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OPTIMAL DIVIDEND STRATEGIES FOR THE COMPOUND POISSON MODEL WITH DEBIT INTEREST

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Abstract: In this paper, we study the optimal dividend strategy for an insurance company with debit interest and dividend payments. By HJB methods, a rule for choosing the strategy that maximizes the expected accumulated discounted dividends until absolute ruin is given. Under the so called threshold strategy, we derive integro-differential equations for the expected accumulated discounted dividends until absolute ruin. Then, explicit expressions for the expected accumulated discounted dividends until absolute ruin with exponential claim amounts are obtained. Finally, based on the explicit expressions, we prove that the optimal strategy is a threshold strategy and the optimal level of threshold is also obtained.

Keywords: debit interest; threshold dividend strategy; optimal strategy; the expected discounted penalty function

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1 Introduction

Let $\{R(t) = x + ct - S(t),\ \ t \geq 0\}$ be the compound Poisson surplus process for an insurer, where $x \geq 0$ is the initial surplus, $c > 0$ is the gross premium rate, the aggregate claims process $\{S(t), t \geq 0\}$ is a compound Poisson process with claim frequency λ and i.i.d. nonnegative random variables with a common distribution function $F(y)$ that satisfies $F(0) = 0$ and has a finite mean μ.

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Generally, when the surplus ever drops below zero we say that the insurer is at ruin. In reality, however, when the surplus is negative, the insurer could borrow an amount of money that equals to the deficit at a debit rate interest force $\delta > 0$ to continue her business. Meanwhile, the insurer will repay the debts continuously from her premium. Thus the surplus of the insurer is driven under the interest force δ when the surplus is negative. The negative surplus may return to a positive level. It is clear that when the negative surplus attains or is below the level $-c/\delta$, the surplus is no longer able to be positive and consequently absolute ruin occurs at that moment.

We denote the compound Poisson surplus process with debit interest force δ by $R_\delta(t)$, which satisfies the following differential equation

$$\mathrm{d}R_\delta(t) = (c + \delta R_\delta(t)\mathbf{1}_{\{R_\delta(t)<0\}})\mathrm{d}t - \mathrm{d}S(t), \quad R_\delta(0) = u. \tag{1.1}$$

In literature, model (1.1) is discussed by Zhou and Zhang [1], Cai [2], and among others.

We now enrich model (1.1). We assume that the insurance company is a stock company, and dividends are paid to the shareholders according to a certain dividend strategy L. Let $D(t)$ denote the accumulated paid dividends up to time t, and $\{U_\delta(t), t \geq 0\}$ be the company's surplus. Thus

$$U_\delta(t) = R_\delta(t) - D(t), \quad t \geq 0. \tag{1.2}$$

In addition, it is reasonable to assume that no dividends should be paid out when the surplus below zero. Now, we define the absolute ruin time by $\tau_\delta = \inf\{t \geq 0 : U_\delta(t) \leq 0\}$ and the present value of all dividends until absolute ruin by $D = \int_0^{\tau_\delta} e^{-\beta t}\mathrm{d}D(t)$, where $\beta > 0$ is the force of interest for valuation. The performance of the strategy $L = \{D(t)\}_{t\geq 0}$ is measured by the expectation of the present value of all dividends until absolute ruin, that is

$$V(u; L) = E[D]. \tag{1.3}$$

In literature, $V(u; L)$ is also said to be the value function associated to a strategy L. Under the criterion of maximizing the value function $V(u; L)$, the company usually wants to choose an optimal dividend strategy which is denoted by L^*. A dividend strategy L^* is said to be optimal if and only if

$$V(u; L^*) = \sup_{L\in\Pi} V(u; L), \tag{1.4}$$

where the supremum is taken over the set Π of all admissible strategies.

The optimal dividend problem goes back to De Finetti [3], who found that the optimal strategy must be a barrier strategy and showed how the optimal level of the barrier can be determined. A first treatment of the optimal dividend strategy in the compound Poisson surplus process can be found in Section 6.4 of Bühlmann [4]. The problem of finding the optimal dividend payment strategy was discussed extensively by Borch [5, 6], Zhao et al.

[7], Thonhausera and Albrechera [8]. Moreover, Gerber and Shiu [9] considered the optimal dividend strategies in the compound Poisson model. They showed that the maximal value function of the dividends can be characterized by the Hamilton-Jacobi-Bellman (HJB) equation. A rule for deciding between plowback and dividend payout is derived. Some recent papers on the compound Poisson risk model with debit interest and dividend payments are Yuan and Hu [10], Yuen et al. [11] and so on.

In this paper, we study this optimization problem under model (1.2) and the constraint that only dividend strategies with dividend rate bounded by a ceiling are admissible. We assume that the ceiling is less than the premium rate. Thus, the constraint is $\mathrm{d}D(t) \le \gamma \mathrm{d}t$, where $\gamma \in (0, c)$ is the dividend-rate ceiling. The theoretical foundations are laid in Section 2. It is shown that the maximal value function can be characterized by the Hamilton-Jacobi-Bellman (HJB) equation. A rule for deciding between plowback and dividend payout is derived. Section 3 and Section 4 discuss the expectation of the discounted dividends until absolute ruin of a threshold strategy, if a threshold strategy is applied. Explicit expression of the value function $V(u; b)$ is given in the case of exponential claim amount distributions. In Section 5, for the case of an exponential claim amount distribution, it is shown that the optimal dividend strategy is a threshold strategy, and the optimal threshold is determined.

2 The HJB Equation and Verification of Optimality

In this section, we first show that the expectation of the present value of all dividends of an optimal strategy satisfies Bellman's dynamic programming principle. And then show that if the value function associated to a strategy L, $V(u; L)$ satisfying the HJB equation, it is indeed an optimal strategy.

Let $B(u, 0) = E[e^{-\beta T(u,0)}]$ denote the Laplace transform of $T(u, 0)$, where $T(u, 0)$ is the first passage time from a given surplus level u $(-c/\delta < u < 0)$ to surplus zero . When $-c/\delta < u < 0$, conditioning on the time $T(u, 0)$, we have $V(u; L) = B(u, 0)V(0; L)$. Under the assumption that no dividends should be paid out when the surplus is below zero, and note that $B(u, 0)$ is independent with the given dividend strategy L. Then, it only needs to discuss the maximization problem $m(u) = \sup\limits_{L\in\Pi} V(u; L)$ for $u \ge 0$.

For $u \ge 0$, consider the infinitesimal time interval $(0, \Delta t]$, $\Delta t > 0$, and the following dividend strategy. Suppose that between the time 0 and time Δt, dividends are paid at rate r, and thereafter, an optimal strategy is applied. By conditioning on whether there is a claim in this time interval and on the amount of the claim if it occurs, we see that the expectation of the present value of all dividends until absolute ruin is

$$r\Delta t + e^{-\beta\Delta t}\left[(1-\lambda\Delta t)m\big(u+(c-r)\Delta t\big) + \lambda\Delta t\int_0^{u+\frac{c}{\delta}} m(u-y)\mathrm{d}F(y)\right] + o(\Delta t).$$

Taylor expansion yields that

$$E[D] = m(u) + \big[r + (c-r)m^{'}(u) - (\lambda+\beta)m(u) + \lambda\int_0^{u+\frac{c}{\delta}} m(u-y)\mathrm{d}F(y)\big]\Delta t + o(\Delta t). \quad (2.1)$$

We maximum the value of (2.1) for $r \in [0, \gamma]$. Then, the fact that $m(u)$ is the optimal value shows that $m(u)$ satisfies the HJB functional equation

$$\max_{0\le r\le\gamma}\left\{r+(c-r)m^{'}(u)\right\}-(\lambda+\beta)m(u)+\lambda\int_0^{u+\frac{c}{\delta}}m(u-y)\mathrm{d}F(y)=0,\ u\ge 0. \tag{2.2}$$

Conversely, for some given dividend strategy, let $v(u) = E[D]$, a function of the initial surplus $u \ge 0$, if $v(u)$ satisfies the HJB equation (2.2), by a similar argument to Section 4 in Gerber and Shiu [9], we know that the given strategy is an optimal dividend strategy.

From (2.2), we know that the expression to be maximized is $r[1-m^{'}(u)]$ for $r \in [0, \gamma]$. Thus, the optimal dividend rate at time 0 is

$$\begin{aligned} r &= 0, \quad \text{if } m^{'}(u) > 1, \\ r &= \gamma, \quad \text{if } m^{'}(u) < 1. \end{aligned}$$

If $m^{'}(u) = 1$, the dividend rate r can be any value between 0 and γ. Then, at time $t \in (0, \tau_\delta)$, the optimal dividend rate is

$$\begin{aligned} r &= 0, \quad \text{if } m^{'}(u) > 1, \\ r &= \gamma, \quad \text{if } m^{'}(u) < 1. \end{aligned} \tag{2.3}$$

3 Threshold Strategy

If the solution of equation (2.2) has the property that $m^{'}(u) > 1$ for $u < b$ and $m^{'}(u) < 1$ for $u > b$ for some number b, then the optimal dividend strategy is particularly appealing: whenever $U_\delta(t) < b$, no dividends are paid, and whenever $U_\delta(t) > b$, dividends are paid at the maximal rate γ. We shall call such a dividend strategy a threshold strategy.

Threshold strategy are of interest, even in cases where the optimal dividend strategy is not of this form. Let L be the threshold strategy, then model (1.2) can be described as

$$\mathrm{d}U_\delta(t)=\begin{cases}(c-\gamma)\mathrm{d}t-\mathrm{d}S(t), & U_\delta(t)\ge b,\\ c\mathrm{d}t-\mathrm{d}S(t), & 0\le U_\delta(t)<b,\\ (\delta U_\delta(t)+c)\mathrm{d}t-\mathrm{d}S(t), & -c/\delta<U_\delta(t)<0\end{cases}$$

with $U_\delta(0) = u$. Let $V(u; b)$ denote the expectation of the present value of all dividend until absolute ruin, where u is the initial surplus and b is the threshold. Note that the path of $V(u; b)$ is different at $-c/\delta < u < 0$, $0 \le u < b$, $u \ge b$, for notational convenience, we write

$$\begin{aligned} V(u;b) &= V(u;b)1_{\{-c/\delta<u<0\}}+V(u;b)1_{\{0\le u<b\}}+V(u;b)1_{\{u\ge b\}} \\ &:= V_1(u;b)+V_2(u;b)+V_3(u;b). \end{aligned}$$

In the following theorem, we derive integro-differential equations for $V(u; b)$.

Theorem 3.1 As a function of u, the expected discounted value of all dividends until ruin $V(u;b)$ satisfies the following integro-differential equations

$$(\delta u+c)V_1^{'}(u;b)=(\lambda+\beta)V_1(u;b)-\lambda\int_0^{u+c/\delta}V_1(u-y;b)\mathrm{d}F(y) \tag{3.1}$$

for $-c/\delta<u<0$,

$$cV_2^{'}(u;b)=(\lambda+\beta)V_2(u;b)-\lambda\int_0^{u+c/\delta}V(u-y;b)\mathrm{d}F(y) \tag{3.2}$$

for $0\le u<b$, and for $u\ge b$,

$$(c-\gamma)V_3^{'}(u;b)=(\lambda+\beta)V_3(u;b)-\lambda\int_0^{u+c/\delta}V(u-y;b)\mathrm{d}F(y)-\gamma. \tag{3.3}$$

Proof For $-c/\delta<u<0$, conditioning on the occurrence of a claim within an infinitesimal time interval $(0,\Delta t]$, we have

$$\begin{aligned}V_1(u;b) \;&=\; (1-\lambda\Delta t)e^{-\beta\Delta t}V_1(h_\delta(\Delta t,u);b)\\ &\quad+\lambda\Delta te^{-\beta\Delta t}\int_0^{h_\delta(\Delta t,u)+\frac{c}{\delta}}V_1(h_\delta(\Delta t,u)-y;b)\mathrm{d}F(y)+o(\Delta t),\end{aligned}$$

where $h_\delta(t,u)=ue^{\delta t}+c(e^{\delta t}-1)/\delta<0$ is the surplus at time Δt if no claim occurs during $(0,\Delta t]$. The Taylor expansion yields

$$\begin{aligned}V_1(u;b) \;&=\; V_1(u;b)-(\lambda+\beta)V_1(u;b)\Delta t+(\delta u+c)V_1^{'}(u;b)\Delta t\\ &\quad+\lambda\int_0^{u+\frac{c}{\delta}}V_1(u-y;b)\mathrm{d}F(y)\Delta t+o(\Delta t).\end{aligned}$$

Subtracting $V_1(u;b)$ from both sides of the above equation, dividing by Δt and then letting $\Delta t\to 0$, we achieve (3.1). Similarly, one can obtain (3.2) and (3.3). This completes the proof of Theorem 3.1.

We mentioned that, in the case when $\gamma=c$, Theorem 3.1 above then reduces to Theorem 2.1 in Yuen et al. [11].

Proposition 3.1 The boundary conditions of $V(u;b)$ are as follows.

$$\lim_{u\downarrow -c/\delta}V_1(u;b) = 0, \tag{3.4}$$

$$\lim_{u\uparrow\infty}V_3(u;b) = \frac{\gamma}{\beta}. \tag{3.5}$$

Moreover, by the continuous assumption, we have

$$V_1(0-;b)=V_2(0+;b),\ V_1^{'}(0-;b)=V_2^{'}(0+;b), \tag{3.6}$$

$$V_2(b-;b)=V_3(b+;b), \tag{3.7}$$

$$cV_2^{'}(b-;b)=(c-\gamma)V_3^{'}(b+;b)+\gamma. \tag{3.8}$$

Proof Condition (3.4) is obviously for the fact that when $U_\delta(0) = -c/\delta$ the surplus stays at the zero until the next claim and at this time ruin occurs. When $u \to \infty$, then the ruin time $\tau_\delta = \infty$, and then $D = \int_0^\infty e^{-\beta t}\mathrm{d}D(t) = \gamma/\beta$, which leads to (3.5). The proof of Proposition 3.1 is completed.

4 Analysis of $V(u;b)$ with Exponential Claim

In this section, we will calculate the explicit expressions of $V(u;b)$ with the assumption that the claim size is exponentially distributed with mean μ.

Substituting $F(y) = 1-e^{-y/\mu}$ into (3.1)–(3.3), respectively, and changing the integration variables, we get

$$(\delta u + c)V_1^{'}(u;b) = (\lambda+\beta)V_1(u;b) - \frac{\lambda}{\mu}e^{-u/\mu}\int_{-c/\delta}^{u} V_1(x;b)e^{x/\mu}\mathrm{d}x \tag{4.1}$$

for $-c/\delta < u < 0$,

$$cV_2^{'}(u;b) = (\lambda+\beta)V_2(u;b) - \frac{\lambda}{\mu}e^{-u/\mu}\int_{-c/\delta}^{u} V(x;b)e^{x/\mu}\mathrm{d}x \tag{4.2}$$

for $0 \le u < b$, and

$$(c-\gamma)V_3^{'}(u;b) = (\lambda+\beta)V_3(u;b) - \gamma - \frac{\lambda}{\mu}e^{-u/\mu}\int_{-c/\delta}^{u} V(x;b)e^{x/\mu}\mathrm{d}x \tag{4.3}$$

for $u \ge b$. Applying the operator $\big(\mathrm{d}/\mathrm{d}u + 1/\mu\big)$ to (4.1)–(4.3), respectively, we have

$$(\delta u + c)V_1^{''}(u;b) + \big[\frac{\delta u + c}{\mu} - (\lambda+\beta-\delta)\big]V_1^{'}(u;b) - \frac{\beta}{\mu}V_1(u;b) = 0 \tag{4.4}$$

for $-c/\delta < u < 0$,

$$cV_2^{''}(u;b) + \big[\frac{c}{\mu} - (\lambda+\beta)\big]V_2^{'}(u;b) - \frac{\beta}{\mu}V_2(u;b) = 0 \tag{4.5}$$

for $0 \le u < b$, and

$$(c-\gamma)V_3^{''}(u;b) + \big[\frac{c-\gamma}{\mu} - (\lambda+\beta)\big]V_3^{'}(u;b) - \frac{\beta}{\mu}V_3(u;b) = -\frac{\gamma}{\mu} \tag{4.6}$$

for $u \ge b$.

When $-c/\delta < u < 0$, let $z = -(\delta u + c)/(\delta\mu)$ and denote $g(z) := V_1(u;b)$, then (4.4) can be rewritten as

$$zg^{''}(z) + \big[1 - \frac{\lambda+\beta}{\delta} - z\big]g^{'}(z) - (-\frac{\beta}{\delta})g(z) = 0, \quad -\frac{c}{\delta\mu} < z < 0. \tag{4.7}$$

By eq.(13.1.15) and eq.(13.1.18) of Abranmowitz and Stegun [12], we know that the solution of (4.7) has the form

$$g(z) = c_0 e^z U(1-\frac{\lambda}{\delta}, 1-\frac{\lambda+\beta}{\delta}; -z) + c_1(-z)^{\frac{\lambda+\beta}{\delta}}e^z M(1+\frac{\beta}{\delta}, 1+\frac{\lambda+\beta}{\delta}; -z),$$

where $U(a,b;x)$ and $M(a,b;x)$ are the confluent hypergeometric function of the first and second kinds, respectively, and c_0 and c_1 are unknown constants to be determined. From the boundary condition (3.4) we know that $c_0 = 0$. Hence

$$\begin{aligned} V_1(u;b) &= g\big(-\frac{\delta u+c}{\delta\mu}\big) = c_1\big(\frac{\delta u+c}{\delta\mu}\big)^{\frac{\lambda+\beta}{\delta}} \cdot \exp\big(-\frac{\delta u+c}{\delta\mu}\big) M\big(1+\frac{\beta}{\delta}, 1+\frac{\lambda+\beta}{\delta}; \frac{\delta u+c}{\delta\mu}\big) \\ &= c_1\big(\frac{\delta u+c}{\delta\mu}\big)^{\frac{\lambda+\beta}{\delta}} M\big(\frac{\lambda}{\delta}, 1+\frac{\lambda+\beta}{\delta}; -\frac{\delta u+c}{\delta\mu}\big) := c_1 f(u). \end{aligned} \tag{4.8}$$

When $0 \le u < b$, it is well known that the general solution to (4.5) is

$$V_2(u;b) = c_2 e^{r_1 u} + c_3 e^{R_1 u}, \tag{4.9}$$

where c_2 and c_3 are unknown constants to be determined, and $r_1 > 0$ $R_1 < 0$ are the roots of the equation $cr^2 - (\lambda+\beta-c/\mu)r - \beta/\mu = 0$.

When $u \ge b$, Condition (3.5) shows that the solution of (4.6) is of the form

$$V_3(u;b) = \frac{\gamma}{\beta} + c_4 e^{R_2 u}, \tag{4.10}$$

where $\frac{\gamma}{\beta}$ is a particular solution of (4.6) and $R_2 < 0$ are the negative root of the equation

$$(c-\gamma)r^2 - (\lambda+\beta-(c-\gamma)/\mu)r - \beta/\mu = 0.$$

Using the fact $M^{'}(a,b;x) = \frac{a}{b}M(a+1,b+1;x)$ and the definition of $f(u)$, we have

$$\begin{aligned} f(0) &= \big(\frac{c}{\delta\mu}\big)^{(\lambda+\beta)/\delta} M\big(\frac{\lambda}{\delta}, 1+\frac{\lambda+\beta}{\delta}; -\frac{c}{\delta\mu}\big) := \big(\frac{c}{\delta\mu}\big)^{(\lambda+\beta)/\delta} M_1, \\ f^{'}(0) &= \big(\frac{c}{\delta\mu}\big)^{(\lambda+\beta)/\delta}\Big[\frac{\lambda+\beta}{c}M_1 - \frac{\lambda}{(\lambda+\beta+\delta)\mu} M\big(1+\frac{\lambda}{\delta}, 2+\frac{\lambda+\beta}{\delta}; -\frac{c}{\delta\mu}\big)\Big] \\ &:= \big(\frac{c}{\delta\mu}\big)^{(\lambda+\beta)/\delta}\big(\frac{\lambda+\beta}{c}M_1 - \frac{\lambda}{(\lambda+\beta+\delta)\mu}M_2\big). \end{aligned}$$

By the boundary conditions (3.6)–(3.8), we have

$$\begin{aligned} &c_1 f(0) = c_2 + c_3, \\ &c_1 f^{'}(0) = c_2 r_1 + c_3 R_1, \\ &c_2 e^{r_1 b} + c_3 e^{R_1 b} = c_4 e^{R_2 b} + \frac{\gamma}{\beta}, \\ &c\big(c_2 r_1 e^{r_1 b} + c_3 R_1 e^{R_1 b}\big) = (c-\gamma)c_4 R_2 e^{R_2 b} + \gamma. \end{aligned}$$

By solving the above equations, we obtain

$$V_1(u;b) = \frac{(r_1-R_1)(\gamma-\frac{\gamma}{\beta}(c-\gamma)R_2)\big(\frac{\delta u+c}{\delta\mu}\big)^{\frac{\lambda+\beta}{\delta}} M(\frac{\lambda}{\delta}, 1+\frac{\lambda+\beta}{\delta}; -\frac{\delta u+c}{\delta\mu})}{\big(cr_1-(c-\gamma)R_2\big)h_1 e^{r_1 b} - \big(cR_1-(c-\gamma)R_2\big)h_2 e^{R_1 b}}, \tag{4.11}$$

$$V_2(u;b) = \frac{(\gamma-\frac{\gamma}{\beta}(c-\gamma)R_2)(h_1 e^{r_1 u} - h_2 e^{R_1 u})}{\big(cr_1-(c-\gamma)R_2\big)h_1 e^{r_1 b} - \big(cR_1-(c-\gamma)R_2\big)h_2 e^{R_1 b}}, \tag{4.12}$$

$$\begin{aligned} V_3(u;b) &= \frac{\gamma}{\beta} + \frac{(\gamma-\frac{\gamma}{\beta}cr_1)h_1 e^{r_1 b} - (\gamma-\frac{\gamma}{\beta}cR_1)h_2 e^{R_1 b}}{\big(cr_1-(c-\gamma)R_2\big)h_1 e^{r_1 b} - \big(cR_1-(c-\gamma)R_2\big)h_2 e^{R_1 b}} \times e^{R_2(u-b)} \\ &= \frac{\gamma}{\beta}\big[1-e^{R_2(u-b)}\big] + V_2(b;b)e^{R_2(u-b)}, \end{aligned} \tag{4.13}$$

where $h_1=(r_1+\frac{1}{\mu})M_1-\frac{\lambda}{(\lambda+\beta+\delta)\mu}M_2$, $h_2=(R_1+\frac{1}{\mu})M_1-\frac{\lambda}{(\lambda+\beta+\delta)\mu}M_2$.

5 Optimal Threshold Strategies

Now, if we assume that the optimal strategy is a threshold strategy and let b^* denote the optimal threshold. We first assume that $b^*>0$, (2.3) implies that

$$V^{'}(u;b^*)>1 \quad \text{for } u<b^*, \tag{5.1}$$

$$V^{'}(u;b^*)<1 \quad \text{for } u>b^*. \tag{5.2}$$

If

$$V^{'}(u;b^*)<1 \quad \text{for all } u>0, \tag{5.3}$$

then one can choose $b^*=0$. Thus, the threshold strategy with $b^*=0$ is the optimal strategy.

Formula (3.8) shows that $V_2^{'}(b-;b)$ is a weighted average of $V_3^{'}(b+;b)$ and 1. Hence, the two quantities $V_2^{'}(b-;b)$ and $V_3^{'}(b+;b)$ are both less than 1, both greater than 1, or both equal to 1. From this fact and inequalities (5.1) and (5.3), we get the boundary conditions

$$V_2^{'}(b^*-;b^*)=1, \tag{5.4}$$

$$V_3^{'}(b^*+;b^*)=1. \tag{5.5}$$

From the boundary conditions (5.4) or (5.5) we can obtain the optimal threshold value b^*.

Next, we show that for an exponential claim amount, the optimal dividend strategy is indeed a threshold strategy.

Theorem 5.1 For an exponential claim distribution, the threshold dividend strategy with

$$b^*=\frac{1}{r_1-R_1}\ln\left(\frac{\big[(\gamma-\frac{\gamma}{\beta}(c-\gamma)R_2)R_1-(cR_1-(c-\gamma)R_2)\big]h_2}{\big[(\gamma-\frac{\gamma}{\beta}(c-\gamma)R_2)r_1-(cr_1-(c-\gamma)R_2)\big]h_1}\right) \tag{5.6}$$

is the optimal strategy in model (1.2).

Proof From (4.13), it follows that

$$V_3^{'}(u;b)=-R_2\big[\frac{\gamma}{\beta}-V_2(b;b)\big]e^{R_2(u-b)}, \quad u\geq b. \tag{5.7}$$

Thus, $b^*=0$ if condition (5.3) is satisfied or, equivalently, if

$$V_3^{'}(u;b)=-R_2\big[\frac{\gamma}{\beta}-V_2(0;0)\big]\leq 1, \tag{5.8}$$

which by (4.12) is

$$V_3^{'}(u;b)=-R_2\frac{\gamma}{\beta}\frac{\lambda M_1-\frac{\lambda c}{(\lambda+\beta+\delta)\mu}M_2}{\big(\lambda+\beta-(c-\gamma)R_2\big)M_1-\frac{\lambda c}{(\lambda+\beta+\delta)\mu}M_2}\leq 1. \tag{5.9}$$

Now, suppose that the inequalities in (5.8) and (5.9) are changed from $\leq$ to $>$. From (5.5) and (5.7), we get the condition for b^*,

$$-R_2\Big[\frac{\gamma}{\beta} - V_2(b^*;b^*)\Big] = 1, \tag{5.10}$$

that is, b^* is the solution of the equation

$$V_2(b^*;b^*) = \frac{\gamma}{\beta} + \frac{1}{R_2}. \tag{5.11}$$

From this and (4.13), it follows that

$$V_3(u;b^*) = \frac{\gamma}{\beta} + \frac{1}{R_2}e^{R_2(u-b^*)}, \quad u \geq b^*. \tag{5.12}$$

It remains to verify conditions (5.2) and (5.3) are satisfied. Condition (5.3) is trivial from (5.10). Note that differentiating (4.12) twice yields

$$V_2^{''}(u;b) = \frac{\big(\gamma - \frac{\gamma}{\beta}(c-\gamma)R_2\big)\big(r_1^2 h_1 e^{r_1 u} - R_1^2 h_2 e^{R_1 u}\big)}{\big(cr_1 - (c-\gamma)R_2\big)h_1 e^{r_1 b} - \big(cR_1 - (c-\gamma)R_2\big)h_2 e^{R_1 b}}. \tag{5.13}$$

The eq.(3.5) in Yuen et al. [11] shows that h_1 and h_2 have the same sign. It is trivial that $M_1 > 0$, $M_2 > 0$, and $h_1 \geq (r_1 + \frac{1}{\mu})M_1 - \frac{1}{\mu}M_1 \geq 0$. Then (5.13) is an increasing function because it is the difference of an increasing and a decreasing function; hence its maximum value for $0 \leq u < b$ is attained at $u = b$. Consequently, inequality (5.2) is equivalent to the condition that

$$V_2^{''}(b^*-;b^*) \leq 0. \tag{5.14}$$

From (4.5) and (5.4), we have

$$cV_2^{''}(b^*-;b^*) = \frac{\beta}{\mu}V_2(b^*-;b^*) - \frac{c}{\mu} - (\lambda+\beta),$$

and from (4.6) and (5.5), we have

$$(c-\gamma)V_3^{''}(b^*+;b^*) = \frac{\beta}{\mu}V_3(b^*+;b^*) - \frac{c}{\mu} - (\lambda+\beta).$$

Hence (5.14) is the same as the condition that $V_3^{''}(b^*+;b^*) \leq 0$, which is certainly true because

$$V_3^{''}(b^*;b^*) = -R_2^2\Big[\frac{\gamma}{\beta} - V_2(b^*;b^*)\Big] = R_2 \tag{5.15}$$

by (5.11).

Finally, we determine the value of b^* by solving (5.4) or (5.5). This leads to (5.6). We complete the proof of Theorem 5.1.

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带借贷利率复合Poisson盈余过程的最优分红策略

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摘 要: 本文研究了保险公司在允许借贷和分红情况下的最优分红策略. 利用HJB方法, 先得到满足使分红总现值最大化的分红策略, 然后针对门槛分红策略得到分红总现值满足的微积分方程, 再在指数索赔分布下获得了分红总现值的精确表达式, 最后证明门槛分红策略是最优的分红策略, 并得到了最优的分红起始值.

关键词: 借贷利率; 门槛分红策略; 最优策略; 期望折扣惩罚函数

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