&\left.+\sum_{i=1}^{k}\overline{y_i}\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)+\sum_{i=1}^{k}y_i\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right]\\
=&I^{2}_{0^+}f(t)+x_1I^{2-\beta}_{0^+}(\frac{t^{\beta-1}}{\Gamma(\beta)})+x_0I^{2-\beta}_{0^+}(\frac{t^{\beta-2}}{\Gamma(\beta-1)})\\
&+\sum_{i=1}^{k}\overline{y_i}\left[I^{2-\beta}_{0^+}(\frac{t^{\beta-1}}{\Gamma(\beta)})-t_iI^{2-\beta}_{0^+}(\frac{t^{\beta-2}}{\Gamma(\beta-1)})\right]+\sum_{i=1}^{k}y_iI^{2-\beta}_{0^+}(\frac{t^{\beta-2}}{\Gamma(\beta-1)})\\
=&I^{2}_{0^+}f(t)+x_1t+x_0+\sum_{i=1}^{k}\overline{y_i}(t-t_i)+\sum_{i=1}^{k}y_i,\\
I^{1-\beta}_{0^+}x(t)=&I^{1-\beta}_{0^+}\left[I^{\beta}_{0^+}f(t)+x_1\frac{t^{\beta-1}}{\Gamma(\beta)}+x_0\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right.\\
&\left.+\sum_{i=1}^{k}\overline{y_i}\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)+\sum_{i=1}^{k}y_i\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right]\\
=&I_{0^+}f(t)+x_1+\sum_{i=1}^{k}\overline{y_i}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\Delta I^{2-\beta}_{0^+}x(t_k)&=I^{2-\beta}_{0^+}x(t_k^+)-I^{2-\beta}_{0^+}x(t_k^-)\\
&=\sum_{i=1}^{k}\overline{y_i}(t_k-t_i)+\sum_{i=1}^{k}y_i-\sum_{i=1}^{k-1}\overline{y_i}(t_k-t_i)-\sum_{i=1}^{k-1}y_i\\
&=y_k,
\end{aligned}$$

$$\Delta I_{0^+}^{1-\beta}x(t_k) = I_{0^+}^{1-\beta}x(t_k^+) - I_{0^+}^{1-\beta}x(t_k^-) = \sum_{i=1}^{k}\overline{y_i} - \sum_{i=1}^{k-1}\overline{y_i} = \overline{y_k}.$$

Consequently, all the conditions of problem (2.1) are satisfied. Hence, (2.2) is a solution of problem (2.1)

Next, based on Theorem, we consider the solutions of the Cauchy problem(1.1)

Definition 2.3 Suppose function $x : (-\propto, T] \to X$. The solution of the fractional differential equation, given by

$$x(t) = \begin{cases} x_0 = \phi \in B_v, \qquad t \in (-\propto, 0], \\ \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} f(s, x_s)ds + g(t, x_t) \\ +\varphi_2 \dfrac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \dfrac{t^{\beta-2}}{\Gamma(\beta-1)}, t \in [0, t_1], \\ \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} f(s, x_s)ds + g(t, x_t) + \varphi_2 \dfrac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \dfrac{t^{\beta-2}}{\Gamma(\beta-1)} \\ +\displaystyle\sum_{i=1}^{k} J_i(x(t_i^-)) \dfrac{t^{\beta-2}}{\Gamma(\beta-1)}(\dfrac{t}{\beta-1} - t_i) \\ +\displaystyle\sum_{i=1}^{k} I_i(x(t_i^-)) \dfrac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}], \end{cases}$$

will be called a fundamental solution of problem (1.1).

Lemma 2.3 (see [10]) Assume $x \in B_v^{'}$, then for $t \in J,\ x_t \in B_v$. Moreover

$$l\|x(t)\| \le \|x_t\|_{B_v} \le \|\phi\|_{B_v} + l \sup_{s\in[0,t]} \|s^{2-\beta}x(s)\|,$$

where $l = \displaystyle\int_{-\propto}^{0} v(t)dt < +\propto, \phi = x_0$.

Next, we consider some definitions and properties of the measures of noncompactness.

The Hausdorff measure of noncompactness $\beta(\cdot)$ defined on each bounded subset $\mathscr{B}$ of Banach space X is given by

$$\beta(\mathscr{B}) = \inf\{\varepsilon > 0; \mathscr{B} \quad \text{has a finite } \ \varepsilon- \ \text{ net in } \ X\}.$$

Some basic properties of $\beta(\cdot)$ are given in the following lemma.

Lemma 2.4 (see [11–13]) If X is a real Banach space and $\mathscr{B}, \mathscr{D} \subset X$ are bounded, then the following properties are satisfied

(1) monotone: if for all bounded subsets $\mathscr{B}, \mathscr{D}$ of X, $\mathscr{B} \subseteq \mathscr{D}$ implies $\beta(\mathscr{B}) \le \beta(\mathscr{D})$;

(2) nonsingular: $\beta(\{x\} \cup \mathscr{B}) = \beta(\mathscr{B})$ for every $x \in X$ and every nonempty subset $\mathscr{B} \subset X$;

(3) regular: $\mathscr{B}$ is precompact if and only if $\beta(\mathscr{B}) = 0$;

(4) $\beta(\mathscr{B} + \mathscr{D}) \le \beta(\mathscr{B}) + \beta(\mathscr{D})$, where $\mathscr{B} + \mathscr{D} = \{x + y; x \in \mathscr{B}, y \in \mathscr{D}\}$;

(5) $\beta(\mathscr{B} \cup \mathscr{D}) \le \max\{\beta(\mathscr{B}), \beta(\mathscr{D})\}$;

(6) $\beta(\lambda\mathscr{B}) \leq |\lambda|\beta(\mathscr{B})$;

(7) if $W \subset C(J; X)$ is bounded and equicontinuous, then $t \to \beta(W(t))$ is continuous on J, and

$$\beta(W) \leq \max_{t\in J} \beta(W(t)), \tag{2.1}$$

$$\beta\left(\int_0^t W(s)ds\right) \leq \int_0^t \beta(W(s))ds \text{ for all } t \in J, \tag{2.2}$$

where

$$\int_0^t W(s)ds = \left\{\int_0^t u(s)ds \text{ for all } u \in W, t \in J\right\};$$

(8) if $\{u_n\}_1^\infty$ is a sequence of Bochner integrable functions from J into X with $\|u_n(t)\| \leq \widehat{m}(t)$ for almost all $t \in J$ and every $n \geq 1$, where $\widehat{m}(t) \in L(J; R^+)$, then the function $\psi(t) = \beta(\{u_n\}_{n=1}^\infty)$ belongs to $L(J; R^+)$ and satisfies

$$\beta\left(\left\{\int_0^t u_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^t \psi(s)ds; \tag{2.3}$$

(9) if W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^\infty \subset W$ such that

$$\beta(W) \leq 2\beta(\{u_n\}_{n=1}^\infty) + \varepsilon. \tag{2.4}$$

The following lemmas about the Hausdorff measure of noncompactness will be used in proving our main results.

Lemma 2.5 (see [14]) Let D be a closed convex subset of a Banach Space X and $0 \in D$. Assume that $F : D \to X$ is a continuous map which satisfies the Mönch's condition, that is, $M \subseteq D$ is countable, $M \subseteq \overline{\text{co}}\,(0 \cup F(M)) \Rightarrow \overline{M}$ is compact. Then F has a fixed point in D.

Next, we consider the Ulam stability for the equation.

Consider the following inequality

$$\|D_{0^+}^\beta[x(t) - g(t, x_t)] - f(t, x_t)\| < \varepsilon.$$

Definition 2.4 Equation (1.1) is Hyers-Ulam stable if, for any $\varepsilon > 0$, there exists a solution $y(t)$ which satisfies the above inequality and has the same initial value as $x(t)$, where $x(t)$ is a solution to (1.1). Then $y(t)$ satisfies $\|y(t) - x(t)\| < K\varepsilon$ in which K is a constant.

3 Existence

To prove our main results, we list the following basic assumptions of this paper.

(H$_1$) The function $f : J \times B_v \to X$ satisfies the following conditions.

(i) $f(\cdot, \phi)$ is measurable for all $\phi \in B_v$ and $f(t, \cdot)$ is continuous for a.e. $t \in J$.

(ii) There exist a constant $\alpha_1 \in (0, \alpha)$, $m \in L^{\frac{1}{\alpha_1}}(J, R^+)$ and a positive integrable function $\Omega : R^+ \to R^+$ such that $\|f(t, \phi)\| \leq m(t)\Omega(\|\phi\|_{B_v})$ for all $(t, \phi) \in J \times B_v$, where Ω satisfies $\liminf\limits_{n\to\propto} \frac{\Omega(n)}{n} = 0$.

(iii) There exist a constant $\alpha_2 \in (0,\alpha)$ and a function $\eta \in L^{\frac{1}{\alpha_2}}(J,R^+)$ such that, for any bounded subset $F_1 \subset B_v$,

$$\beta(f(t,F_1)) \le \eta(t)[\sup_{\theta\in(-\propto,0]} \beta(F_1(\theta))]$$

for a.e. $t \in J$, where $F_1(\theta)=\{v(\theta): v\in F_1\}$ and β is the Hausdorff MNC.

(H_2) The function $g: J\times B_v \to X$ satisfies the following conditions.

(i) g is continuous and there exist a constant $H_1>0$ and

$$\|t^{2-\beta}g(t,x)\| \le H_1(1+\|x\|_{B_v}).$$

(ii) There exist a constant $\alpha_3 \in (0,\alpha)$ and $g^* \in L^{\frac{1}{\alpha_3}}(J,R^+)$ such that, for any bounded subset $F_2 \subset B_v$,

$$\beta(g(t,F_2)) \le g^*(t) \sup_{\theta\in(-\propto,0]} \beta(F_2(\theta)), \quad G=\sup_{t\in J} g^*(t).$$

(H_3) $I_k, J_k: X\to X, k=1,2,\cdots,m$ are continuous functions and satisfy

$$\begin{aligned}&\|I_k(x)\|_X \le c_k\|x\|_{B_v'}, \quad \|J_k(x)\|_X \le f_k\|x\|_{B_v'},\\ &\beta(t^{\beta-2}I_k(F_3)) \le K_k \sup_{\theta\in(-\propto,T]}\beta(F_3(\theta)),\\ &\beta(t^{\beta-2}J_k(F_4)) \le M_k \sup_{\theta\in(-\propto,T]}\beta(F_4(\theta)),\end{aligned}$$

where $c_k, f_k, K_k, M_k>0$. $F_3, F_4 \subset B_v^{'}$.

(H_4): $H_1 l + \frac{T^*}{\Gamma^*}\sum\limits_{i=1}^{m}(f_i+c_i) < 1$,

$$M^* = \frac{2T^\beta}{\Gamma(\beta+1)}\|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)} + G + \frac{T^*}{\Gamma^*}\sum_{i=1}^{m}(M_i+K_i) < 1,$$

where $T^*=\max\{1,T,T^2\}$, $\Gamma^*=\min\{\Gamma(\beta+1),\Gamma(\beta),\Gamma(\beta-1)\}$.

Theorem 3.1 Suppose conditions $(\mathrm{H}_1)-(\mathrm{H}_4)$ are satisfied. Then system(1.1) has at least one solution on J.

Proof We define the operator $\Gamma: B_v^{'} \to B_v^{'}$ by

$$\Gamma x(t) = \begin{cases} x_0=\phi\in B_v, \qquad t\in(-\propto,0] \\ \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1}f(s,x_s)ds + g(t,x_t) + \varphi_2\frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1\frac{t^{\beta-2}}{\Gamma(\beta-1)}, t\in[0,t_1], \\ \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1}f(s,x_s)ds + g(t,x_t) + \varphi_2\frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ +\displaystyle\sum_{i=1}^{k} J_i(x(t_i^-))\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i) \\ +\displaystyle\sum_{i=1}^{k} I_i(x(t_i^-))\frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t\in(t_k,t_{k+1}]. \end{cases}$$

The operator Γ has a fixed point if and only if system (1.1) has a solution. For $\phi \in B_v$, denote

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\propto, 0], \\ 0, & t \in J. \end{cases}$$

Then $\hat{\phi}(t) \in B_v^{'}$. Let

$$x(t) = y(t) + \hat{\phi}(t), \quad -\propto < t \le T.$$

It is easy to see that y satisfies $y_0 = 0, t \in (-\propto, 0]$ and

$$y(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)ds + g(t, y_t + \hat{\phi}_t) \\ +\varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)ds + g(t, y_t + \hat{\phi}_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ +\sum_{i=1}^{k} J_i(y(t_i^-) + \hat{\phi(t_i^-)}) \frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1} - t_i) \\ +\sum_{i=1}^{k} I_i(y(t_i^-) + \hat{\phi(t_i^-)}) \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}] \end{cases}$$

if and only if $x(t)$ satisfies $x(t) = \phi(t), t \in (-\propto, 0]$ and

$$x(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, x_s)ds + g(t, x_t) \\ +\varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, x_s)ds + g(t, x_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ +\sum_{i=1}^{k} J_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1} - t_i) \\ +\sum_{i=1}^{k} I_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}]. \end{cases}$$

Define the Banach space $(B_v^{''}, \|\cdot\|_{B_v^{''}})$, induced by $B_v^{'}$

$$B_v^{''} = \{y : y \in B_v^{'}, y_0 = 0\}$$

with the norm

$$\|y(t)\|_{B_v^{''}} = \sup\{s^{2-\beta}\|y(s)\|_X, s \in [0, T]\}.$$

Let $B_r = \{y \in B_v^{''} : \|y\|_{B_v^{'}} \le r\}$. Then for each r, B_r is a bounded, close and convex subset. For any $y \in B_r$, it follows from Lemma 2.3 that

$$\begin{aligned} \|y_t + \hat{\phi}_t\|_{B_v} &\le \|y_t\|_{B_v} + \|\hat{\phi}_t\|_{B_v} \\ &\le l \sup_{s\in[0,t]} s^{2-\beta}\|x(s)\| + \|\phi\|_{B_v} \\ &\le lr + \|\phi\|_{B_v} = r'. \end{aligned}$$

We define the operator $N : B_v^{''} \to B_v^{''}$ by

$$Ny(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)ds + g(t, y_t + \hat{\phi}_t) \\ +\varphi_2 \dfrac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \dfrac{t^{\beta-2}}{\Gamma(\beta-1)}, t \in [0, t_1], \\ \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)ds \\ +g(t, y_t + \hat{\phi}_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ +\displaystyle\sum_{i=1}^{k} J_i(y(t_i^-) + \phi(\hat{t_i^-}))\dfrac{t^{\beta-2}}{\Gamma(\beta-1)}(\dfrac{t}{\beta-1} - t_i) \\ +\displaystyle\sum_{i=1}^{k} I_i(y(t_i^-) + \phi(\hat{t_i^-}))\dfrac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}]. \end{cases}$$

Step 1 We prove that there exists some $r > 0$ such that $N(B_r) \subset B_r$. If this is not true, then, for each positive integer r, there exist $y_r \in B_r$ and $t_r \in (-\propto, T]$ such that $\|(Ny_r)(t_r)\| > r$. On the other hand, it follows from the assumption that

$$\begin{aligned} t_r^{2-\beta}\|N(y_r(t_r))\| \le & \left\| \frac{1}{\Gamma(\beta)}\int_0^{t_r} (t_r - s)^{\beta-1} t_r^{2-\beta} f(s, (y_r)_s + \hat{\phi}_s)ds \right\| \\ & + \left\| t_r^{2-\beta} g(t_r, (y_r)_{t_r} + \hat{\phi}_{t_r}) \right\| \\ & + \left\| \varphi_2 \frac{t_r^{2-\beta+\beta-1}}{\Gamma(\beta)} \right\| + \left\| \varphi_1 \frac{t_r^{2-\beta+\beta-2}}{\Gamma(\beta-1)} \right\| \\ & + \left\| \sum_{i=1}^{k} J_i(y_r(t_i^-) + \phi(\hat{t_i^-}))\frac{t_r^{2-\beta+\beta-2}}{\Gamma(\beta-1)}(\frac{t_r}{\beta-1} - t_i) \right\| \\ & + \left\| \sum_{i=1}^{k} I_i(y_r(t_i^-) + \phi(\hat{t_i^-}))\frac{t_r^{2-\beta+\beta-2}}{\Gamma(\beta-1)} \right\| \\ \le & \left\| \frac{T^{2-\beta}}{\Gamma(\beta)}\int_0^{t_r} (t_r - s)^{\beta-1} f(s, (y_r)_s + \hat{\phi}_s)ds \right\| + H_1(1 + \left\| (y_r)_{t_r} + \hat{\phi}_t \right\|_{B_v}) \\ & + \frac{T}{\Gamma(\beta)}\|\varphi_2\| + \frac{1}{\Gamma(\beta-1)}\|\varphi_1\| \\ & + \frac{T}{\Gamma(\beta)}\left\| \sum_{i=1}^{k} J_i(y_r(t_i^-) + \phi(\hat{t_i^-})) \right\| \\ & + \frac{1}{\Gamma(\beta-1)}\left\| \sum_{i=1}^{k} I_i(y_r(t_i^-) + \phi(\hat{t_i^-})) \right\| \\ \le & \frac{T^2}{\Gamma(\beta+1)} m(t)\Omega(r') + H_1 + H_1 r' \\ & + \frac{T^*}{\Gamma^*}(\|\varphi_2\| + \|\varphi_1\|) + \left(\frac{T^*}{\Gamma^*}\sum_{i=1}^{m} f_i + c_i \right) r. \end{aligned}$$

So we have

$$
\begin{aligned}
r &< \|Ny_r(t_r)\|_{B_v''} \\
&\le \frac{T^2}{\Gamma(\beta+1)} m(t)\Omega(r') + H_1 + H_1 r' \\
&\quad + \frac{T^*}{\Gamma^*}(\|\varphi_2\| + \|\varphi_1\|) + \left(\frac{T^*}{\Gamma^*}\sum_{i=1}^{m} f_i + c_i\right) r.
\end{aligned}
$$

Dividing both sides by r and taking $r \to +\propto$ from

$$
\lim_{r\to\propto} \frac{r'}{r} = \lim_{r\to\propto} \frac{lr + \|\phi\|_{B_v}}{r} = l
$$

and

$$
\lim_{n\to\propto} \inf \frac{\Omega(n)}{n} = 0
$$

yields

$$
H_1 l + \frac{T^*}{\Gamma^*}\sum_{i=1}^{m}(f_i + c_i) < 1.
$$

This contradicts (H_4). Thus, for some number $r, N(B_r) \subset B_r$.

Step 2 N is continuous on B_r. Let $\{y^n\}_{n=1}^{+\propto} \subset B_r$, with $y^n \to y$ in B_r as $n \to +\propto$. Then, by using hypotheses (H_1),(H_2) and (H_3), we have

(i)

$$
f(s, y_s^n + \hat{\phi}_s) \to f(s, y_s + \hat{\phi}_s), \quad n \to\propto .
$$

(ii)

$$
g\left(t, y_t^n + \hat{\phi}_t\right) \to g\left(t, y_t + \hat{\phi}_t\right), \quad n \to\propto .
$$

(iii)

$$
\|I_i(y^n(t_i^-) + \hat{\phi}(t_i^-)) - I_i(y(t_i^-) + \hat{\phi}(t_i^-))\| \to 0,
$$

$$
\|J_i(y^n(t_i^-) + \hat{\phi}(t_i^-)) - J_i(y(t_i^-) + \hat{\phi}(t_i^-))\| \to 0, \quad n \to\propto, i = 1, 2, \cdots, m.
$$

Now, for every $t \in [0, t_1]$, we have

$$
\begin{aligned}
t^{2-\beta}\|Ny^n(t) - Ny(t)\| \le& \left\| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} t^{2-\beta} [f(s, y_s^n + \hat{\phi}_s) - f(s, y_s + \hat{\phi}_s)] ds \right\| \\
&+ \left\| t^{2-\beta}[g(t, y_t^n + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t) \right\| \\
\le& \frac{T^2}{\Gamma(\beta+1)} \left\| f(s, y_s^n + \hat{\phi}_s) - f(s, y_s + \hat{\phi}_s) \right\| \\
&+ T^{2-\beta} \left\| g(t, y_t^n + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t) \right\| \\
\to& 0 \quad (n \to\propto).
\end{aligned}
$$

Moreover, for all $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, we have

$$\begin{aligned} t^{2-\beta}\|Ny^n(t) - Ny(t)\| \leq & \frac{T^2}{\Gamma(\beta+1)} \left\| f(s, y_s^n + \hat{\phi}_s) - f(s, y_s + \hat{\phi}_s) \right\| \\ & + T^{2-\beta} \left\| g(t, y_t^n + \hat{\phi}_t) - g(t, y_t + \hat{\phi}_t) \right\| \\ & + \frac{T}{\Gamma(\beta)} \sum_{i=1}^{k} \left\| J_i(y^n(t_i^-) + \hat{\phi}(t_i^-)) - J_i(y(t_i^-) + \hat{\phi}(t_i^-)) \right\| \\ & + \frac{1}{\Gamma(\beta-1)} \sum_{i=1}^{k} \left\| I_i(y^n(t_i^-) + \hat{\phi}(t_i^-)) - I_i(y(t_i^-) + \hat{\phi}(t_i^-)) \right\| \\ \to & 0 \quad (n \to \propto). \end{aligned}$$

We thus obtain

$$\|Ny^n - Ny\|_{B_v''} \to 0 \text{ as } n \to\propto$$

implying that N is continuous on B_r.

Step 3 The map $N(B_r)$ is equicontinuous on J. The functions $\{Ny : y \in B_r\}$ are equicontinuous at $t = 0$. For $t_1, t_2 \in J_k, t_1 < t_2, k = 0, 1, 2, \cdots, m$ and $y \in B_r$ we have

$$\begin{aligned} t^{2-\beta}\|Ny(t_1) - Ny(t_2)\| \leq & C_1(t_1) t_2^{2-\beta} \|Ny(t_1) - Ny(t_2)\| \\ \leq & C_1(t_1) \|t_1^{2-\beta} Ny(t_1) - t_2^{2-\beta} Ny(t_2)\| \\ & + C_1(t_1) \|t_2^{2-\beta} Ny(t_2) - t_1^{2-\beta} Ny(t_2)\| \\ \leq & C_1(t_1) \|t_1^{2-\beta} Ny(t_1) - t_2^{2-\beta} Ny(t_2)\| \\ & + C_1(t_1) \|Ny(t_2)\| \|t_2^{2-\beta} - t_1^{2-\beta}\|, \end{aligned}$$

where there exist $C_1(t_1) > 0$. The right side is independent of $y \in B_r$ and tend to zero as $t_1 \to t_2$ since $t^{2-\beta} Ny(t) \in C(J_{k,X})$ and $\|t_2^{2-\beta} - t_1^{2-\beta}\| \to 0$ as $t_1 \to t_2$. So $\|Ny(t_1) - Ny(t_2)\|_{B_v''} \to 0$ as $t_1 \to t_2$. Hence, $N(B_r)$ is equicontinuous on J.

Step 4 Mönch's condition holds.

Let $N = N_1 + N_2 + N_3$, where

$$\begin{aligned} N_1 y(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s) ds, \\ N_2 y(t) = & g(t, y_t + \hat{\phi}_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \\ N_3 y(t) = & \sum_{i=1}^{k} J_i(y(t_i^-) + \hat{\phi(t_i^-)}) \frac{t^{\beta-2}}{\Gamma(\beta-1)} (\frac{t}{\beta-1} - t_i) \\ & + \sum_{i=1}^{k} I_i(y(t_i^-) + \hat{\phi(t_i^-)}) \frac{t^{\beta-2}}{\Gamma(\beta-1)}. \end{aligned}$$

Assume $W \subseteq B_r$ is countable and $W \subseteq \overline{\text{co}}(\{0\} \cup N(W))$. We show that $\beta(W) = 0$, where β is the Hausdorff MNC. Without loss of generality, we may suppose that $W = \{y^n\}_{n=1}^{\propto}$.

Since $N(W)$ is equicontinuous on J_k, $W \subseteq \overline{\text{co}}\,(\{0\} \cup N(W))$ is equicontinuous on J_k as well. Using Lemma 2.4, (H$_1$)(iii), (H$_2$)(ii),(H$_3$), we have

$$\begin{aligned}
\beta\left(\{N_1 y^n(t)\}_{n=1}^{\propto}\right) \leq & \frac{2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \eta(s) [\sup_{-\propto<\theta\leq 0} \beta(\{y_s^n(\theta)\}_{n=1}^{\propto})] ds \\
\leq & \frac{2T^\beta}{\Gamma(\beta+1)} \|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)} \sup_{-\propto<\theta\leq 0} \beta(\{y_s^n(\theta)\}_{n=1}^{\propto}), \\
\beta\left(\{N_2 y^n(t)\}_{n=1}^{\propto}\right) \leq & \beta(g(t, y_t^n + \hat{\phi}_t)) \\
\leq & G \sup_{-\propto<\theta\leq 0} \beta(\{y_t^n(\theta)\}_{n=1}^{\propto}), \\
\beta(\{N_3 y^n(t)\}_{n=1}^{\propto}) \leq & \frac{T}{\Gamma(\beta)} \beta(\{\sum_{i=1}^{k} t^{\beta-2} J_i(y^n(t_i^-) + \phi(\hat{t_i^-}))\}_{n=1}^{\propto}) \\
& + \frac{1}{\Gamma(\beta-1)} \beta(\{\sum_{i=1}^{k} t^{\beta-2} I_i(y^n(t_i^-) + \phi(\hat{t_i^-}))\}_{n=1}^{\propto}) \\
\leq & \frac{T^*}{\Gamma^*} (\beta(\{\sum_{i=1}^{k} t^{\beta-2} J_i(y^n(t_i^-) + \phi(\hat{t_i^-}))\}_{n=1}^{\propto}) \\
& + \beta(\{\sum_{i=1}^{k} t^{\beta-2} I_i(y^n(t_i^-) + \phi(\hat{t_i^-}))\}_{n=1}^{\propto})) \\
\leq & \frac{T^*}{\Gamma^*} \sum_{i=1}^{m} (M_i + K_i) \sup_{-\propto<\theta\leq 0} \beta(\{y^n(\theta)\}_{n=1}^{\propto}).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\beta\left(\{N y^n(t)\}_{n=1}^{\propto}\right) \leq & \beta\left(\{N_1 y^n(t)\}_{n=1}^{\propto}\right) + \beta\left(\{N_2 y^n(t)\}_{n=1}^{\propto}\right) + \beta\left(\{N_3 y^n(t)\}_{n=1}^{\propto}\right) \\
\leq & \frac{2T^\beta}{\Gamma(\beta+1)} \|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)} \sup_{-\propto<\theta\leq 0} \beta(\{y_s^n(\theta)\}_{n=1}^{\propto}) \\
& + G \sup_{-\propto<\theta\leq 0} \beta(\{y_t^n(\theta)\}_{n=1}^{\propto}) \\
& + \frac{T^*}{\Gamma^*} \sum_{i=1}^{m} (M_i + K_i) \sup_{-\propto<\theta\leq 0} \beta(\{y^n(\theta)\}_{n=1}^{\propto}) \\
\leq & \left(\frac{2T^\beta}{\Gamma(\beta+1)} \|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)} + G \right. \\
& \left. + \frac{T^*}{\Gamma^*} \sum_{i=1}^{m} (M_i + K_i) \right) \beta(\{y^n(t)\}_{n=1}^{\propto}) \\
= & M^* \beta\left(\{y^n(t)\}_{n=1}^{\propto}\right),
\end{aligned}$$

where M^* is defined in assumption (H$_4$). Since W and $N(W)$ are equicountinuous on every J_k, it follows from Lemma 2.4 that the inequality implies $\beta(NW) \leq M^* \beta(W)$. Thus, from Mönch's condition, we have

$$\beta(W) \leq \beta(\overline{\text{co}}\{0\} \cup N(W)) = \beta(NM) \leq M^* \beta(W).$$

Since $M^* < 1$, we get $\beta(W) = 0$. It follows that W is relatively compact. Using Lemma 2.5, we know that N has a fixed point y in W. So the theorem is proved.

4 Ulam Stability

(H_5) The function $g(t,x)$ satisfies the condition that $|g(t,x) - g(t,y)| \leqslant L|x-y|$, L is a constant and $0 < Ll < 1$.

Theorem 3.2 Suppose conditions (H_1)(H_3)(H_4)(H_5) are satisfied. Then system(1.1) has at least one solution on J and this solution is Ulam stable.

Proof It is easy to see that the solution satisfies condition (H_2) when the solution satisfies condition (H_5). By using Theorem 3.1, we can prove the existence of this solution. Then we consider the inequality

$$E\|D_{0^+}^{\beta}[x(t) - g(t,x_t)] - f(t,x_t)\|^2 < \varepsilon.$$

Suppose there exists a function $f_1(t,y_t)$ satisfies $\|f(t,x_t) - f_1(t,y_t)\| < \varepsilon$, Then for the equation

$$\begin{cases} D_{0^+}^{\beta}[x(t) - g(t,x_t)] = f(t,x_t), \quad t \in [0,T], \quad t \neq t_k, \\ \Delta I_{0^+}^{2-\beta} x(t_k) = I_k(x(t_k^-)), \quad \Delta I_{0^+}^{1-\beta} x(t_k) = J_k(x(t_k^-)), \\ I_{0^+}^{2-\beta}[x(0) - g(0,x_0)] = \varphi_1 \in B_v, \quad I_{0^+}^{1-\beta}[x(0) - g(0,x_0)] = \varphi_2 \in B_v. \end{cases}$$

We have the fundamental solution of this equation as

$$y(t) = \begin{cases} y_0 = \phi \in B_v, \qquad t \in (-\propto, 0], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} f(s,y_s)ds + g(t,y_t) \\ +\varphi_2 \dfrac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \dfrac{t^{\beta-2}}{\Gamma(\beta-1)}, t \in [0,t_1], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} f(s,y_s)ds + g(t,y_t) + \varphi_2 \dfrac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \dfrac{t^{\beta-2}}{\Gamma(\beta-1)} \\ + \displaystyle\sum_{i=1}^{k} J_i(y(t_i^-)) \dfrac{t^{\beta-2}}{\Gamma(\beta-1)} (\dfrac{t}{\beta-1} - t_i) \\ + \displaystyle\sum_{i=1}^{k} I_i(y(t_i^-)) \dfrac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}]. \end{cases}$$

It is obvious to see that the solution is Ulam stable in the interval $(-\propto, 0]$, so, first, let's have a look at the interval $t \in (0, t_1]$,

$$\begin{aligned} t^{2-\beta}\|x(t) - y(t)\| &= |\frac{t^{2-\beta}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}(f(s,x_s) - f(s,y_s))ds| + t^{2-\beta}|g(t,x_t) - g(t,y_t)| \\ &\leq \frac{T^{2-\beta}\varepsilon}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} ds + Llt^{2-\beta}\|x(t) - y(t)\|. \end{aligned}$$

So, we have

$$t^{2-\beta}\|x(t)-y(t)\| \leq \frac{T^{2-\beta}\varepsilon}{\Gamma(\beta)(1-Ll)}\int_0^{t_1}(t_1-s)^{\beta-1}ds.$$

Here $K=\frac{T^{2-\beta}}{\Gamma(\beta)(1-Ll)}\int_0^{t_1}(t_1-s)^{\beta-1}ds$.

Second, consider the interval $t\in(t_1,t_2]$,

$$\begin{aligned}
t^{2-\beta}\|x(t)-y(t)\| =&\frac{t^{2-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}(f(s,x_s)-f(s,y_s))ds+t^{2-\beta}|g(t,x_t)-g(t,y_t)|\\
&+\sum_{i=1}^k(J_i(x(t_i^-))-J_i(y(t_i^-)))\frac{1}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)\\
&+\sum_{i=1}^k(I_i(x(t_i^-))-I_i(y(t_i^-)))\frac{1}{\Gamma(\beta-1)}\\
\leq&\frac{T^{2-\beta}\varepsilon}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}ds+Llt^{2-\beta}\|x(t)-y(t)\|\\
&+\sum_{i=1}^k(J_i(x(t_i^-))-J_i(y(t_i^-)))\frac{1}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)\\
&+\sum_{i=1}^k(I_i(x(t_i^-))-I_i(y(t_i^-)))\frac{1}{\Gamma(\beta-1)}.
\end{aligned}$$

As we have had the conclusion that in the interval $t\in(0,t_1]$ that $|y(t)-x(t)|<K\varepsilon$, so we have

$$|I_i(x(t_i^-))-I_i(y(t_i^-))|<K_1\varepsilon,$$

$$|J_i(x(t_i^-))-J_i(y(t_i^-))|<K_2\varepsilon,$$

due to I_k,J_k are continuous functions.

So

$$\begin{aligned}
t^{2-\beta}\|x(t)-y(t)\| \leq&\frac{T^{2-\beta}\varepsilon}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}ds+t^{2-\beta}Ll\|x(t)-y(t)\|\\
&+\sum_{i=1}^k(J_i(x(t_i^-))-J_i(y(t_i^-)))\frac{1}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_i)\\
&+\sum_{i=1}^k(I_i(x(t_i^-))-I_i(y(t_i^-)))\frac{1}{\Gamma(\beta-1)}\\
\leq&\frac{T^{2-\beta}\varepsilon}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}ds+Llt^{2-\beta}\|x(t)-y(t)\|\\
&+\sum_{i=1}^k\frac{K_1\varepsilon}{\Gamma(\beta-1)}(\frac{t}{\beta-1}-t_1)+\frac{K_2k\varepsilon}{\Gamma(\beta-1)}.
\end{aligned}$$

So

$$
\begin{aligned}
t^{2-\beta}\|x(t)-y(t)\| \leq & \frac{T^{2-\beta}\varepsilon}{\Gamma(\beta)(1-Ll)} \int_0^{t_2} (t_2-s)^{\beta-1} ds \\
& + \sum_{i=1}^{k} \frac{K_1\varepsilon}{\Gamma(\beta-1)(1-Ll)} (\frac{t_2}{\beta-1} - t_1) \\
& + \frac{K_2 k\varepsilon}{\Gamma(\beta-1)(1-Ll)}.
\end{aligned}
$$

So, in the interval $t \in (t_1, t_2]$,

$$
\begin{aligned}
K = & \frac{T^{2-\beta}}{\Gamma(\beta)(1-Ll)} \int_0^{t_2} (t_2-s)^{\beta-1} ds \\
& + \sum_{i=1}^{k} \frac{K_1}{\Gamma(\beta-1)(1-Ll)} (\frac{t_2}{\beta-1} - t_1) \\
& + \frac{K_2 k}{\Gamma(\beta-1)(1-Ll)}.
\end{aligned}
$$

In this way, when t is in the interval $t \in (t_{i-1}, t_i]$ can be proved.

References

[1] Podlubny I. Fractional differential equations[M]. San Diego: Academic Press, 1999.

[2] Liu Z, Sun J. Nonlinear boundary value problem of fractional differential systems[J]. Comput. Appl. Math., 2012, 64: 463–475.

[3] Liu Z, Lu L. A class of BVPs for nonlinear fractional differential equations with p-Laplacian operator[J]. Electron. J. Qual. Theory Differ. Equ., 2012, 70: 1–16.

[4] Wang J, Zhou Y. Completely controllability of fractional evolution systems[J]. Commun. Nonlinear Sci. Numer. Simul., 2012, 17: 4346–4355.

[5] Shu X, Lai Y, Chen Y. The existence of mild solutions for impulsive fractional partial differential equations[J]. Nonlinear Anal, 2011, 74(5): 2003–2011.

[6] Shu X, Shi Y. A study on the mild solution of impulsive fractional evolution equations[J]. Appl. Math. Comp., 2016, 273: 465–476.

[7] Heymans N, Podlubny I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives[J]. Rheol. Acta., 2006, 45: 765–771.

[8] Weera Y, SotirisK N, Jessada T. Impulsive multiorders Riemann-Liouville fractional differential equation[J]. Discrete Dynamics in Nature and Society, 2015, 2015: 1–9.

[9] Kilbas A, Srivastava H, Trujillo J. Theory and applications of fractional differential equations[M]. Amsterdam: Elsevier, 2006.

[10] Yan B. Boundary value problems on the half-line with impulses and infinite delay[J]. J. Math. Anal. Appl., 2001, 259(1): 94–114

[11] Deinz H. On the behavious of measure of noncompactness with respect to differentiation and integration of vector-valued functions[J]. Nonliear Anal.: TMA, 1983, 7: 1351–1371

[12] Guo D, Lakshmikantham V, Liu X. Nonliear integral equations in abstract spaces[M]. Dordrecht: Kluwer Academic, 1996

[13] Mönch H. Boundary value problems for nonlinear ordinary differential equations of second order in Banach space[J]. Nonlinear Anal.: TMA, 1980, 4: 985–999

[14] Bellman R. The stability of solutions of linear differential equations[J]. Duke Math. J., 1943, 10(4): 643–647

关于分数阶微分方程解的存在性与Ulam稳定性探究

郭育辰, 舒小保

(湖南大学数学与计量经济学院, 湖南 长沙 410082)

摘要: 本文主要研究了带有脉冲的无限时滞的中立型黎曼刘维尔型分数阶微分方程. 通过使用不动点理论以及非紧性测度, 证明了方程解的存在性和Ulam稳定性.

关键词: 脉冲黎曼刘维尔型分数阶微分方程; 不动点理论; Hausdorff非紧性测度; Ulam稳定性.

MR(2010)**主题分类号:** 35R10; 34K45; 26A33 **中图分类号:** O193