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扩张的圈 Schrödinger-Virasoro 代数的二上同调群

王 松, 王晓明
(上海海洋大学信息学院, 上海 201306)

摘 要: 本文研究了扩张的圈 Schrödinger-Virasoro 代数, 给出了这类李代数的所有二上同调群, 同时得到了这类李代数的所有泛中心扩张.

关键词: Schrödinger-Virasoro 代数; 二上循环; 中心扩张
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1 引言

众所周知, Schrödinger 代数和 Virasoro 代数与非平衡统计物理密切相关, 它们在数学和物理学 (如统计物理学) 的许多领域中都有着重要的作用. Schrödinger-Virasoro 代数 $\mathfrak{sb}$ 最初是由 Henkel 在研究自由 Schrödinger 方程的不变性时被引入文献 [1], 其结构和表示理论被 Roger 和 Unterberger 在文献 [2] 深入研究. 比如, Henkel 在文献 [1] 中给出了 $\mathfrak{sb}$ 只有一维中心扩张. Roger 和 Unterberger 在研究 $\mathfrak{sb}$ 的同调理论时得出了有三维外导子 [2]. $\mathfrak{sb}$ 上的有限维不可约权模在文献 [3] 中被分类. 广义的 Schrödinger-Virasoro 代数的自同构群及 Verma 模被完全确定 [4]. 最近几年 Schrödinger-Virasoro 代数及其变形的结构和表示理论被许多学者广泛研究 [5–8]. 为了研究 $\mathfrak{sb}$ 的顶点表示, Unterberger 介绍了一类新的无限维李代数 [9], 称之为扩张的 Schrödinger-Virasoro 代数 $\widetilde{\mathfrak{sb}}$, 该李代数是复数域 $\mathbb{C}$ 上的向量空间, 带有一组基 $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$, 满足李积关系

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n}, [M_m, M_n] = 0, [N_m, N_n] = 0, [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1},\\
&[L_m, M_n] = nM_{m+n}, [L_m, N_n] = nN_{m+n}, [L_m, Y_{n+\frac{1}{2}}] = (n+\frac{1-m}{2})Y_{m+n+\frac{1}{2}},\\
&[N_m, M_n] = 2M_{m+n}, [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}}, [M_m, Y_{n+\frac{1}{2}}] = 0, \quad \forall m, n \in \mathbb{Z}.
\end{aligned}
$$

该无限维李代数的导子、自同构群及中心扩张等结构理论在文献 [10] 中被完全刻画.

本文将考虑一类与扩张的 Schrödinger-Virasoro 代数 $\widetilde{\mathfrak{sb}}$ 相关的无限维李代数. 通过 $\widetilde{\mathfrak{sb}}$ 与洛朗多项式代数 $\mathbb{C}[t, t^{-1}]$ 张量成一个新的无限维李代数 $\widetilde{\mathfrak{sb}} \otimes \mathbb{C}[t, t^{-1}]$, 记作 $\widetilde{\mathscr{W}}$, 称之为扩张的圈 Schrödinger-Virasoro 代数, 满足以下李积关系

$$
\begin{aligned}
&[L_{m,i}, L_{n,j}] = (n-m)L_{m+n,i+j}, [M_{m,i}, M_{n,j}] = 0, [N_{m,i}, N_{n,j}] = 0, [N_{m,i}, M_{n,j}] = 2M_{m+n,i+j},\\
&[L_{m,i}, M_{n,j}] = nM_{m+n,i+j}, [L_{m,i}, N_{n,j}] = nN_{m+n,i+j}, [L_{m,i}, Y_{n+\frac{1}{2},j}] = (n+\frac{1-m}{2})Y_{m+n+\frac{1}{2},i+j},\\
&[M_{m,i}, Y_{n+\frac{1}{2},j}] = 0, [N_{m,i}, Y_{n+\frac{1}{2},j}] = Y_{m+n+\frac{1}{2},i+j}, [Y_{m+\frac{1}{2},i}, Y_{n+\frac{1}{2},j}] = (m-n)M_{m+n+1,i+j}.
\end{aligned}
$$

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作者简介: 王松 (1981–), 男, 江苏扬州, 讲师, 主要研究方向: 李代数及其表示理论的研究.

对任意的 $m,n,i,j\in\mathbb{Z}$, 其中 $L_{m,i}$ 表示 $L_m\otimes t^i$, 其他定义类似.

由于李代数的二上循环在其中心扩张方面起着关键作用, 可以借助它构造许多无限维李代数, 并且可以进一步刻画所得李代数的结构及表示. 同时上同调群和李代数的结构密切相关, 比如一阶同调群和李代数的导子代数及李双代数的联系, 从而上同调群的计算就显得比较重要. 本文主要确定了扩张的圈 Schrödinger-Virasoro 代数 $\widetilde{\mathscr{W}}$ 的所有二上同调群, 并且给出了它的泛中心扩张. 我们希望借助于中心扩张能够进一步深刻理解 $\widetilde{\mathscr{W}}$ 的结构及其表示.

2 扩张的圈 Schrödinger-Virasoro 代数的二上同调群

李代数 $\widetilde{\mathscr{W}}$ 上的双线性型 $\psi:\widetilde{\mathscr{W}}\times\widetilde{\mathscr{W}}\to\mathbb{C}$ 若满足以下关系

$$\begin{aligned}&\psi(x,y)=-\psi(y,x),\\&\psi(x,[y,z])+\psi(y,[z,x])+\psi(z,[x,y])=0,\end{aligned}$$

对任意的 $x,y\in\widetilde{\mathscr{W}}$, 称 ψ 为 $\widetilde{\mathscr{W}}$ 上的二上循环. 记 $C^2(\widetilde{\mathscr{W}},\ \mathbb{C})$ 为 $\widetilde{\mathscr{W}}$ 上的所有二上循环构成的向量空间. 对于任意的线性函数 $f:\widetilde{\mathscr{W}}\to\mathbb{C}$, 可以定义一个二上循环 ψ_f,

$$\psi_f(x,y)=f([x,y]),\qquad\forall\,x,\,y\in\widetilde{\mathscr{W}},\tag{2.1}$$

称这样的二上循环为 $\widetilde{\mathscr{W}}$ 的二上边缘或平凡的二上循环. 记 $B^2(\widetilde{\mathscr{W}},\ \mathbb{C})$ 为 $\widetilde{\mathscr{W}}$ 上的所有二上边缘构成的向量空间. 如果 $\phi-\psi$ 是一个平凡的二上循环, 则称二上循环 ϕ 和 ψ 等价. 记 $\bar{\phi}$ 为所有与二上循环 ϕ 等价的等价类. 由所有这样的等价类构成的商空间

$$H^2(\widetilde{\mathscr{W}},\ \mathbb{C})=C^2(\widetilde{\mathscr{W}},\ \mathbb{C})/B^2(\widetilde{\mathscr{W}},\ \mathbb{C})$$

称为 $\widetilde{\mathscr{W}}$ 上的二上同调群.

设 ψ 是 $\widetilde{\mathscr{W}}$ 上的任意一个二上循环, 可以利用 ψ 定义一个 $\mathbb{C}$ 上的线性映射 $f:\widetilde{\mathscr{W}}\to\mathbb{C}$ 满足

$$\begin{aligned}&f(L_{m,i})=\frac{1}{m}\psi(L_{0,0},L_{m,i}),\quad\forall\,m\neq0,\\&f(L_{0,i})=-\frac{1}{2}\psi(L_{1,0},L_{-1,i}),\\&f(M_{m,i})=\frac{1}{m}\psi(L_{0,0},M_{m,i}),\quad\forall\,m\neq0,\\&f(M_{0,i})=-\psi(L_{1,0},M_{-1,i}),\\&f(N_{m,i})=\frac{1}{m}\psi(L_{0,0},N_{m,i}),\quad\forall\,m\neq0,\\&f(N_{0,i})=-\psi(L_{1,0},N_{-1,i}),\\&f(Y_{n+\frac{1}{2},i})=\frac{1}{n+\frac{1}{2}}\psi(L_{0,0},Y_{n+\frac{1}{2},i}),\quad\forall\,n,i\in\mathbb{Z}.\end{aligned}$$

令 $\phi=\psi-\psi_f$, 其中 ψ_f 即为 (2.1) 式中的定义, 显然很容易验证

$$\phi(L_{0,0},L_{m,i})=0,\quad\forall\,m\neq0,\tag{2.2}$$

$$\phi(L_{1,0},L_{-1,i})=0,\tag{2.3}$$

$$\phi(L_{0,0},M_{m,i})=0,\quad \forall\ m\neq 0, \tag{2.4}$$

$$\phi(L_{1,0},M_{-1,i})=0, \tag{2.5}$$

$$\phi(L_{0,0},N_{m,i})=0,\quad \forall\ m\neq 0, \tag{2.6}$$

$$\phi(L_{1,0},N_{-1,i})=0, \tag{2.7}$$

$$\phi(L_{0,0},Y_{n+\frac{1}{2},i})=0,\quad \forall\ n,i\in\mathbb{Z}. \tag{2.8}$$

下面通过几个引理给出主要结果.

引理 2.1 $\phi(L_{m,i},Y_{n+\frac{1}{2},j})=0,\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由二上循环 ϕ 的关系知

$$\begin{aligned}(n+\frac{1}{2})\phi(L_{m,i},Y_{n+\frac{1}{2},j}) &= \phi(L_{m,i},[L_{0,0},Y_{n+\frac{1}{2},j}])\\ &= \phi([L_{m,i},L_{0,0}],Y_{n+\frac{1}{2},j})+\phi(L_{0,0},[L_{m,i},Y_{n+\frac{1}{2},j}])\\ &= -m\phi(L_{m,i},Y_{n+\frac{1}{2},j})+(n+\frac{1-m}{2})\phi(L_{0,0},Y_{n+m+\frac{1}{2},j}).\end{aligned}$$

整理得

$$(n+m+\frac{1}{2})\phi(L_{m,i},Y_{n+\frac{1}{2},j})=(n+\frac{1-m}{2})\phi(L_{0,0},Y_{n+m+\frac{1}{2},j}).$$

又由 (2.8) 式, 则上式可得 $(n+m+\frac{1}{2})\phi(L_{m,i},Y_{n+\frac{1}{2},j})=0$, 从而有

$$\phi(L_{m,i},Y_{n+\frac{1}{2},j})=0\quad \forall m,n,i,j\in\mathbb{Z}.$$

引理 2.2 $\phi(N_{m,i},Y_{n+\frac{1}{2},j})=0,\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由二上循环 ϕ 的关系知

$$\begin{aligned}(n+\frac{1}{2})\phi(N_{m,i},Y_{n+\frac{1}{2},j}) &= \phi(N_{m,i},[L_{0,0},Y_{n+\frac{1}{2},j}])\\ &= \phi([N_{m,i},L_{0,0}],Y_{n+\frac{1}{2},j})+\phi(L_{0,0},[N_{m,i},Y_{n+\frac{1}{2},j}])\\ &= -m\phi(N_{m,i},Y_{n+\frac{1}{2},j})+\phi(L_{0,0},Y_{m+n+\frac{1}{2},j}).\end{aligned}$$

整理得

$$(n+m+\frac{1}{2})\phi(N_{m,i},Y_{n+\frac{1}{2},j})=\phi(L_{0,0},Y_{n+m+\frac{1}{2},j}).$$

又由 (2.8) 式, 则上式可得 $(n+m+\frac{1}{2})\phi(N_{m,i},Y_{n+\frac{1}{2},j})=0$, 即有

$$\phi(N_{m,i},Y_{n+\frac{1}{2},j})=0,\quad \forall m,n,i,j\in\mathbb{Z}.$$

引理 2.3 $\phi(M_{m,i},Y_{n+\frac{1}{2},j})=0,\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由二上循环 ϕ 的关系知

$$\begin{aligned}(n+\frac{1}{2})\phi(M_{m,i},Y_{n+\frac{1}{2},j}) &= \phi(M_{m,i},[L_{0,0},Y_{n+\frac{1}{2},j}])\\ &= \phi([M_{m,i},L_{0,0}],Y_{n+\frac{1}{2},j})+\phi(L_{0,0},[M_{m,i},Y_{n+\frac{1}{2},j}])\\ &= -m\phi(M_{m,i},Y_{n+\frac{1}{2},j})+\phi(L_{0,0},0).\end{aligned}$$

整理得

$$(n+m+\frac{1}{2})\phi(M_{m,i},Y_{n+\frac{1}{2},j})=0.$$

则有

$$\phi(M_{m,i},Y_{n+\frac{1}{2},j})=0,\quad \forall\, m,n,i,j\in\mathbb{Z}.$$

引理 2.4 $\phi(L_{m,i},M_{n,j})=\phi(Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j})=0,\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由关系

$$\begin{aligned}
n\phi(L_{m,i},M_{n,j}) &= \phi(L_{m,i},[L_{0,0},M_{n,j}])\\
&= \phi([L_{m,i},L_{0,0}],M_{n,j})+\phi(L_{0,0},[L_{m,i},M_{n,j}])\\
&= -m\phi(L_{m,i},M_{n,j})+n\phi(L_{0,0},M_{m+n,i+j}).
\end{aligned}$$

整理得

$$(n+m)\phi(L_{m,i},M_{n,j})=n\phi(L_{0,0},M_{m+n,i+j}).$$

若 $m+n\neq 0$, 又由 (2.4) 式, 则有

$$\phi(L_{m,i},M_{n,j})=0. \tag{2.9}$$

又由关系

$$\phi([L_{m,i},L_{n,j}],M_{-(m+n),k})+\phi([L_{n,j},M_{-(m+n),k}],L_{m,i})+\phi([M_{-(m+n),k},L_{m,i}],L_{n,j})=0,$$

整理得

$$(n-m)\phi(L_{m+n,i+j},M_{-(m+n),k})+(m+n)\phi(L_{m,i},M_{-m,j+k})-(m+n)\phi(L_{n,j},M_{-n,k+i})=0. \tag{2.10}$$

取 $n=m$, 则有

$$m\phi(L_{m,i},M_{-m,j+k})=m\phi(L_{m,j},M_{-m,k+i}).$$

从而当 $m\neq 0$ 时, 有

$$\phi(L_{m,i},M_{-m,j+k})=\phi(L_{m,j},M_{-m,k+i}). \tag{2.11}$$

在 (2.10) 式中令 $m=-n$, 则有

$$2n\phi(L_{0,i+j},M_{0,k})=0,$$

从而

$$\phi(L_{0,i+j},M_{0,k})=0. \tag{2.12}$$

由 (2.11) 和 (2.12) 式, 得到

$$\phi(L_{m,i},M_{-m,j+k})=\phi(L_{m,j},M_{-m,k+i}),\quad \forall\, m,i,j,k\in\mathbb{Z}. \tag{2.13}$$

上式说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关, 从而不妨设 $A_{m,i+j}=\phi(L_{m,i+j},M_{-m,0})$. 在 (2.10) 式中取 $m=1$, 又由 (2.5) 式有

$$(n-1)A_{n+1,i+j+k}-(n+1)A_{n,i+j+k}=0. \tag{2.14}$$

在 (2.14) 式中用 $n-1$ 替换 n, 则有

$$(n-2)A_{n,i+j+k}-nA_{n-1,i+j+k}=0. \tag{2.15}$$

在 (2.10) 式中取 $n=n-1, m=2$, 可得

$$(n-3)A_{n+1,i+j+k}-(n+1)A_{n-1,i+j+k}=-(n+1)A_{2,i+j+k}. \tag{2.16}$$

将 (2.14),(2.15) 和 (2.16) 式联立方程, 解得

$$A_{n,i+j+k}=\frac{n(n-1)}{2}A_{2,i+j+k}, \quad n\neq -1. \tag{2.17}$$

在 (2.10) 式中令 $m=2, n=-1$, 从而

$$A_{2,i+j+k}=A_{-1,i+j+k}. \tag{2.18}$$

由 (2.17) 和 (2.18) 式, 得到

$$A_{n,i+j}=\frac{n(n-1)}{2}A_{2,i+j}, \quad \forall\, n,i,j\in\mathbb{Z}. \tag{2.19}$$

考虑关系

$$\begin{aligned}(n+\frac{1}{2})\phi(Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j}) &= \phi(Y_{m+\frac{1}{2},i},[L_{0,0},Y_{n+\frac{1}{2},j}])\\ &= \phi([Y_{m+\frac{1}{2},i},L_{0,0}],Y_{n+\frac{1}{2},j})+\phi(L_{0,0},[Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j}])\\ &= -(m+\frac{1}{2})\phi(Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j})+(m-n)\phi(L_{0,0},M_{m+n+1,i+j}).\end{aligned}$$

若 $m+n+1\neq 0$, 由 (2.4) 式, 则上式可得

$$\phi(Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j})=0. \tag{2.20}$$

又由关系

$$\begin{aligned}\phi([Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j}],L_{-(m+n+1),k}) &+ \phi([Y_{n+\frac{1}{2},j},L_{-(m+n+1),k}],Y_{m+\frac{1}{2},i})\\ &+ \phi([L_{-(m+n+1),k},Y_{m+\frac{1}{2},i}],Y_{n+\frac{1}{2},j})=0,\end{aligned}$$

整理得到

$$\begin{aligned}(n-m)\phi(L_{-(m+n+1),k},M_{m+n+1,i+j})+\frac{3n+m+2}{2}\phi(Y_{m+\frac{1}{2},i},Y_{-(m+\frac{1}{2}),j+k})\\ -\frac{3m+n+2}{2}\phi(Y_{n+\frac{1}{2},j},Y_{-(n+\frac{1}{2}),i+k})=0.\end{aligned} \tag{2.21}$$

在 (2.21) 式中取 $m=0$, 则有

$$\frac{n+2}{2}\phi(Y_{n+\frac{1}{2},j},Y_{-(n+\frac{1}{2}),i+k})=n\phi(L_{-(n+1),k},M_{n+1,i+j})+\frac{3n+2}{2}\phi(Y_{\frac{1}{2},i},Y_{-\frac{1}{2},j+k}). \tag{2.22}$$

在上式取 $n=-2$, 结合 (2.5) 式, 从而有

$$\phi(Y_{\frac{1}{2},i},Y_{-\frac{1}{2},j+k})=0. \tag{2.23}$$

由 (2.19),(2.22) 及 (2.23) 式, 可得

$$\phi(Y_{n+\frac{1}{2},j},Y_{-(n+\frac{1}{2}),i+k})=n(n+1)A_{2,i+j+k},\quad n\neq-2. \tag{2.24}$$

上式也说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关. 当 $n=-2$ 时, 由 ϕ 的反对称性及上式知

$$\phi(Y_{-2+\frac{1}{2},j},Y_{-(-2+\frac{1}{2}),i+k})=-\phi(Y_{\frac{3}{2},i+k},Y_{-\frac{3}{2},j})=-2A_{2,i+j+k}. \tag{2.25}$$

在 (2.24) 式中分别取 $n=2,-3$, 则有

$$\phi(Y_{\frac{5}{2},j},Y_{-\frac{5}{2},i+k})=6A_{2,i+j+k},\quad \phi(Y_{-\frac{5}{2},j},Y_{\frac{5}{2},i+k})=6A_{2,i+j+k}.$$

由 ϕ 的反对称性知 $A_{2,i+j+k}=0$. 从而由 (2.19), (2.24) 和 (2.25) 式知

$$\phi(L_{m,i},M_{n,j})=\phi(Y_{m+\frac{1}{2},i},Y_{n+\frac{1}{2},j})=0,\quad \forall\, m,n,i,j\in\mathbb{Z}.$$

引理 2.5 $\phi(M_{m,i},M_{n,j})=0,\quad \forall\, m,n,i,j\in\mathbb{Z}.$

证 由于

$$\begin{aligned} n\phi(M_{m,i},M_{n,j}) &= \phi(M_{m,i},[L_{0,0},M_{n,j}]) \\ &= \phi([M_{m,i},L_{0,0}],M_{n,j}])+\phi(L_{0,0},[M_{m,i},M_{n,j}]) \\ &= -m\phi(M_{m,i},M_{n,j}), \end{aligned}$$

从而有

$$\phi(M_{m,i},M_{n,j})=0,\quad m+n\neq0. \tag{2.26}$$

又由

$$\phi([L_{m,i},M_{n,j}],M_{-(m+n),k})+\phi([M_{n,j},M_{-(m+n),k}],L_{m,i})+\phi([M_{-(m+n),k},L_{m,i}],M_{n,j})=0,$$

则有

$$n\phi(M_{m+n,i+j},M_{-(m+n),k})-(m+n)\phi(M_{n,j},M_{-n,i+k})=0. \tag{2.27}$$

在 (2.27) 式中令 $m=0$, 则有

$$\phi(M_{n,i+j},M_{-n,j})=\phi(M_{n,j},M_{-n,i+k}),\quad n\neq0. \tag{2.28}$$

在 (2.27) 式中令 $n=0$, 可得

$$\phi(M_{0,j},M_{0,i+k})=0. \tag{2.29}$$

综合 (2.28) 和 (2.29) 式, 有

$$\phi(M_{n,i+j},M_{-n,k})=\phi(M_{n,j},M_{-n,i+k}),\quad \forall n,i,j,k\in\mathbb{Z}.$$

上式说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关. 从而不妨令 $B_{n,i+j} = \phi(M_{n,i+j}, M_{-n,0})$, 在式 (2.27) 式中取 $n=1$, 则有

$$B_{m+1,i+j+k} = (m+1)B_{1,i+j+k}, \tag{2.30}$$

又由

$$\begin{aligned}\phi([Y_{m+\frac{1}{2},i}, Y_{n+\frac{1}{2},j}], M_{-(m+n+1),k}) &+ \phi([Y_{n+\frac{1}{2},j}, M_{-(m+n+1),k}], Y_{m+\frac{1}{2},i}) \\ &+ \phi([M_{-(m+n+1),k}, Y_{m+\frac{1}{2},i}], Y_{n+\frac{1}{2},j}) = 0,\end{aligned}$$

从而

$$(m-n)B_{m+n+1,i+j+k} = 0.$$

在上式中取 $m=2, n=-2$, 可得 $B_{1,i+j+k}=0$, 由 (2.30) 式知

$$B_{m+1,i+j+k} = 0, \quad \forall\, m,i,j,k \in \mathbb{Z}. \tag{2.31}$$

综合 (2.26) 和 (2.31) 式, 得到

$$\phi(M_{m,i}, M_{n,j}) = 0, \quad \forall\ m,n,i,j \in \mathbb{Z}.$$

引理 2.6 $\phi(M_{m,i}, N_{n,j}) = 0,\ \forall\ m,n,i,j \in \mathbb{Z}$.

证 由关系

$$\begin{aligned} n\phi(M_{m,i}, N_{n,j}) &= \phi(M_{m,i}, [L_{0,0}, N_{n,j}]) \\ &= \phi([M_{m,i}, L_{0,0}], N_{n,j}) + \phi(L_{0,0}, [M_{m,i}, N_{n,j}]) \\ &= -m\phi(M_{m,i}, N_{n,j}) - 2\phi(L_{0,0}, M_{m+n,i+j}),\end{aligned}$$

从而由 (2.4) 式得

$$\phi(M_{m,i}, N_{n,j}) = 0, \quad m+n \neq 0. \tag{2.32}$$

又由关系

$$\phi([L_{m,i}, M_{n,j}], N_{-(m+n),k}) + \phi([M_{n,j}, N_{-(m+n),k}], L_{m,i}) + \phi([N_{-(m+n),k}, L_{m,i}], M_{n,j}) = 0,$$

结合引理 2.4, 整理得

$$n\phi(M_{m+n,i+j}, N_{-(m+n),k}) - (m+n)\phi(M_{n,j}, N_{-n,k+i}) = 0. \tag{2.33}$$

若 $m=0$, 则有

$$\phi(M_{n,i+j}, N_{-n,k}) = \phi(M_{n,j}, M_{-n,k+i}), \quad n \neq 0. \tag{2.34}$$

在 (2.33) 式中令 $n=0$, 可得

$$\phi(M_{0,j}, N_{0,k+i}) = 0. \tag{2.35}$$

由 (2.33) 和 (2.35) 式, 有

$$\phi(M_{n,i+j}, N_{-n,k}) = \phi(M_{n,j}, M_{-n,k+i}), \quad \forall\, n,i,j,k \in \mathbb{Z}. \tag{2.36}$$

上式说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关. 不妨设 $C_{n,i+j}=\phi(M_{n,i+j},N_{-n,0})$, 在 (2.33) 中令 $n=1$, 则有

$$C_{m+1,i+j+k}=(m+1)C_{1,i+j+k}. \tag{2.37}$$

又由关系

$$\begin{aligned}\phi([Y_{n+\frac{1}{2},i},Y_{m+\frac{1}{2},j}],N_{-(m+n+1),k}) &+ \phi([Y_{m+\frac{1}{2},j},N_{-(m+n+1),k}],Y_{n+\frac{1}{2},i})\\ &+ \phi([N_{-(m+n+1),k},Y_{n+\frac{1}{2},i}],Y_{m+\frac{1}{2},j})=0,\end{aligned}$$

又由引理 2.4, 整理得

$$(n-m)C_{m+n+1,i+j+k}=0.$$

在上式令 $n=1,m=-1$, 则有 $C_{1,i+j+k}=0$, 从而由 (2.37) 式, 知

$$C_{m+1,i+j+k}=0,\quad \forall\, m,i,j,k\in\mathbb{Z}. \tag{2.38}$$

由上式结合 (2.32) 式, 就可得到

$$\phi(M_{m,i},N_{n,j})=0,\quad \forall\ m,n,i,j\in\mathbb{Z}.$$

引理 2.7 $\phi(L_{m,i},N_{n,j})=\delta_{m+n,0}\frac{m(m-1)}{2}\phi(L_{2,i+j},N_{-2,0}),\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由关系

$$\begin{aligned}n\phi(L_{m,i},N_{n,j}) &= \phi(L_{m,i},[L_{0,0},N_{n,j}])\\ &= \phi([L_{m,i},L_{0,0}],N_{n,j})+\phi(L_{0,0},[L_{m,i},N_{n,j}])\\ &= -m\phi(L_{m,i},N_{n,j})+n\phi(L_{0,0},N_{m+n,i+j}).\end{aligned}$$

由 (2.6) 式, 则有

$$\phi(L_{m,i},N_{n,j})=0,\quad m+n\neq 0. \tag{2.39}$$

又由关系

$$\phi([L_{m,i},L_{n,j}],N_{-(m+n),k})+\phi([L_{n,j},N_{-(m+n),k}],L_{m,i})+\phi([N_{-(m+n),k},L_{m,i}],L_{n,j})=0,$$

整理得

$$(n-m)\phi(L_{m+n,i+j},N_{-(m+n),k})+(m+n)\phi(L_{m,i},N_{-m,j+k})-(m+n)\phi(L_{n,j},N_{-n,k+i})=0. \tag{2.40}$$

在上式中取 $m=n$, 则有

$$\phi(L_{n,i},N_{-n,j+k})=\phi(L_{n,j},N_{-n,k+i}),\quad n\neq 0. \tag{2.41}$$

在 (2.40) 式中取 $m=-n$, 可得

$$\phi(L_{0,i+j},N_{0,k})=0. \tag{2.42}$$

由 (2.41) 和 (2.42) 式, 则有

$$\phi(L_{n,i},N_{-n,j+k})=\phi(L_{n,j},N_{-n,k+i})\quad \forall\, n,i,j,k\in\mathbb{Z}. \tag{2.43}$$

上式说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关. 不妨设 $D_{n,i+j}=\phi(L_{n,i+j},N_{-n,0})$, 在 (2.40) 式中取 $n=1$, 结合 (2.7) 式, 则有

$$(1-m)D_{m+1}+(m+1)D_m=0. \tag{2.44}$$

在上式用 $m-1$ 替换 m, 有

$$(2-m)D_m+mD_{m-1}=0. \tag{2.45}$$

在 (2.40) 式中令 $n=2, m=m-1$, 得到

$$(3-m)D_{m+1}+(m+1)D_{m-1}=(m+1)D_2. \tag{2.46}$$

将式 (2.44),(2.45) 和 (2.46) 式联立方程, 解得

$$D_m=\frac{m(m-1)}{2}D_2,\quad m\neq -1. \tag{2.47}$$

在 (2.40) 式中取 $m=2, n=-1$, 结合 (2.7) 式, 易得 $D_2=D_{-1}$, 从而由 (2.47) 式就有

$$D_m=\frac{m(m-1)}{2}D_2,\quad \forall\, m\in\mathbb{Z}.$$

由上式和 (2.39) 式, 有

$$\phi(L_{m,i},N_{n,j})=\delta_{m+n,0}\frac{m(m-1)}{2}\phi(L_{2,i+j},N_{-2,0}),\quad \forall\ m,n,i,j\in\mathbb{Z}.$$

引理 2.8 $\phi(N_{m,i},N_{n,j})=\delta_{m+n,0}m\phi(N_{1,i+j},N_{-1,0}),\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由关系

$$\begin{aligned} n\phi(N_{m,i},N_{n,j}) &= \phi(N_{m,i},[L_{0,0},N_{n,j}]) \\ &= \phi([N_{m,i},L_{0,0}],N_{n,j})+\phi(L_{0,0},[N_{n,j},N_{m,i}]) \\ &= -m\phi(N_{m,i},N_{n,j}), \end{aligned}$$

整理得

$$\phi(N_{m,i},N_{n,j})=0,\quad m+n\neq 0. \tag{2.48}$$

又由关系

$$\phi([L_{m,i},N_{n,j}],N_{-(m+n),k})+\phi([N_{n,j},N_{-(m+n),k}],L_{m,i})+\phi([N_{-(m+n),k},L_{m,i}],N_{n,j})=0,$$

从而有

$$n\phi(N_{m+n,i+j},N_{-(m+n),k})-(m+n)\phi(N_{n,j},N_{-n,k+i})=0. \tag{2.49}$$

在上式取 $m=0$, 则有

$$\phi(N_{n,i+j},N_{-n,k})=\phi(N_{n,j},N_{-n,k+i}),\quad n\neq 0,\tag{2.50}$$

在 (2.49) 式中取 $m=-n$, 可得

$$\phi(N_{0,i+j},N_{0,k})=0,\tag{2.51}$$

由 (2.50) 和 (2.51) 式, 有

$$\phi(N_{n,i+j},N_{-n,k})=\phi(N_{n,j},N_{-n,k+i}),\quad \forall\, n,i,j,k\in\mathbb{Z}.\tag{2.52}$$

上式说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关. 不妨设 $E_{n,i+j}=\phi(N_{n,i+j},N_{-n,0})$, 在 (2.49) 式中令 $n=1$, 就有

$$E_{m+1,i+j+k}=(m+1)E_{1,i+j+k},\quad \forall\, m,i,j,k\in\mathbb{Z}.\tag{2.53}$$

由 (2.48) 和 (2.53) 式, 则有 $\phi(N_{m,i},N_{n,j})=\delta_{m+n,0}m\phi(N_{1,i+j},N_{-1,0}),\quad \forall\ m,n,i,j\in\mathbb{Z}.$

引理 2.9 $\phi(L_{m,i},L_{n,j})=\delta_{m+n,0}\frac{m^3-m}{6}\phi(L_{2,i+j},L_{-2,0}),\quad \forall\ m,n,i,j\in\mathbb{Z}.$

证 由关系

$$\begin{aligned}
n\phi(L_{m,i},L_{n,j}) &= \phi(L_{m,i},[L_{0,0},L_{n,j}])\\
&= \phi([L_{m,i},L_{0,0}],L_{n,j})+\phi(L_{0,0},[L_{m,i},L_{n,j}])\\
&= -m\phi(L_{m,i},L_{n,j})+(n-m)\phi(L_{0,0},L_{m+n,i+j}).
\end{aligned}$$

由上式及 (2.2) 式可得

$$\phi(L_{m,i},L_{n,j})=0,\quad m+n\neq 0.\tag{2.54}$$

考虑关系

$$\phi([L_{m,i},L_{n,j}],L_{-(m+n),k})+\phi([L_{n,j},L_{-(m+n),k}],L_{m,i})+\phi([L_{-(m+n),k},L_{m,i}],L_{n,j})=0,$$

整理得

$$(n-m)\phi(L_{m+n,i+j},L_{-(m+n),k})+(m+2n)\phi(L_{m,i},L_{-m,j+k})-(2m+n)\phi(L_{n,j},L_{-n,k+i})=0.\tag{2.55}$$

在上式中令 $n=m$, 则有

$$\phi(L_{m,i},L_{-m,j+k})=0,\quad m\neq 0.\tag{2.56}$$

在 (2.55) 式中取 $m=-n=1$, 结合上式可得

$$\phi(L_{0,i+j},L_{0,k})=0,\tag{2.57}$$

由 (2.56) 和 (2.57) 式, 有

$$\phi(L_{m,i},L_{-m,j+k})=0,\quad \forall\, m,j,k\in\mathbb{Z}.\tag{2.58}$$

上式说明仅与第二个指标的和 $i+j+k$ 有关, 而与位置无关. 不妨设 $F_{n,i+j}=\phi(L_{m,i+j},L_{-m,0})$, 在 (2.55) 式中令 $m=1$, 结合 (2.3) 式, 则有

$$(n-1)F_{n+1,i+j+k}-(n+2)F_{n,i+j+k}=0. \tag{2.59}$$

将上式 n 由 $n-1$ 替代, 从而

$$(n-2)F_{n,i+j+k}-(n+1)F_{n-1,i+j+k}=0. \tag{2.60}$$

在 (2.55) 式中取 $m=2,n=n-1$, 可得

$$(n-3)F_{n+1,i+j+k}-(n+3)F_{n-1,i+j+k}=-2nF_{2,i+j+k}. \tag{2.61}$$

将 (2.59),(2.60) 和 (2.61) 式联立方程组, 解得

$$F_{n,i+j+k}=\frac{n^3-n}{6}F_{2,i+j+k}.$$

由上式及 (2.54) 式, 就得到 $\phi(L_{m,i},L_{n,j})=\delta_{m+n,0}\frac{m^3-m}{6}\phi(L_{2,i+j},L_{-2,0})$, $\forall\, m,n,i,j\in\mathbb{Z}$.

由引理 2.1–2.9, 我们就可以得到本文的主要定理.

定理 2.1 二上同调群

$$H^2(\widetilde{\mathscr{W}},\ \mathbb{C})=\prod_{k\in\mathbb{Z},\ i\in I}\mathbb{C}\bar{\phi}_{k,i},$$

其中 $I=\{1,2,3\}$, $\mathbb{C}\bar{\phi}_{k,i}$ 表示所有的直积, $\bar{\phi}_{k,i}$ 是 $\phi_{k,i}$ 的同调类, 满足

$$\begin{aligned}
\phi_{k,1}(L_{m,i},L_{n,j}) &= \delta_{m+n,0}\delta_{i+j,k}\frac{m^3-m}{12},\\
\phi_{k,2}(L_{m,i},N_{n,j}) &= \delta_{m+n,0}\delta_{i+j,k}(m^2-m),\\
\phi_{k,3}(N_{m,i},N_{n,j}) &= \delta_{m+n,0}\delta_{i+j,k}m,
\end{aligned}$$

其他关系均为 0, 对于 $m,n,i,j,k\in\mathbb{Z}$.

注 考虑中心扩张 $\widehat{\mathscr{W}}=\widetilde{\mathscr{W}}\bigoplus C^1\otimes\mathbb{C}[t,t^{-1}]\bigoplus C^2\otimes\mathbb{C}[t,t^{-1}]\bigoplus C^3\otimes\mathbb{C}[t,t^{-1}]$, 满足李积关系: 对于任意给定的 $k\in\mathbb{Z}$, 有

$$\begin{aligned}
&[L_{m,i},L_{n,j}]=(n-m)L_{m+n,i+j}+\delta_{m+n,0}\delta_{i+j,k}\frac{m^3-m}{12}C_k^1,\\
&[L_{m,i},N_{n,j}]=nN_{m+n,i+j}+\delta_{m+n,0}\delta_{i+j,k}(m^2-m)C_k^2,\\
&[N_{m,i},N_{n,j}]=2M_{m+n,i+j}+\delta_{m+n,0}\delta_{i+j,k}mC_k^3,\\
&[\widehat{\mathscr{W}},C_k^t]=0,\quad \forall\, m,n,i,j\in\mathbb{Z},
\end{aligned}$$

其中 $C_k^t=C^t\otimes t^k, t=1,2,3$, 其他李积关系与 $\widetilde{\mathscr{W}}$ 一致. 定理 2.1 表明 $\widehat{\mathscr{W}}$ 是 $\widetilde{\mathscr{W}}$ 的泛中心扩张.

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SECOND COHOMOLOGY GROUPS OF THE EXTENDED LOOP SCHÖDINGER-VIRASORO ALGEBRAS

WANG Song , WANG Xiao-ming

(*College of Information Technology, Shanghai Ocean University, Shanghai 201306, China*)

Abstract: In this paper, we study the extended loop Schrödinger-Virasoro algebras and give the second cohomology groups of the extended loop Schrödinger-Virasoro algebras are determined. Moreover, we obtain the universal central extensions of the extended loop Schrödinger-Virasoro algebras are given.

Keywords: Schrödinger-Virasoro algebra; 2-cocycle; central extension

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